7.2 Analysis of Three Dimensional Stress and Strain

The concept of traction and stress was introduced and discussed in Book I, §3. For the most part, the discussion was confined to two-dimensional states of stress. Here, the fully three dimensional stress state is examined. There will be some repetition of the earlier analyses.

7.2.1 The Traction Vector and Stress Components

Consider a traction vector **t** acting on a surface element, Fig. 7.2.1. Introduce a Cartesian coordinate system with base vectors $\mathbf{e}_i$ so that one of the base vectors is a normal to the surface and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 7.1.1, the $\mathbf{e}_3$ direction is taken to be normal to the plane, and a superscript on **t** denotes this normal:

$$\mathbf{t}^{(\mathbf{e}_3)} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3 \tag{7.2.1}$$

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the direction of the normal and the second denotes the direction of the component to the plane. Thus the three components of the traction vector shown in Fig. 7.2.1 are $\sigma_{31}, \sigma_{32}, \sigma_{33}$:

$$\mathbf{t}^{(\mathbf{e}_3)} = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \tag{7.2.2}$$

The first two stresses, the components acting tangential to the surface, are shear stresses whereas σ_{33}, acting normal to the plane, is a normal stress.

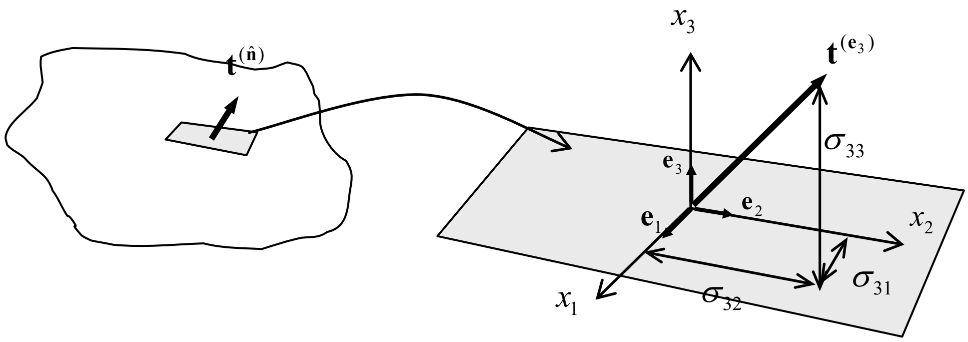

Figure 7.2.1: components of the traction vector

Consider the three traction vectors $\mathbf{t}^{(\mathbf{e}_1)}, \mathbf{t}^{(\mathbf{e}_2)}, \mathbf{t}^{(\mathbf{e}_3)}$ acting on the surface elements whose outward normals are aligned with the three base vectors $\mathbf{e}_j$, Fig. 7.2.2a. The three (or six) surfaces can be amalgamated into one diagram as in Fig. 7.2.2b.

In terms of stresses, the traction vectors are

$$\begin{aligned}\mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3 \qquad \text{or} \qquad \mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ij}\mathbf{e}_j \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3\end{aligned} \tag{7.2.3}$$

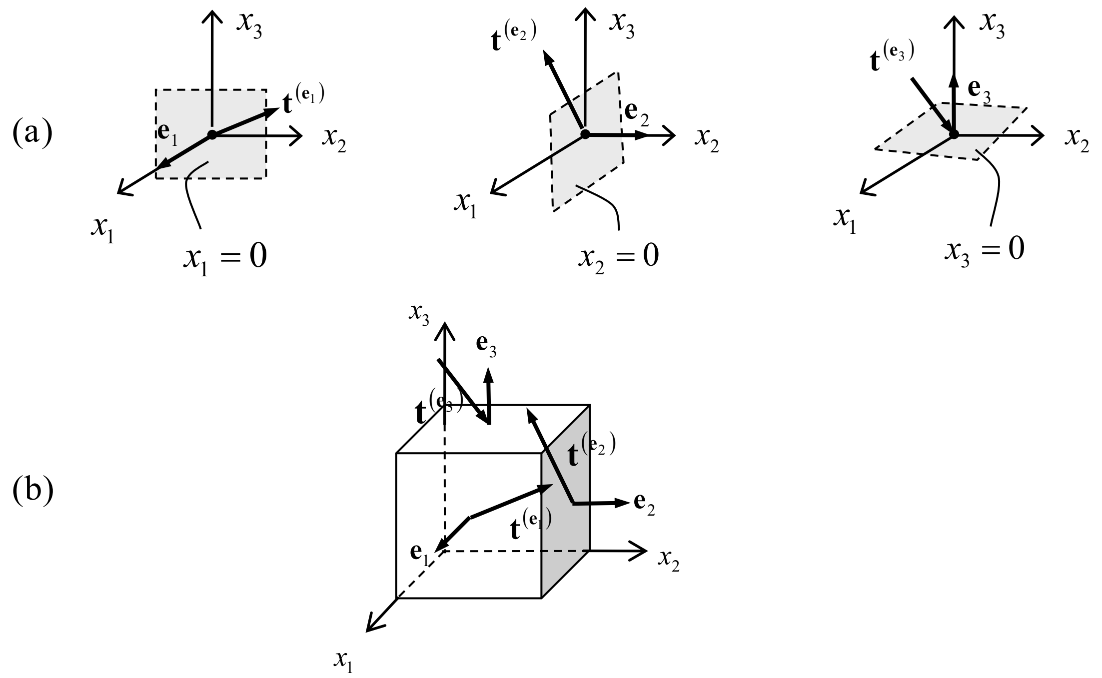

Figure 7.2.2: the three traction vectors acting at a point; (a) on mutually orthogonal planes, (b) the traction vectors illustrated on a box element

The components of the three traction vectors, i.e. the stress components, can now be displayed on a box element as in Fig. 7.2.3. Note that the stress components will vary slightly over the surfaces of an elemental box of finite size. However, it is assumed that the element in Fig. 7.2.3 is small enough that the stresses can be treated as constant, so that they are the stresses acting *at* the origin.

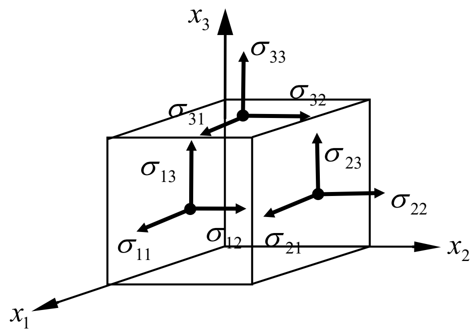

Figure 7.2.3: the nine stress components with respect to a Cartesian coordinate system

The nine stresses can be conveniently displayed in 3×3 matrix form:

$$\left[\sigma_{ij}\right]=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} \tag{7.2.4}$$

It is important to realise that, if one were to take an element at some different orientation to the element in Fig. 7.2.3, but at the *same material particle*, for example aligned with the axes x_1', x_2', x_3' shown in Fig. 7.2.4, one would then have different tractions acting and the nine stresses would be different also. The stresses acting in this new orientation can be represented by a new matrix:

$$\left[\sigma'_{ij}\right]=\begin{bmatrix}\sigma'_{11} & \sigma'_{12} & \sigma'_{13}\\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23}\\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33}\end{bmatrix} \tag{7.2.5}$$

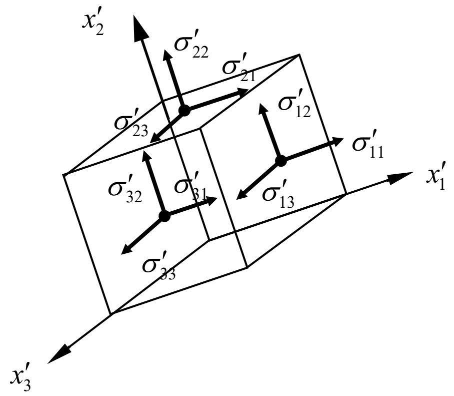

Figure 7.2.4: the stress components with respect to a Cartesian coordinate system different to that in Fig. 7.2.3

7.2.2 Cauchy's Law

Cauchy's Law, which will be proved below, states that the normal to a surface, $\mathbf{n}=n_i\mathbf{e}_i$, is related to the traction vector $\mathbf{t}^{(\mathbf{n})}=t_i\mathbf{e}_i$ acting on that surface, according to

$$t_i=\sigma_{ji}n_j \tag{7.2.6}$$

Writing the traction and normal in vector form and the stress in 3×3 matrix form,

$$\left[t_i^{(\mathbf{n})}\right]=\begin{bmatrix}t_1^{(\mathbf{n})}\\ t_2^{(\mathbf{n})}\\ t_3^{(\mathbf{n})}\end{bmatrix},\quad \left[\sigma_{ij}\right]=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix},\quad \left[n_i\right]=\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix} \tag{7.2.7}$$

and Cauchy's law in matrix notation reads

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad (7.2.8)$$

Note that it is the transpose stress matrix which is used in Cauchy's law. Since the stress matrix is symmetric, one can express Cauchy's law in the form

$$\boxed{t_i = \sigma_{ij} n_j} \qquad \textbf{Cauchy's Law} \qquad (7.2.9)$$

Cauchy's law is illustrated in Fig. 7.2.5; in this figure, positive stresses σ_{ij} are shown.

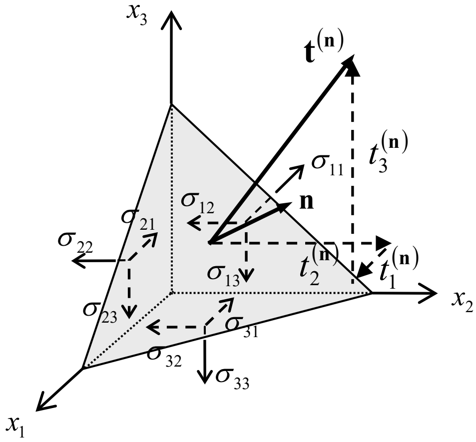

Figure 7.2.5: Cauchy's Law; given the stresses and the normal to a plane, the traction vector acting on the plane can be determined

Normal and Shear Stress

It is useful to be able to evaluate the normal stress σ_N and shear stress σ_S acting on any plane, Fig. 7.2.6. For this purpose, note that the stress acting normal to a plane is the projection of $\mathbf{t}^{(\mathbf{n})}$ in the direction of $\mathbf{n}$,

$$\sigma_N = \mathbf{n} \cdot \mathbf{t}^{(\mathbf{n})} \qquad (7.2.10)$$

The magnitude of the shear stress acting on the surface can then be obtained from

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} \qquad (7.2.11)$$

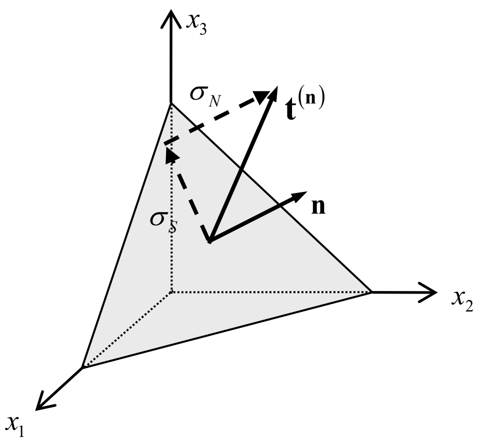

Figure 7.2.6: the normal and shear stress acting on an arbitrary plane through a point

Example

The state of stress at a point with respect to a Cartesian coordinates system $0x_1x_2x_3$ is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine:

(a) the traction vector acting on a plane through the point whose unit normal is $\mathbf{n} = (1/3)\mathbf{e}_1 + (2/3)\mathbf{e}_2 - (2/3)\mathbf{e}_3$

(b) the component of this traction acting perpendicular to the plane

(c) the shear component of traction on the plane

<u>Solution</u>

(a) From Cauchy's law,

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (-2/3)\mathbf{e}_1 + 3\mathbf{e}_2 - \hat{\mathbf{e}}_3$.

(b) The component normal to the plane is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

(c) The shearing component of traction is

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \left\{\left[\left(-\tfrac{2}{3}\right)^2 + (3)^2 + (-1)^2\right] - \left[\left(\tfrac{22}{9}\right)^2\right]\right\}^{1/2} \approx 2.1$$

■

Proof of Cauchy's Law

Cauchy's law can be proved using force equilibrium of material elements. First, consider a tetrahedral free-body, with vertex at the origin, Fig. 7.2.7. It is required to determine the traction **t** in terms of the nine stress components (which are all shown positive in the diagram).

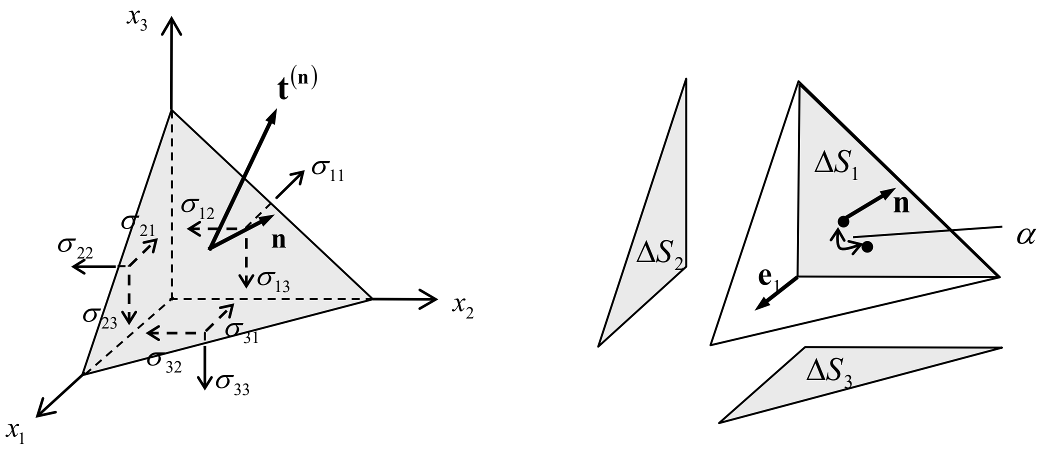

Figure 7.2.7: proof of Cauchy's Law

The components of the unit normal, n_i, are the direction cosines of the normal vector, i.e. the cosines of the angles between the normal and each of the coordinate directions:

$$\cos(\mathbf{n}, \mathbf{e}_i) = \mathbf{n} \cdot \mathbf{e}_i = n_i \tag{7.2.12}$$

Let the area of the base of the tetrahedron, with normal **n**, be ΔS. The area ΔS_1 is then $\Delta S \cos\alpha$, where α is the angle between the planes, as shown to the right of Fig. 7.2.7; this angle is the same as that between the vectors **n** and $\mathbf{e}_1$, so $\Delta S_1 = n_1 \Delta S$, and similarly for the other surfaces:

$$\Delta S_i = n_i \Delta S \tag{7.2.13}$$

The resultant surface force on the body, acting in the x_i direction, is then

$$\sum F_i = t_i \Delta S - \sigma_{ji} \Delta S_j = t_i \Delta S - \sigma_{ji} n_j \Delta S \tag{7.2.14}$$

For equilibrium, this expression must be zero, and one arrives at Cauchy's law.

<u>Note:</u>
As proved in Book III, this result holds also in the general case of accelerating material elements in the presences of body forces.

7.2.3 The Stress Tensor

Cauchy's law 7.2.9 is of the same form as 7.1.24 and so by definition the stress is a tensor. Denote the stress tensor in symbolic notation by $\boldsymbol{\sigma}$. Cauchy's law in symbolic form then reads

$$\mathbf{t} = \boldsymbol{\sigma}\,\mathbf{n} \tag{7.2.15}$$

Further, the transformation rule for stress follows the general tensor transformation rule 7.1.31:

$$\boxed{\begin{aligned} \sigma_{ij} &= Q_{ip}Q_{jq}\sigma'_{pq} \qquad \ldots \quad [\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}] \\ \sigma'_{ij} &= Q_{pi}Q_{qj}\sigma_{pq} \qquad \ldots \quad [\boldsymbol{\sigma}'] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}][\mathbf{Q}] \end{aligned}} \quad \textbf{Stress Transformation Rule} \tag{7.2.16}$$

As with the normal and traction vectors, the components and hence matrix representation of the stress changes with coordinate system, as with the two different matrix representations 7.2.4 and 7.2.5. However, there is only one stress tensor $\boldsymbol{\sigma}$ at a point. Another way of looking at this is to note that an infinite number of planes pass through a point, and on each of these planes acts a traction vector, and each of these traction vectors has three (stress) components. *All* of these traction vectors taken together define the complete **state of stress** at a point.

Example

The state of stress at a point with respect to an $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) What are the stress components with respect to axes $0x'_1x'_2x'_3$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_2 axis, Fig. 7.2.8?

(b) Use Cauchy's law to evaluate the normal and shear stress on a plane with normal $\mathbf{n} = \left(1/\sqrt{2}\right)\mathbf{e}_1 + \left(1/\sqrt{2}\right)\mathbf{e}_3$ and relate your result with that from (a)

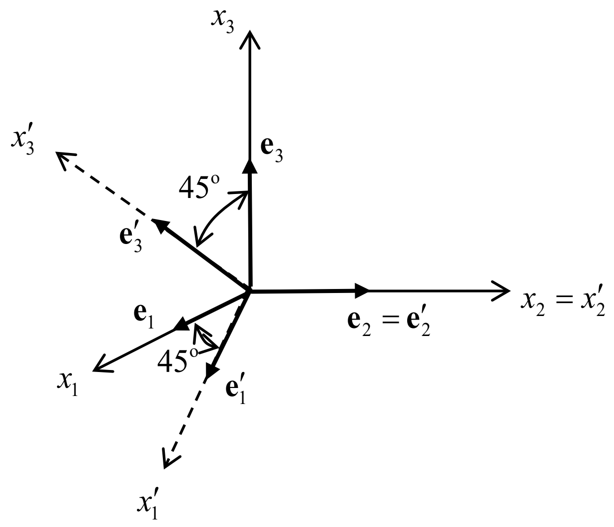

Figure 7.2.8: two different coordinate systems at a point

Solution

(a) The transformation matrix is

$$[Q_{ij}] = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and $\mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$ as expected. The rotated stress components are therefore

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{\sqrt{2}} & \frac{1}{2} \\ \frac{3}{\sqrt{2}} & 3 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{3}{2} \end{bmatrix}$$

and the new stress matrix is symmetric as expected.

(b) From Cauchy's law, the traction vector is

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (\sqrt{2})\mathbf{e}_1 - (1/\sqrt{2})\mathbf{e}_2 + (1/\sqrt{2})\mathbf{e}_3$. The normal and shear stress on the plane are

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = 3/2$$

and

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \sqrt{3 - (3/2)^2} = \sqrt{3}/2$$

The normal to the plane is equal to $\mathbf{e}_3'$ and so σ_N should be the same as σ_{33}' and it is. The stress σ_S should be equal to $\sqrt{(\sigma_{31}')^2 + (\sigma_{32}')^2}$ and it is. The results are displayed in Fig. 7.2.9, in which the traction is represented in different ways, with components $\left(t_1^{(\mathbf{n})}, t_2^{(\mathbf{n})}, t_3^{(\mathbf{n})}\right)$ and $\left(\sigma'_{31}, \sigma'_{32}, \sigma'_{33}\right)$.

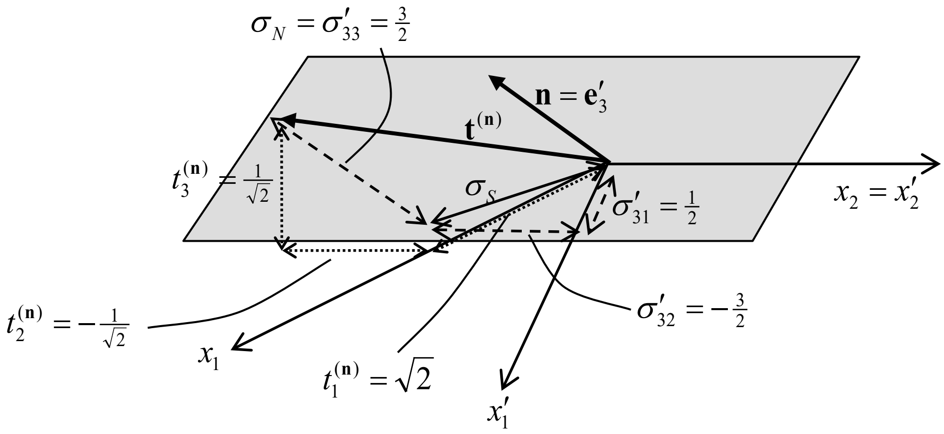

Figure 7.2.9: traction and stresses acting on a plane

Isotropic State of Stress

Suppose the state of stress in a body is

$$\sigma_{ij} = \sigma_0 \delta_{ij} \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \tag{7.2.17}$$

One finds that the application of the stress tensor transformation rule yields the very same components no matter what the new coordinate system {▲Problem 3}. In other words, no shear stresses act, no matter what the orientation of the plane through the point. This is termed an **isotropic state of stress**, or a **spherical state of stress**. One example of isotropic stress is the stress arising in a fluid at rest, which cannot support shear stress, in which case

$$[\boldsymbol{\sigma}] = -p[\mathbf{I}] \tag{7.2.18}$$

where the scalar p is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic state of stress**.

7.2.4 Principal Stresses

For certain planes through a material particle, there are traction vectors which act normal to the plane, as in Fig. 7.2.10. In this case the traction can be expressed as a scalar multiple of the normal vector, $\mathbf{t}^{(\mathbf{n})} = \sigma \mathbf{n}$.

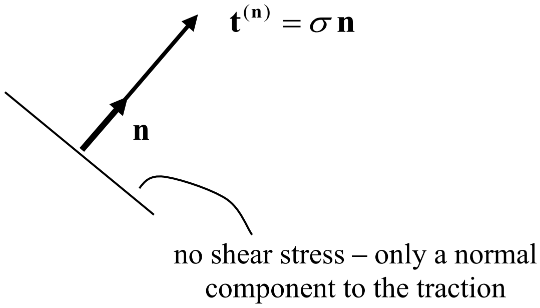

Figure 7.2.10: a purely normal traction vector

From Cauchy's law then, for these planes,

$$\boldsymbol{\sigma}\,\mathbf{n} = \sigma\,\mathbf{n}, \quad \sigma_{ij}n_j = \sigma n_i, \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.19}$$

This is a standard **eigenvalue problem** from Linear Algebra: given a matrix $\lfloor \sigma_{ij} \rfloor$, find the **eigenvalues** σ and associated **eigenvectors n** such that Eqn. 7.2.19 holds.
To solve the problem, first re-write the equation in the form

$$(\boldsymbol{\sigma} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}, \quad (\sigma_{ij} - \sigma\delta_{ij})n_j = 0, \quad \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.2.20}$$

or

$$\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.2.21}$$

This is a set of three homogeneous equations in three unknowns (if one treats σ as known). From basic linear algebra, this system has a solution (apart from $n_i = 0$) if and only if the determinant of the coefficient matrix is zero, i.e. if

$$\det(\boldsymbol{\sigma} - \sigma\mathbf{I}) = \det \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} = 0 \tag{7.2.22}$$

Evaluating the determinant, one has the following cubic **characteristic equation** of the stress tensor $\boldsymbol{\sigma}$,

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \qquad \textbf{Characteristic Equation} \tag{7.2.23}$$

and the **principal scalar invariants** of the stress tensor are

$$\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{31}
\end{aligned} \tag{7.2.24}$$

(I_3 is the determinant of the stress matrix.) The characteristic equation 7.2.23 can now be solved for the eigenvalues σ and then Eqn. 7.2.21 can be used to solve for the eigenvectors **n**.

Now another theorem of linear algebra states that the eigenvalues of a real (that is, the components are real), symmetric matrix (such as the stress matrix) are all real and further that the associated eigenvectors are mutually orthogonal. This means that the three roots of the characteristic equation are real and that the three associated eigenvectors form a mutually orthogonal system. This is illustrated in Fig. 7.2.11; the eigenvalues are called **principal stresses** and are labelled $\sigma_1, \sigma_2, \sigma_3$ and the three corresponding eigenvectors are called **principal directions**, the directions in which the principal stresses act. The planes on which the principal stresses act (to which the principal directions are normal) are called the **principal planes**.

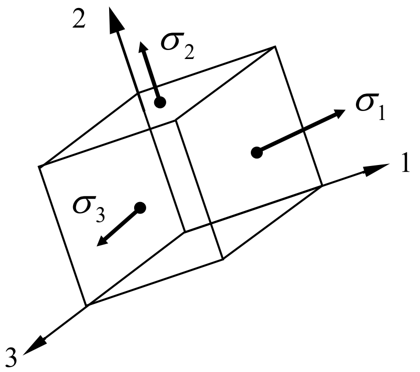

Figure 7.2.11: the three principal stresses acting at a point and the three associated principal directions 1, 2 and 3

Once the principal stresses are found, as mentioned, the principal directions can be found by solving Eqn. 7.2.21, which can be expressed as

$$\begin{aligned}
(\sigma_{11} - \sigma)n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 &= 0 \\
\sigma_{21}n_1 + (\sigma_{22} - \sigma)n_2 + \sigma_{23}n_3 &= 0 \\
\sigma_{31}n_1 + \sigma_{32}n_2 + (\sigma_{33} - \sigma)n_3 &= 0
\end{aligned} \tag{7.2.25}$$

Each principal stress value in this equation gives rise to the three components of the associated principal direction vector, n_1, n_2, n_3. The solution also requires that the magnitude of the normal be specified: for a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$. The directions of the normals are also chosen so that they form a right-handed set.

Example

The stress at a point is given with respect to the axes $Ox_1x_2x_3$ by the values

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}.$$

Determine (a) the principal values, (b) the principal directions (and sketch them).

<u>Solution:</u>
(a)
The principal values are the solution to the characteristic equation

$$\begin{vmatrix} 5-\sigma & 0 & 0 \\ 0 & -6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{vmatrix} = (-10+\sigma)(5-\sigma)(15+\sigma) = 0$$

which yields the three principal values $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$.
(b)
The eigenvectors are now obtained from Eqn. 7.2.25. First, for $\sigma_1 = 10$,

$$\begin{aligned} -5n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 16n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 9n_3 &= 0 \end{aligned}$$

and using also the equation $n_1^2 + n_2^2 + n_3^2 = 1$ leads to $\mathbf{n}_1 = -(3/5)\mathbf{e}_2 + (4/5)\mathbf{e}_3$. Similarly, for $\sigma_2 = 5$ and $\sigma_3 = -15$, one has, respectively,

$$\begin{aligned} 0n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 11n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 4n_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 20n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 + 9n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 + 16n_3 &= 0 \end{aligned}$$

which yield $\mathbf{n}_2 = \mathbf{e}_1$ and $\mathbf{n}_3 = (4/5)\mathbf{e}_2 + (3/5)\mathbf{e}_3$. The principal directions are sketched in Fig. 7.2.12. Note that the three components of each principal direction, n_1, n_2, n_3, are the direction cosines: the cosines of the angles between that principal direction and the three coordinate axes. For example, for σ_1 with $n_1 = 0, n_2 = -3/5, n_3 = 4/5$, the angles made with the coordinate axes x_1, x_2, x_3 are, respectively, 90, 126.87° and 36.87°.

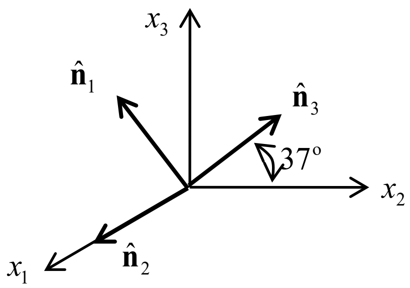

Figure 7.2.12: principal directions

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Invariants

The principal stresses $\sigma_1, \sigma_2, \sigma_3$ are independent of any coordinate system; the $0x_1x_2x_3$ axes to which the stress matrix in Eqn. 7.2.19 is referred can have any orientation – the same principal stresses will be found from the eigenvalue analysis. This is expressed by using the symbolic notation for the problem: $\boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n}$, which is independent of any coordinate system. Thus the principal stresses are intrinsic properties of the stress state at a point. It follows that the functions I_1, I_2, I_3 in the characteristic equation Eqn. 7.2.23 are also independent of any coordinate system, and hence the name principal scalar invariants (or simply **invariants**) of the stress.

The stress invariants can also be written neatly in terms of the principal stresses:

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{7.2.26}$$

Also, if one chooses a coordinate system to coincide with the principal directions, Fig. 7.2.12, the stress matrix takes the simple form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{7.2.27}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions – the stress matrix has the form 7.2.27 no matter what orientation the planes through the point.

Example

The two stress matrices from the Example of §7.2.3, describing the stress state at a point with respect to different coordinate systems, are

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \qquad [\sigma'_{ij}] = \begin{bmatrix} 3/2 & 3/\sqrt{2} & 1/2 \\ 3/\sqrt{2} & 3 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 3/2 \end{bmatrix}$$

The first invariant is the sum of the normal stresses, the diagonal terms, and is the same for both as expected:

$$I_1 = 2 + 3 + 1 = \tfrac{3}{2} + 3 + \tfrac{3}{2} = 6$$

The other invariants can also be obtained from either matrix, and are

$$I_2 = 6, \;\; I_3 = -3$$

■

7.2.5 Maximum and Minimum Stress Values

Normal Stresses

The three principal stresses include the maximum and minimum normal stress components acting at a point. To prove this, first let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions*. Consider next an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. From Cauchy's law (see Fig. 7.2.13 – the stress matrix in Cauchy's law is now with respect to the principal directions 1, 2 and 3), the normal stress acting on the plane with normal $\mathbf{n}$ is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}, \qquad \sigma_N = \sigma_{ij} n_j n_i \tag{7.2.28}$$

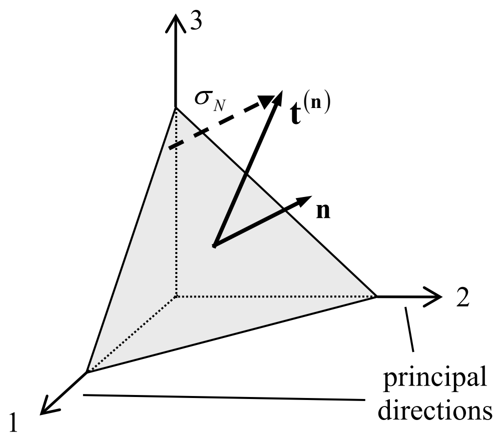

Figure 7.2.13: normal stress acting on a plane defined by the unit normal n

Thus

$$\sigma_N = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.29}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \tag{7.2.30}$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_3 \tag{7.2.31}$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45° to the principal planes and that they have magnitude equal to half the difference between the principal stresses. First, again, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors in the principal directions and consider an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. The normal stress is given by Eqn. 7.2.29,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.32}$$

Cauchy's law gives the components of the traction vector as

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{bmatrix} \tag{7.2.33}$$

and so the shear stress on the plane is, from Eqn. 7.2.11,

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \tag{7.2.34}$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right) n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right) n_2^2 + \sigma_3^2 - \left[(\sigma_1 - \sigma_3) n_1^2 + (\sigma_2 - \sigma_3) n_2^2 + \sigma_3\right]^2 \tag{7.2.35}$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned} \frac{\partial\left(\sigma_S^2\right)}{\partial n_1} &= n_1 (\sigma_1 - \sigma_3)\left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) n_1^2 + (\sigma_2 - \sigma_3) n_2^2\right]\right\} = 0 \\ \frac{\partial\left(\sigma_S^2\right)}{\partial n_2} &= n_2 (\sigma_2 - \sigma_3)\left\{\sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) n_1^2 + (\sigma_2 - \sigma_3) n_2^2\right]\right\} = 0 \end{aligned} \tag{7.2.36}$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 7.2.36 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned}
\mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|
\end{aligned} \tag{7.2.37}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) \tag{7.2.38}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 7.2.14.

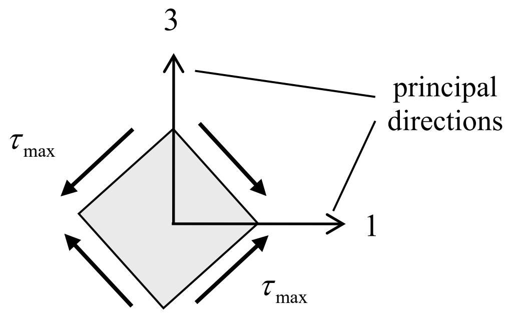

Figure 7.2.14: maximum shear stress at a point

Example

Consider the stress state examined in the Example of §7.2.4:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

The principal stresses were found to be $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{25}{2}$$

One of the planes upon which they act is shown in Fig. 7.2.15 (see Fig. 7.2.12)

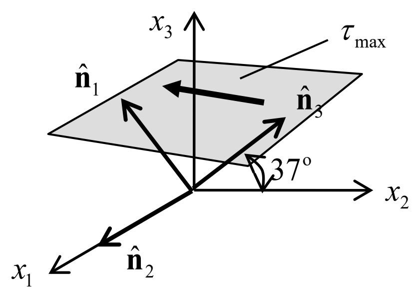

Figure 7.2.15: maximum shear stress

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7.2.6 Mohr's Circles of Stress

The Mohr's circle for 2D stress states was discussed in Book I, §3.5.5. For the 3D case, following on from section 7.2.5, one has the conditions

$$\begin{aligned}
\sigma_N &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \\
\sigma_S^2 + \sigma_N^2 &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \\
n_1^2 + n_2^2 + n_3^2 &= 1
\end{aligned} \tag{7.2.39}$$

Solving these equations gives

$$\begin{aligned}
n_1^2 &= \frac{(\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\
n_2^2 &= \frac{(\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\
n_3^2 &= \frac{(\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}
\end{aligned} \tag{7.2.40}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and noting that the squares of the normal components must be positive, one has that

$$\begin{aligned}
(\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2 \geq 0 \\
(\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2 \leq 0 \\
(\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2 \geq 0
\end{aligned} \tag{7.2.41}$$

and these can be re-written as

$$\begin{aligned}
\sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_2 + \sigma_3)\right]^2 \geq \left[\tfrac{1}{2}(\sigma_2 - \sigma_3)\right]^2 \\
\sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right]^2 \leq \left[\tfrac{1}{2}(\sigma_1 - \sigma_3)\right]^2 \\
\sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right]^2 \geq \left[\tfrac{1}{2}(\sigma_1 - \sigma_2)\right]^2
\end{aligned} \tag{7.2.42}$$

If one takes coordinates (σ_N, σ_S), the equality signs here represent circles in (σ_N, σ_S) stress space, Fig. 7.2.16. Each point (σ_N, σ_S) in this stress space represents the stress on a particular plane through the material particle in question. Admissible (σ_N, σ_S) pairs are given by the conditions Eqns. 7.2.42; they must lie inside a circle of centre $\left(\frac{1}{2}(\sigma_1 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_3)$. This is the large circle in Fig. 7.2.16. The points must lie outside the circle with centre $\left(\frac{1}{2}(\sigma_2 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_2 - \sigma_3)$ and also outside the circle with centre $\left(\frac{1}{2}(\sigma_1 + \sigma_2), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_2)$; these are the two smaller circles in the figure. Thus the admissible points in stress space lie in the shaded region of Fig. 7.2.16.

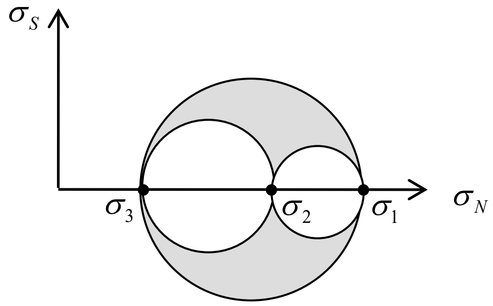

Figure 7.2.16: admissible points in stress space

7.2.7 Three Dimensional Strain

The strain ε_{ij}, in symbolic form $\boldsymbol{\varepsilon}$, is a tensor and as such it follows the same rules as for the stress tensor. In particular, it follows the general tensor transformation rule 7.2.16; it has principal values ε which satisfy the characteristic equation 7.2.23 and these include the maximum and minimum normal strain at a point. There are three principal strain invariants given by 7.2.24 or 7.2.26 and the maximum shear strain occurs on planes oriented at 45° to the principal directions.

7.2.8 Problems

1. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -2 \end{bmatrix}$$

Use Cauchy's law to determine the traction vector acting on a plane trough this point whose unit normal is $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$. What is the normal stress acting on the plane? What is the shear stress acting on the plane?

2. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

What are the stress components with respect to axes $0x_1'x_2'x_3'$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_3 axis

3. Show, in both the index and matrix notation, that the components of an isotropic stress state remain unchanged under a coordinate transformation.

4. Consider a two-dimensional problem. The stress transformation formulae are then, in full,
$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' \\ \sigma_{21}' & \sigma_{22}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
Multiply the right hand side out and use the fact that the stress tensor is symmetric ($\sigma_{12} = \sigma_{21}$ - not true for all tensors). What do you get? Look familiar?

5. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by
$$\left[\sigma_{ij}\right] = \begin{bmatrix} 5/2 & -1/2 & 0 \\ -1/2 & 5/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Evaluate the principal stresses and the principal directions. What is the maximum shear stress acting at the point?