6.9 Strain Energy in Plates

6.9.1 Strain Energy due to Plate Bending and Torsion

Here, the elastic strain energy due to plate bending and twisting is considered.

Consider a plate element bending in the *x* direction, Fig. 6.9.1. The radius of curvature is $R = \partial^2 w / \partial x^2$. The strain energy due to bending through an angle $\Delta\theta$ by a moment $M_x \Delta y$ is

$$\Delta U = \frac{1}{2}(M_x \Delta y)\frac{\partial^2 w}{\partial x^2}\Delta x \tag{6.9.1}$$

Considering also contributions from M_y and M_{xy}, one has

$$\Delta U = \frac{1}{2}\left(M_x \frac{\partial^2 w}{\partial x^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} + M_y \frac{\partial^2 w}{\partial y^2} \right)\Delta x \Delta y \tag{6.9.2}$$

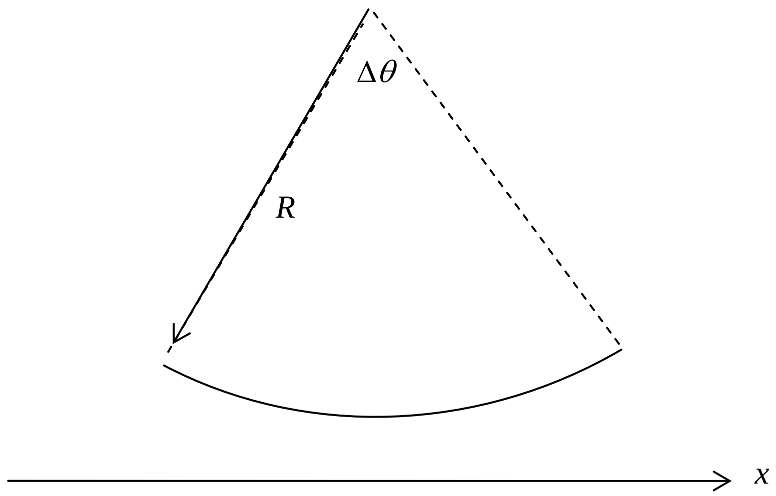

Figure 6.9.1: a bending plate element

Using the moment-curvature relations, one has

$$\begin{aligned}\Delta U &= \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right]\Delta x \Delta y \\ &= \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right) \right]\Delta x \Delta y\end{aligned} \tag{6.9.3}$$

This can now be integrated over the complete plate surface to obtain the total elastic strain energy.

6.9.2 The Principle of Minimum Potential Energy

Plate problems can be solved using the principle of minimum potential energy (see Book I, §8.6). Let $V = -W_{ext}$ be the potential energy of the loads, equivalent to the negative of the work done by those loads, and so the potential energy of the system is $\Pi(w) = U(w) + V(w)$. The solution is then the deflection which minimizes $\Pi(w)$.

When the load is a uniform lateral pressure q, one has

$$\Delta V = -\Delta W_{ext} = +q\, w(x, y)\Delta x \Delta x \tag{6.9.4}$$

and

$$\Delta\Pi = \left\{ \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right] + qw \right\} \Delta x \Delta y \tag{6.9.5}$$

As an example, consider again the simply supported rectangular plate subjected to a uniform load q. Use the same trial function 6.5.38 which satisfies the boundary conditions:

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.9.6}$$

Substituting into 6.9.5 and integrating over the plate gives

$$\begin{aligned} \Pi = \int_0^b\int_0^a & \left\{ \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} \right.\right. \\ & \left. - 2(1-\nu)\left(\frac{m^2 n^2 \pi^4}{a^2 b^2}\left(\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} - \cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b} \right) \right) \right] \\ & \left. + q\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \right\} dxdy \end{aligned} \tag{6.9.7}$$

Carrying out the integration leads to

$$\Pi = \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \times \frac{1}{4}ab \right] + q\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} A_{mn} \times \frac{4ab}{mn\pi^2} \tag{6.9.8}$$

To minimize the total potential energy, one sets

$$\frac{\partial \Pi}{\partial A_{mn}} = DA_{mn} \frac{ab\pi^4}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + q\frac{4ab}{mn\pi^2} = 0$$
$$\rightarrow \quad A_{mn} = -\frac{16q}{\pi^6 Dmn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2} \tag{6.9.9}$$

which is the same result as 6.5.50.

6.9.3 Strain Energy in Polar Coordinates

For circular plates, one can transform the strain energy expression 6.9.3 into polar coordinates, giving {▲Problem 1}

$$\Delta U = \frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 - 2(1-\nu)\times \left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)-\left(\frac{1}{r^2}\frac{\partial w}{\partial \theta}-\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right)^2\right]\right\}\Delta x \Delta y \tag{6.9.10}$$

For an axisymmetric problem, the strain energy is

$$\Delta U = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]\Delta x \Delta y \tag{6.9.11}$$

6.9.4 Vibration of Plates

For vibrating plates, one needs to include the kinetic energy of the plate. The kinetic energy of a plate element of dimensions $\Delta x, \Delta y$ and moving with velocity $\partial \omega / \partial t$ is

$$\Delta K = \frac{1}{2}\rho h\left(\frac{\partial w}{\partial t}\right)^2 \Delta x \Delta y \tag{6.9.12}$$

According to Hamilton's principle, then, the quantity to be minimized is now $U(w)+V(w)-K(w)$.

Consider again the problem of a circular plate undergoing axisymmetric vibrations. The potential energy function is

$$D\pi\int_0^a \left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right] r dr - \pi\rho h \int_0^a \left(\frac{\partial w}{\partial t}\right)^2 r dr \tag{6.9.13}$$

Assume a solution of the form

$$w(r,t) = W(r)\cos(\omega t + \phi) \tag{6.9.14}$$

Substituting this into 6.9.13 leads to

$$D\pi\int_0^a \left[\left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} \right)^2 - 2(1-\nu)\frac{d^2W}{dr^2}\left(\frac{1}{r}\frac{dW}{dr} \right) \right] rdr - \pi\rho h\omega \int_0^a W^2 rdr \tag{6.9.15}$$

Examining the clamped plate, assume a solution, an assumption based on the known static solution 6.6.20, of the form

$$W(r) = A\left(a^2 - r^2\right)^2 \tag{6.9.16}$$

Substituting this into 6.9.15 leads to

$$\begin{aligned} &32D\pi A^2 \int_0^a \left[2\left(a^4 - 4a^2r^2 + 4r^4\right) - (1-\nu)\left(a^4 - 4a^2r^2 + 3r^4\right) \right] rdr \\ &\qquad - \pi\rho\omega h A^2 \int_0^a r\left(a^2 - r^2\right)^4 dr \end{aligned} \tag{6.9.17}$$

Evaluating the integrals leads to

$$\pi A^2 \left(\frac{32}{3} D a^6 - \frac{1}{10}\rho h \omega a^{10} \right) \tag{6.9.18}$$

Minimising this function, setting $\partial / \partial A\{\ \} = 0$, then gives

$$\omega = \alpha \frac{1}{a^2}\sqrt{\frac{D}{\rho h}}, \qquad \alpha = \sqrt{\frac{320}{3}} \approx 10.328 \tag{6.9.19}$$

This simple one-term solution is very close to the exact result given in Table 6.8.1, 10.2158. The result 6.9.19 is of course greater than the actual frequency.

6.9.5 Problems

1. Derive the strain energy expression in polar coordinates, Eqn. 6.9.10.