6.8 Plate Vibrations

In this section, the problem of a vibrating circular plate will be considered. Vibrating plates will be re-examined again in the next section, using a strain energy formulation.

6.8.1 Vibrations of a Clamped Circular Plate

When a plate vibrates with velocity $\partial w / \partial t$, the third equation of equilibrium, Eqn. 6.6.2c becomes the equation of motion

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{6.8.1}$$

With this adjustment, the term q is replaced with $q + \rho h \partial^2 w / \partial t^2$ in the relevant equations; the acceleration term is treated as a transverse load of intensity $\rho h \partial^2 w / \partial t^2$.

Regarding the circular plate, one has from the axisymmetric governing equation 6.6.10 (with $q = 0$),

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right)^2 w = -\frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} \tag{6.8.2}$$

Assume a solution of the form

$$w(r,t) = W(r)\cos(\omega t + \phi) \tag{6.8.3}$$

Substituting into 6.8.2 gives

$$\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right)^2 - k^4 \right] W = 0 \tag{6.8.4}$$

where

$$k^2 = \omega \sqrt{\frac{\rho h}{D}} \tag{6.8.5}$$

Eqn. 6.8.4 gives the two differential equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 \right) W = 0, \qquad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 \right) W = 0 \tag{6.8.6}$$

The solution to these equations are

$$W = C_1 J_0(kr) + C_2 Y_0(kr), \qquad W = C_3 I_0(kr) + C_4 K_0(kr) \tag{6.8.7}$$

where J_0 and Y_0 are, respectively, the Bessel functions of order zero of the first kind and of the second kind; I_0 and K_0 are, respectively, the Modified Bessel functions of order zero of the first kind and of the second kind[1]. These functions are plotted in Fig. 6.8.1 below. For a solid plate with no hole at $r = 0$, one requires that $C_2 = C_4 = 0$, since Y_0 and K_0 become unbounded as $r \to 0$. The general solution is thus

$$W(r) = \overline{A} J_0(kr) + \overline{B} I_0(kr) \tag{6.8.8}$$

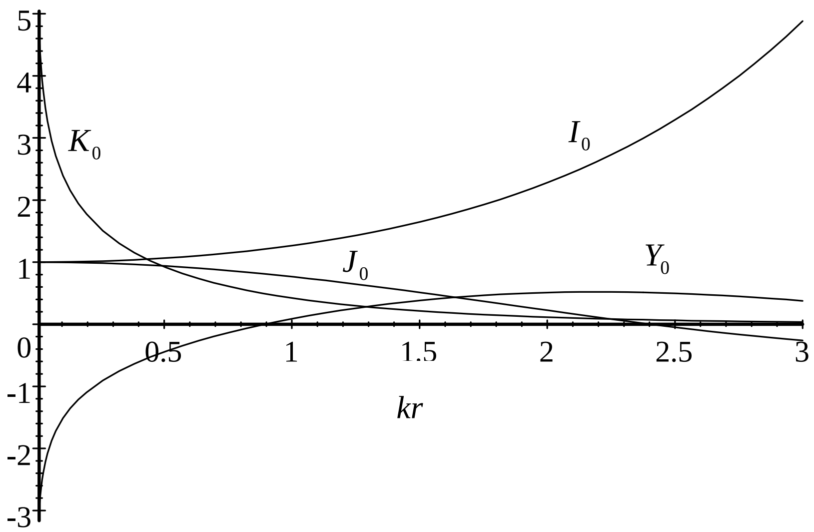

Figure 6.8.1: Bessel Functions

For a clamped plate, the boundary conditions give

$$\begin{aligned} W(a) &= \overline{A} J_0(ka) + \overline{B} I_0(ka) = 0 \\ \left.\frac{dW}{dr}\right|_{r=a} &= \overline{A} J_0'(ka) + \overline{B} I_0'(ka) = 0 \end{aligned} \tag{6.8.9}$$

where the dash means $J_0'(x) = dJ_0(x)/dx$ and $I_0'(x) = dI_0(x)/dx$. Using the relations

$$J_0'(x) = -J_1(x), \quad I_0'(x) = +I_1(x) \tag{6.8.10}$$

where J_1, I_1 are Bessel functions of order one, one has

$$\frac{J_0(ka)}{I_0(ka)} = -\frac{J_1(ka)}{I_1(ka)} \tag{6.8.11}$$

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[1] by *definition*, these Bessel functions are the solution of the differential equations 6.8.6.

The roots ka give the frequencies of vibration of the plate. The function

$$J_0(ka)I_1(ka) + I_0(ka)J_1(ka) \tag{6.8.12}$$

is plotted in Fig. 6.8.2 below. The smallest root is found to be 3.1962. Eqn. 6.8.5 then gives for the frequency,

$$\omega = \alpha \frac{1}{a^2}\sqrt{\frac{D}{\rho h}} \tag{6.8.13}$$

where $\alpha = 10.2158$.

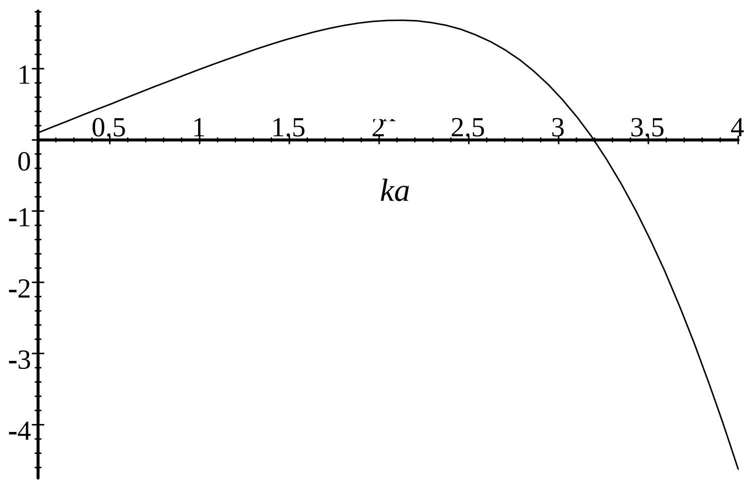

Figure 6.8.2: The Function 6.8.12

Further roots ka of 6.8.12 are given in Table 6.8.1. For each of these roots there is a corresponding frequency ω given by Eqn. 6.8.13, for which the value of α is also tabulated.

	ka	α	nodal circle
1	3.1962	10.2158	
2	6.3064	39.7711	0.3790
3	9.4395	89.1041	0.2548, 0.5833

Table 6.8.1: Roots of Eqn. 6.8.11, frequency factors and nodal circle roots

From 6.8.3, 6.8.8-9, the solution for the deflection is

$$w(r,t) = \bar{A}\left[J_0(kr) - \frac{J_0(ka)}{I_0(ka)} I_0(kr) \right] \cos(\omega t + \phi) \tag{6.8.14}$$

These are an infinite number of deflections, each one corresponding to a root ka. The actual deflection will be a superposition of these individual solutions.

The term inside the square brackets gives the mode shape of the plate during the vibration. The first three (normalized) mode shapes, corresponding to the first three roots, are shown in Fig. 6.8.3.

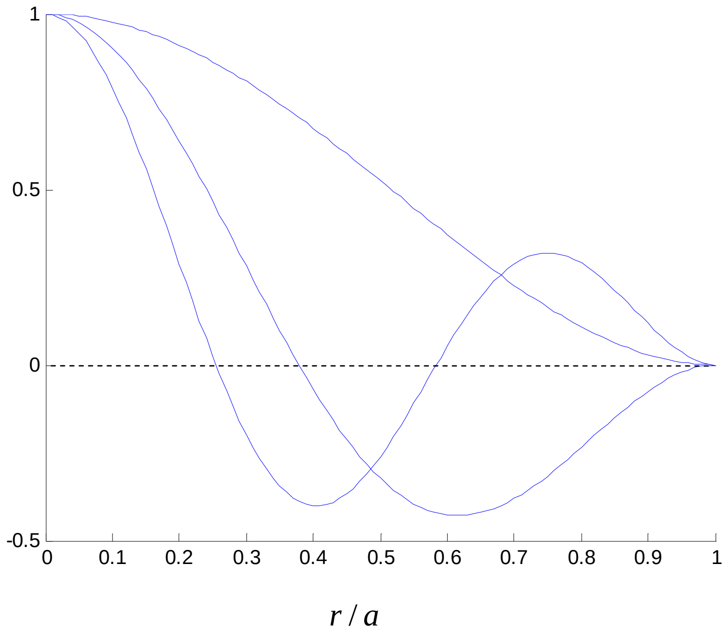

Figure 6.8.3: Mode shapes for the Clamped Circular Plate

The point r/a where these mode-shapes change sign are the positions of the so-called **nodal circles**. These roots of the mode shapes are given in the last column of Table 6.8.1

The General Problem

For circular plates not constrained to an axisymmetric response, one must use the more general differential equation 6.6.5

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w=-\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} \qquad (6.8.15)$$

This time, instead of 6.8.3, assume a solution of the form

$$w(r,\theta,t)=\sum W_n(r)\cos(n\theta)\sin(\omega t+\phi) \qquad (6.8.16)$$

Then 6.8.4-6 become

$$\left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{d}{\partial r}-\frac{n^2}{r^2}\right)^2-k^4\right]W=0 \qquad (6.8.17)$$

where k is again given by 6.8.5, and 6.8.6 becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+k^2\right)W=0, \qquad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-k^2\right)W=0 \tag{6.8.18}$$

The solution to these equations are

$$W=C_1J_n(kr)+C_2Y_n(kr), \qquad W=C_3I_n(kr)+C_4K_n(kr) \tag{6.8.19}$$

where one now has Bessel functions of order n. Proceeding as before, one now needs to find roots of the equation

$$J_n(ka)I_{n+1}(ka)+I_n(ka)J_{n+1}(ka)=0 \tag{6.8.20}$$

and the deflection is

$$w(r,t)=\sum \overline{A}\left[J_n(kr)-\frac{J_n(ka)}{I_n(ka)}I_n(kr)\right]\cos(n\theta)\sin(\omega t+\phi) \tag{6.8.21}$$

The solution for $n=0$ has been given already. For other values of n, there are n so-called **nodal diameters**. For example, for $n=1$ there is one nodal diameter along $\theta=\pm\pi/2$, along which the deflection is zero. The roots of 6.8.20 for this case are given in Table 6.8.2, together with the nodal circle locations.

	ka	α	nodal circle
1	4.6109	21.2604	
2	7.7993	60.8287	0.4897
3	10.9581	120.0792	0.3497, 0.6390

Table 6.8.2: Roots of Eqn. 6.8.20 (*n*=1), frequency factors and nodal circle roots

The mode shapes for half the plate for this case of one nodal diameter are shown in Fig. 6.8.4, corresponding to the first two roots in Table 6.8.2. The frequencies corresponding to these solutions are again given by 6.8.13 with the frequency factor α given in the table.

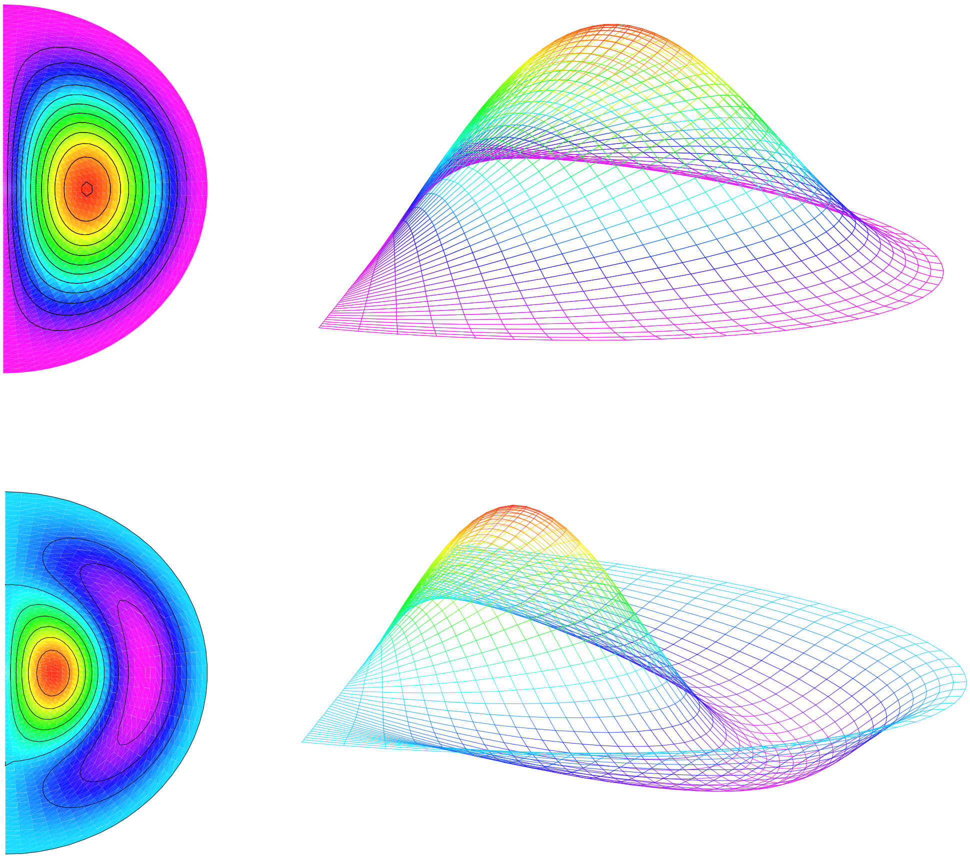

Figure 6.8.4: Mode shapes for the case of one nodal diameter