6.6 Plate Problems in Polar Coordinates

6.6.1 Plate Equations in Polar Coordinates

To examine directly plate problems in polar coordinates, one can first transform the Cartesian plate equations considered in the previous sections into ones in terms of polar coordinates.

First, the definitions of the moments and forces are now

$$M_r = -\int_{-h/2}^{+h/2} z\sigma_{rr}dz, \quad M_\theta = -\int_{-h/2}^{+h/2} z\sigma_{\theta\theta}dz, \quad M_{r\theta} = \int_{-h/2}^{+h/2} z\sigma_{r\theta}dz \tag{6.6.1}$$

and

$$V_r = -\int_{-h/2}^{+h/2} \sigma_{zr}dz, \qquad V_\theta = -\int_{-h/2}^{+h/2} \sigma_{z\theta}dz \tag{6.6.2}$$

The strain-curvature relations, Eqns. 6.2.27, can be transformed to polar coordinates using the transformations from Cartesian to polar coordinates detailed in §4.2 (in particular, §4.2.6). One finds that {▲Problem 1}

$$\begin{aligned} \varepsilon_{rr} &= -z\frac{\partial^2 w}{\partial r^2} \\ \varepsilon_{\theta\theta} &= -z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \\ \varepsilon_{r\theta} &= -z\left(-\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right) \end{aligned} \tag{6.6.3}$$

The moment-curvature relations 6.2.31 become {▲Problem 2}

$$\begin{aligned} M_r &= D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \\ M_\theta &= D\left[\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + \nu\frac{\partial^2 w}{\partial r^2}\right] \\ M_{r\theta} &= -D(1-\nu)\left(-\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right) \end{aligned} \tag{6.6.4}$$

The governing differential equation 6.4.9 now reads

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w = -\frac{q}{D} \tag{6.6.5}$$

The shear forces in terms of deflection, Eqn 6.4.12, now read {▲Problem 3}

$$V_r = D\frac{\partial}{\partial r}\left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right], \quad V_\theta = D\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \qquad (6.6.6)$$

Finally, the stresses are {▲Problem 4}

$$\sigma_{rr} = -\frac{12z}{h^3}M_r, \quad \sigma_{\theta\theta} = -\frac{12z}{h^3}M_\theta, \quad \sigma_{r\theta} = \frac{12z}{h^3}M_{r\theta} \qquad (6.6.7)$$

and

$$\sigma_{zr} = -\frac{3V_r}{2h}\left[1-\left(\frac{z}{h/2}\right)^2\right], \qquad \sigma_{z\theta} = -\frac{3V_\theta}{2h}\left[1-\left(\frac{z}{h/2}\right)^2\right] \qquad (6.6.8)$$

The differential equation 6.6.5 can be solved using a method similar to the Airy stress function method for problems in polar coordinates (the Mitchell solution), that is, a solution is sought in the form of a Fourier series. Here, however, only axisymmetric problems will be considered in detail.

6.6.2 Plate Equations for Axisymmetric Problems

When the loading and geometry of the plate are axisymmetric, the plate equations given above reduce to

$$\begin{aligned} M_r &= D\left[\frac{d^2 w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right] \\ M_\theta &= D\left[\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2 w}{dr^2}\right] \\ M_{r\theta} &= 0 \end{aligned} \qquad (6.6.9)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = -\frac{q(r)}{D} \qquad (6.6.10)$$

$$V_r = D\frac{d}{dr}\left[\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right], \quad V_\theta = 0 \qquad (6.6.11)$$

$$\sigma_{rr} = -\frac{12z}{h^3}M_r, \quad \sigma_{\theta\theta} = -\frac{12z}{h^3}M_\theta, \quad \sigma_{r\theta} = 0 \qquad (6.6.12)$$

and

$$\sigma_{zr} = -\frac{3V_r}{2h}\left[1-\left(\frac{z}{h/2}\right)^2\right], \qquad \sigma_{z\theta} = 0 \tag{6.6.13}$$

Note that there is no twisting moment, so the problem of dealing with non-zero twisting moments on free boundaries seen with rectangular plate does not arise here.

6.6.3 Axisymmetric Plate Problems

For uniform q, direct integration of 6.6.10 leads to

$$w = -\frac{qr^4}{64D} + \frac{1}{4}\overline{A}r^2(\ln r - 1) + \frac{1}{4}\overline{B}r^2 + \overline{C}\ln r + \overline{D} \tag{6.6.14}$$

with

$$\begin{aligned}
\frac{dw}{dr} &= -\frac{qr^3}{16D} + \frac{1}{4}\overline{A}r(2\ln r - 1) + \frac{1}{2}\overline{B}r + \overline{C}\frac{1}{r} \\
\frac{d^2w}{dr^2} &= -\frac{3qr^2}{16D} + \frac{1}{4}\overline{A}(2\ln r + 1) + \frac{1}{2}\overline{B} - \overline{C}\frac{1}{r^2} \\
\frac{d^3w}{dr^3} &= -\frac{3qr}{8D} + \frac{1}{2}\overline{A}\frac{1}{r} + 2\overline{C}\frac{1}{r^3}
\end{aligned} \tag{6.6.15}$$

and

$$V_r = D\left[\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right] = -\frac{qr}{2} + \frac{\overline{A}}{r}D \tag{6.6.16}$$

There are two classes of problem to consider, plates with a central hole and plates with no hole. For a plate with no hole in it, the condition that the stresses remain finite at the plate centre requires that d^2w/dr^2 remains finite, so $\overline{A} = \overline{C} = 0$. Thus immediately one has $V_r = -qr/2$. The boundary conditions at the outer edge $r = a$ give $\overline{B}$ and $\overline{D}$.

1. Solid Plate – Uniform Bending

The simplest case is pure bending of a plate, $M_r = M_0$, with no transverse pressure, $q = 0$. The plate is solid so $\overline{A} = \overline{C} = 0$ and one has $w = \overline{B}r^2/4 + \overline{D}$. The applied moment is

$$M_0 = D\left[\frac{d^2\omega}{dr^2} + \nu\frac{1}{r}\frac{d\omega}{dr}\right] = \frac{1}{2}\overline{B}D[1+\nu] \tag{6.6.17}$$

so $\overline{B} = 2M_0/D(1+\nu)$. Taking the deflection to be zero at the plate-centre, the solution is

$$w = \frac{M_0}{2D(1+\nu)} r^2 \tag{6.6.18}$$

2. Solid Plate Clamped – Uniform Load

Consider next the case of clamped plate under uniform loading. The boundary conditions are that $w = dw/dr = 0$ at $r = a$, leading to

$$\overline{B} = \frac{qa^2}{8D}, \qquad \overline{D} = -\frac{qa^4}{64D} \tag{6.6.19}$$

and hence

$$w = -\frac{q}{64D}\left(r^2 - a^2\right)^2 \tag{6.6.20}$$

which is the same as 6.5.10.

The reaction force at the outer rim is $V_r(a) = -qa/2$. This is a force per unit length; the force acting on an element of the outer rim is $-qa(a\Delta\theta)/2$ and the total reaction force around the outer rim is $-qa^2\pi$, which balances the same applied force.

3. Solid Plate Simply Supported – Uniform Load

For a simply supported plate, $w = 0$ and $M_r = 0$ at $r = a$. Using 6.6.9a, one then has {▲Problem 5}

$$\overline{B} = \frac{3+\nu}{1+\nu}\frac{qa^2}{8D}, \qquad \overline{D} = -\frac{5+\nu}{1+\nu}\frac{qa^4}{64D} \tag{6.6.21}$$

and hence

$$w = -\frac{q}{64D}\left(\frac{5+\nu}{1+\nu}a^2 - r^2\right)\left(a^2 - r^2\right) \tag{6.6.22}$$

The deflection for the clamped and simply supported cases are plotted in Fig. 6.6.1 (for $\nu = 0.3$).

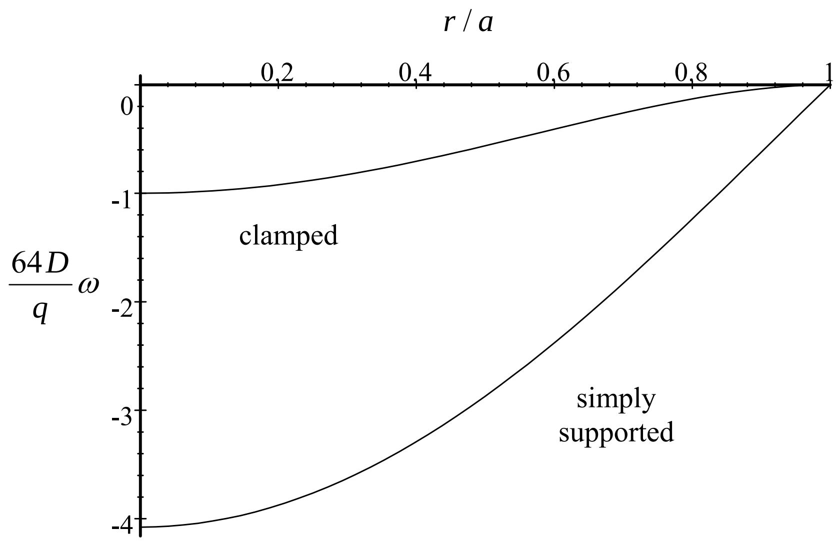

Figure 6.6.1: deflection for a circular plate under uniform loading

4. Solid Plate with a Central Concentrated Force

Consider now the case of a plate subjected to a single concentrated force *F* at $r = 0$. The resultant shear force acting on any cylindrical portion of the plate with radius *r* about the plate-centre is $2\pi r V_r(r)$. As $r \to 0$, one must have an infinite V_r so that this resultant is finite and equal to the applied force *F*. An infinite shear force implies infinite stresses. It is possible for the stresses at the centre of the plate to be infinite. However, although the stresses and strain might be infinite, the displacements, which are obtained from the strains through integration, can remain, and should remain, finite. Although the solution will be "unreal" at the plate-centre, one can again use Saint-Venant's principle to argue that the solution obtained will be valid everywhere except in a small region near where the force is applied.

Thus, seek a solution which has finite displacement in which case, by symmetry, the slope at $r = 0$ will be zero. From the general axisymmetric solution 6.6.15a,

$$\left.\frac{dw}{dr}\right|_{r=0} = \left.\overline{C}\frac{1}{r}\right|_{r=0} \tag{6.6.23}$$

so $\overline{C} = 0$.

From 6.6.16

$$2\pi r V_r\big|_{r=0} = 2\pi \overline{A} D \equiv F \tag{6.6.24}$$

Thus $\overline{A} = F / 2\pi D$ and the moments and shear force become infinite at the plate-centre.

The other two constants can be obtained from the boundary conditions. For a clamped plate, $w = dw/dr = 0$, and one finds that {▲Problem 7}

$$w = \frac{F}{16\pi D}\left[\left(a^2 - r^2\right) + 2r^2 \ln\left(r/a\right)\right] \qquad (6.6.25)$$

This solution results in $\ln(r/a)$ terms in the expressions for moments, giving logarithmically infinite in-plane stresses at the plate-centre.

5. Plate with a Hole

For a plate with a hole in it, there will be four boundary conditions to determine the four constants in Eqn. 6.6.14. For example, for a plate which is simply supported around the outer edge $r = b$ and free on the inner surface $r = a$, one has

$$\begin{aligned} M_r(a) = 0, \quad F_r(a) = 0 \\ w(b) = 0, \quad M_r(b) = 0 \end{aligned} \qquad (6.6.26)$$

6.6.4 Problems

1. Use the expressions 4.2.11-12, which relate second partial derivatives in the Cartesian and polar coordinate systems, together with the strain transformation relations 4.2.17, to derive the strain-curvature relations in polar coordinates, Eqn. 6.6.3.

2. Use the definitions of the moments, 6.6.1, and again relations 4.2.11-12, together with the stress transformation relations 4.2.18, to derive the moment-curvature relations in polar coordinates, Eqn. 6.6.4.

3. Derive Eqns. 6.6.6.

4. Use 6.2.33, 6.4.15-16 to derive the stresses in terms of moments and shear forces, Eqns. 6.6.7-8.

5. Solve the simply supported solid plate problem and hence derive the constants 6.6.21.

6. Show that the solution for a simply supported plate (with no hole), Eqn. 6.6.22, can be considered a superposition of the clamped solution, Eqn. 6.6.20, and a pure bending, by taking an appropriate deflection at the plate-centre in the pure bending case.

7. Solve for the deflection in the case of a clamped solid circular plate loaded by a single concentrated force, Eqn. 6.6.25.