6.5 Plate Problems in Rectangular Coordinates

In this section, a number of important plate problems will be examined using Cartesian coordinates.

6.5.1 Uniform Pressure producing Bending in One Direction

Consider first the case of a plate which bends in one direction only. From 6.3.11 the deflection and moments are

$$w = f(x), \quad M_x(x) = D\frac{d^2w}{dx^2}, \quad M_y(x) = -\nu D\frac{d^2w}{dx^2} \tag{6.5.1}$$

The differential equation 6.4.9 reads

$$\frac{d^4w}{dx^4} = -\frac{q(x)}{D} \tag{6.5.2}$$

The corresponding equation for a beam is $d^4w/dx^4 = p(x)/EI$. If $p(x)/b = -q(x)$, with b the depth of the beam, with $I = bh^3/12$, the plate will respond more stiffly than the beam by a factor of $1/(1-\nu^2)$, a factor of about 10% for $\nu = 0.3$, since

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{1}{1-\nu^2}\frac{EI}{b} \tag{6.5.3}$$

The extra stiffness is due to the constraining effect of M_y, which is not present in the beam.

6.5.2 Deflection of a Circular Plate by a Uniform Lateral Load

A solution for a circular plate problem is presented next. This problem will be examined again in the section which follows using the more natural polar coordinates.

Consider a circular plate with boundary

$$x^2 + y^2 = a^2, \tag{6.5.4}$$

clamped at its edges and subjected to a uniform lateral load q, Fig. 6.5.1.

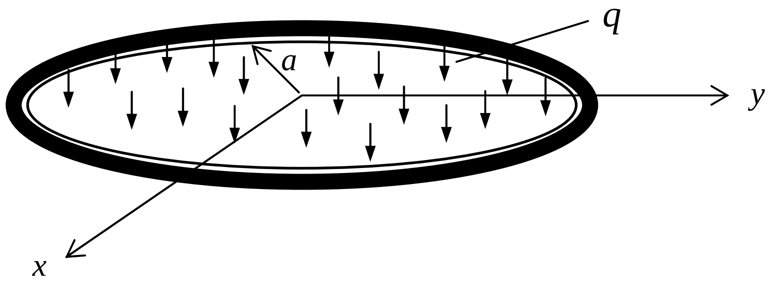

Figure 6.5.1: a clamped circular plate subjected to a uniform lateral load

The differential equation for the problem is given by 6.4.9. The boundary conditions are that the slope and deflection are zero at the boundary:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0 \qquad \text{along} \quad x^2 + y^2 = a^2 \tag{6.5.5}$$

It will be shown that the deflection

$$w = c(x^2 + y^2 - a^2)^2 \tag{6.5.6}$$

is a solution to the problem. First, this function certainly satisfies 6.5.5. Further, letting

$$f(x, y) = x^2 + y^2 - a^2, \tag{6.5.7}$$

the relevant partial derivatives are

$$\begin{aligned}
&\frac{\partial w}{\partial x} = 4cxf, \quad \frac{\partial w}{\partial y} = 4cyf \\
&\frac{\partial^2 w}{\partial x^2} = 4c\left(2x^2 + f\right), \quad \frac{\partial^2 w}{\partial x \partial y} = 8cxy, \quad \frac{\partial^2 w}{\partial y^2} = 4c\left(2y^2 + f\right) \\
&\frac{\partial^3 w}{\partial x^3} = 24cx, \quad \frac{\partial^3 w}{\partial x^2 \partial y} = 8cy, \quad \frac{\partial^3 w}{\partial x \partial y^2} = 8cx, \quad \frac{\partial^3 w}{\partial y^3} = 24cy \\
&\frac{\partial^4 w}{\partial x^4} = 24c, \quad \frac{\partial^4 w}{\partial x^2 \partial y^2} = 8c, \quad \frac{\partial^4 w}{\partial y^4} = 24c
\end{aligned} \tag{6.5.8}$$

Substituting these into the differential equation now yields

$$c = -\frac{q}{64D} \tag{6.5.9}$$

so the deflection is

$$w = -\frac{q}{64D}(x^2 + y^2 - a^2)^2 \tag{6.5.10}$$

This is plotted in Fig. 6.5.2. The maximum deflection occurs at the plate centre, where

$$w_{\max} = -\frac{qa^4}{64D}. \qquad (6.5.11)$$

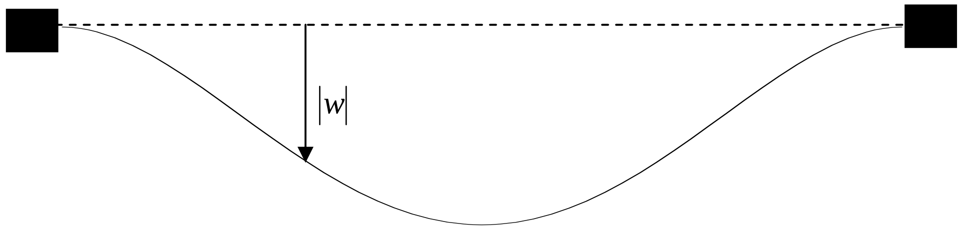

Figure 6.5.2: mid-plane deflection of the clamped circular plate

The curvature $\partial^2 w / \partial x^2$ along a radial line $y = 0$ is displayed in Fig. 6.5.3. The curvature is positive toward the centre of the plate (the plate curves upward) and is negative towards the edge of the plate (the plate curves downward).

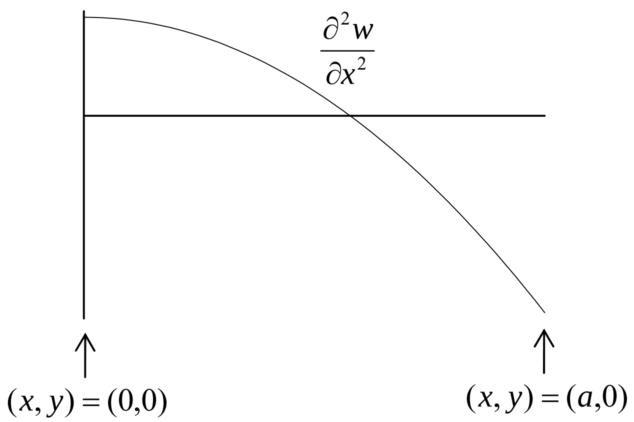

Figure 6.5.3: curvature in the clamped circular plate

The moments occurring in the plate are, from the moment-curvature equations 6.2.31 and 6.5.8,

$$\begin{aligned} M_x &= -\frac{q}{16}\left[(3+\nu)x^2 + (3\nu+1)y^2 - (1+\nu)a^2\right] \\ M_y &= -\frac{q}{16}\left[(3\nu+1)x^2 + (3+\nu)y^2 - (1+\nu)a^2\right] \\ M_{xy} &= +\frac{q}{8}(1-\nu)xy \end{aligned} \qquad (6.5.12)$$

The moment M_x along a radial line $y = 0$ is of the same character as the curvature displayed in Fig. 6.5.3.

The out-of-plane shear forces are, from 6.4.5,

$$V_x = -\frac{qx}{2}, \qquad V_y = -\frac{qy}{2} \tag{6.5.13}$$

At the plate centre, the expressions become

$$M_x = M_y = \frac{q}{16}(1+\nu)a^2, \qquad M_{xy} = V_x = V_y = 0 \tag{6.5.14}$$

Stresses in the Plate

From 6.5.12-13 and 6.2.33, 6.4.15-16, the stresses in the plate are

$$\begin{aligned}
\sigma_{xx} &= \frac{3qz}{4h^3}\left[(3+\nu)x^2 + (3\nu+1)y^2 - (1+\nu)a^2\right] \\
\sigma_{yy} &= \frac{3qz}{4h^3}\left[(3\nu+1)x^2 + (3+\nu)y^2 - (1+\nu)a^2\right] \\
\sigma_{xy} &= \frac{3qz}{2h^3}(1-\nu)xy \\
\sigma_{zx} &= \frac{3qx}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right] \\
\sigma_{zy} &= \frac{3qy}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right]
\end{aligned} \tag{6.5.15}$$

Converting to polar coordinates (r,θ) through

$$x = r\cos\theta,\ y = r\sin\theta \tag{6.5.16}$$

and using a stress transformation,

$$\begin{aligned}
\sigma_{rr} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\
\sigma_{\theta\theta} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\
\sigma_{r\theta} &= \cos\theta\sin\theta\left(\sigma_{yy} - \sigma_{xx}\right) + \cos 2\theta\sigma_{xy}
\end{aligned} \tag{6.5.17}$$

leads to the axisymmetric stress field {▲Problem 1}

$$\begin{aligned}
\sigma_{rr} &= \frac{3qz}{4h^3}\left[(3+\nu)r^2 - (1+\nu)a^2\right] \\
\sigma_{\theta\theta} &= \frac{3qz}{4h^3}\left[(3\nu+1)r^2 - (1+\nu)a^2\right] \\
\sigma_{r\theta} &= 0
\end{aligned} \tag{6.5.18}$$

At the plate centre,

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{3qza^2}{4h^3}(1+\nu) \tag{6.5.19}$$

At the plate edge $r = a$,

$$\sigma_{rr} = \frac{3qza^2}{2h^3}, \qquad \sigma_{\theta\theta} = \frac{3qza^2}{2h^3}\nu \tag{6.5.20}$$

For the shear stress, the traction acting on a surface parallel to the $x-y$ plane can be expressed as (see Fig. 6.5.4)

$$\begin{aligned} \mathbf{t} &= \sigma_{zr}\mathbf{e}_r + \sigma_{z\theta}\mathbf{e}_\theta \\ &= \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y \\ &= \sigma_{zx}(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta) + \sigma_{zy}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta) \end{aligned} \tag{6.5.21}$$

where $\mathbf{e}_i$ is a unit vector in the direction *i*. Thus

$$\begin{aligned} \sigma_{zr} &= \cos\theta\sigma_{zx} + \sin\theta\sigma_{zy} = \frac{3qr}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right] \\ \sigma_{z\theta} &= -\sin\theta\sigma_{zx} + \cos\theta\sigma_{zy} = 0 \end{aligned} \tag{6.5.22}$$

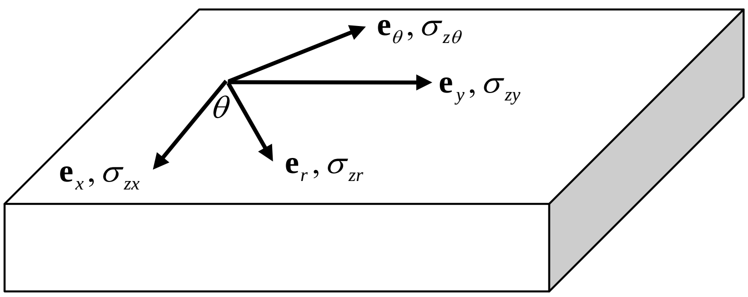

Figure 6.5.4: stress components acting on a surface

Note that the maximum stress in the plate is

$$\sigma_{\max} = \sigma_{rr}(a, h/2) = \frac{3q}{4}\left(\frac{a}{h}\right)^2 \tag{6.5.23}$$

The maximum shear stress, on the other hand, is $\sigma_{zr}(a,0) = 3q/4 \times (a/h)$. Thus the shear stress is of an order h/a smaller than the normal stress.

6.5.3 An Infinite Plate with Sinusoidal Deflection

Consider next the classic plate problem addressed by Navier in 1820. It consists of an infinite plate with an undulating "up/down" sinusoidal deflection, Fig. 6.5.5,

$$w(x,y) = w_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{6.5.24}$$

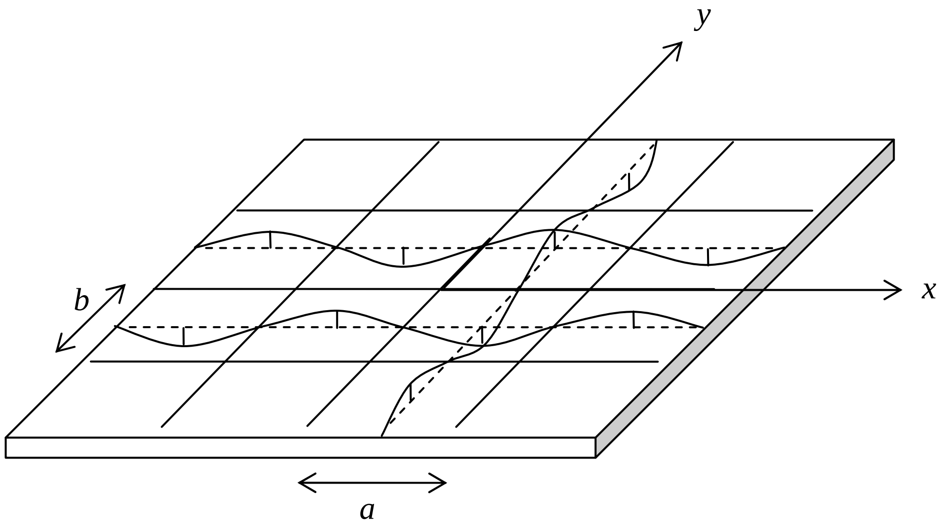

Figure 6.5.5: A plate with sinusoidal deflection

Differentiation of the deflection leads to the curvatures

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= -w_0 \frac{\pi^2}{a^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\frac{\partial^2 w}{\partial y^2} &= -w_0 \frac{\pi^2}{b^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\frac{\partial^2 w}{\partial xy} &= w_0 \frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned} \tag{6.5.25}$$

and hence the pressure

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial xy} + \frac{\partial^2 w}{\partial y^2} = \pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 w(x,y) \equiv -\frac{q(x,y)}{D} \tag{6.5.26}$$

The pressure thus varies like the deflection. There is no need for supports for the plate since the "up" loads balance the "down" loads.

From the moment-curvature relations,

$$M_x = -w_0 D\pi^2 \left(\frac{1}{a^2} + \frac{\nu}{b^2} \right) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$
$$M_y = -w_0 D\pi^2 \left(\frac{\nu}{a^2} + \frac{1}{b^2} \right) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \qquad (6.5.27)$$
$$M_{xy} = -w_0 D(1-\nu)\frac{\pi^2}{ab} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

and, from 6.4.12, the shear forces are

$$V_x = -w_0 D\pi^3 \frac{1}{a}\left(\frac{1}{a^2} + \frac{1}{b^2} \right) \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$
$$V_y = -w_0 D\pi^3 \frac{1}{b}\left(\frac{1}{a^2} + \frac{1}{b^2} \right) \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} \qquad (6.5.28)$$

Note that both q / w and M_x / M_y are constant throughout the plate.

6.5.4 A Simply Supported Plate with Sinusoidal Deflection

Following on from the previous example, consider now a *finite* plate of dimensions a and b with the same sinusoidal deflection 6.5.24, simply supported along the edges $x = 0$, $x = a$, $y = 0$, $y = b$. In what follows, take w_0 in 6.5.24 to be negative, so that the plate is pushed down towards the centre.

According to 6.5.24 and 6.5.27, the deflection and slope is zero along the supported edges, as required. The vertical reactions at the supports are given by 6.5.28. However, according to Eqn. 6.5.27c, there are varying non-zero twisting moments over the ends of the plate. Thus the solution given by 6.5.24-28 is not quite the solution to the simply supported finite-plate problem, unless one can somehow apply the exact required twisting moments over the edges of the plate.

It turns out, however, that the solution 6.5.24-28 is a correct solution, except in a region close to the edges of the plate. This is explained in what follows.

Twisting Moments over "Free" Surfaces

Consider an element of material of width dy, Fig. 6.5.6. The element is subjected to a twisting moment $M_{xy} dy$, Fig. 6.5.6a. This twisting moment is due to shear stresses acting parallel to the plate surface (see Fig. 6.1.8). This system of horizontal forces can be replaced by the statically equivalent system of vertical forces shown in Fig. 6.5.6b – two *forces* of magnitude M_{xy} separate by a distance dy. Recalling Saint-Venant's principle, the difference between the statically equivalent systems of forces of Fig. 6.5.6a and 6.5.6b will lead to differences in the stress field within the plate only in a small region very close to the plate-edges.

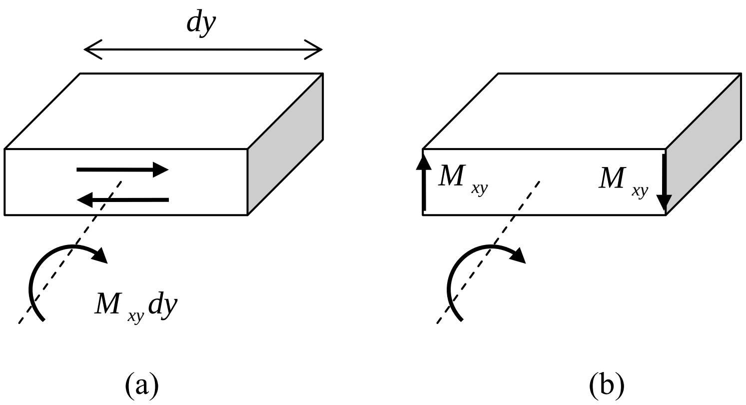

Figure 6.5.6: Equivalent systems of forces leading to the same twisting moment; (a) horizontal forces, (b) vertical forces

Consider next a distribution of twisting moment along the plate edge, Fig. 6.5.7. As can be seen, this distribution is equivalent to a distribution of shearing forces (per unit length) of magnitude

$$\overline{V}_x(y) = -\frac{\partial M_{xy}}{\partial y} \tag{6.5.29}$$

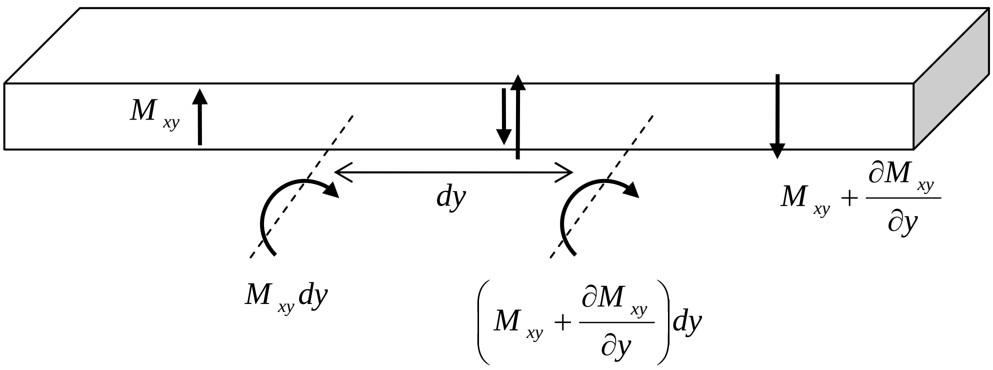

Figure 6.5.7: A distribution of twisting moments along a plate edge

The total vertical reaction along the edges can now be taken to be

$$V_x - \frac{\partial M_{xy}}{\partial y} \tag{6.5.30}$$

(and $V_y - \partial M_{xy} / \partial x$ along the other edges) and this gives a correct solution to the problem. From 6.5.27-28, these reactions are

$$
\begin{aligned}
F_{x0} &= \left(V_x - \frac{\partial M_{xy}}{\partial y} \right)_{(0,y)} = -w_0 D\pi^3 \frac{1}{a}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{b^2}\right]\sin\frac{\pi y}{b} \\
F_{xa} &= \left(V_x - \frac{\partial M_{xy}}{\partial y} \right)_{(a,y)} = +w_0 D\pi^3 \frac{1}{a}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{b^2}\right]\sin\frac{\pi y}{b} \\
F_{y0} &= \left(V_y - \frac{\partial M_{xy}}{\partial x} \right)_{(x,0)} = -w_0 D\pi^3 \frac{1}{b}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{a^2}\right]\sin\frac{\pi x}{a} \\
F_{yb} &= \left(V_y - \frac{\partial M_{xy}}{\partial x} \right)_{(x,b)} = +w_0 D\pi^3 \frac{1}{b}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{a^2}\right]\sin\frac{\pi x}{a}
\end{aligned}
\tag{6.5.31}
$$

Corner Forces

Integrating 6.5.31 over the four edges, the resultant *upward* forces on the four edges (with $w_0 < 0$, they are all four upward) are

$$
\begin{aligned}
+\overline{F}_{x0} &= -\overline{F}_{xa} = -2w_0 D\pi^2 \frac{b}{a}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{b^2}\right] \\
+\overline{F}_{y0} &= -\overline{F}_{yb} = -2w_0 D\pi^2 \frac{a}{b}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{a^2}\right]
\end{aligned}
\tag{6.5.32}
$$

and the resultant of these may be expressed as

$$
F_{\text{up}} = -4w_0 D\pi^2 ab\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 + \frac{2(1-\nu)}{a^2b^2}\right] \tag{6.5.33}
$$

The resultant downward force is, using 6.5.26,

$$
\begin{aligned}
F_{\text{down}} &= \int_0^b\int_0^a q(x,y)dxdy = -w_0 D\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 \int_0^b\int_0^a \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}dxdy \\
&= -4w_0 D\pi^2 ab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2
\end{aligned}
\tag{6.5.34}
$$

The difference between F_{up} and F_{down} is due to the re-distributed twisting moment, and is explained a follows: consider again Fig. 6.5.7, where the edge twisting moments have been replaced with a statically equivalent distribution of shear forces. It can be seen that there results shear forces at the ends of the plate-edge (the "corners"), where the shear forces M_{xy} have no neighbouring shear force of opposite sign with which to "cancel out". There are concentrated forces (per unit length) at the plate-corners of magnitude M_{xy}. Examining Fig. 6.5.7, which shows the edge $x = a$, the force $M_{xy}(a,0)$ is positive up whereas the force $M_{xy}(a,b)$ is positive down. There are also contributions to the corner forces at $(a,0)$ and (a,b) from the adjacent edges, shown in Fig. 6.5.8. One finds that the *downward* concentrated forces at the corner are

$$\begin{aligned} P_{00} &= +2M_{xy}(0,0) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\ P_{a0} &= -2M_{xy}(a,0) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\ P_{ab} &= +2M_{xy}(a,b) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\ P_{0b} &= -2M_{xy}(0,b) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \end{aligned} \tag{6.5.35}$$

Adding these to F_{down} of Eqn. 6.5.34 now gives the F_{up} of Eqn. 6.5.33.

Physically, if one applies a pressure to a simply supported plate, the plate will tend to rise at the four corners, in a twisting action. The corner forces 6.5.35 are necessary to keep the corners down and so produce the deflection 6.5.24.

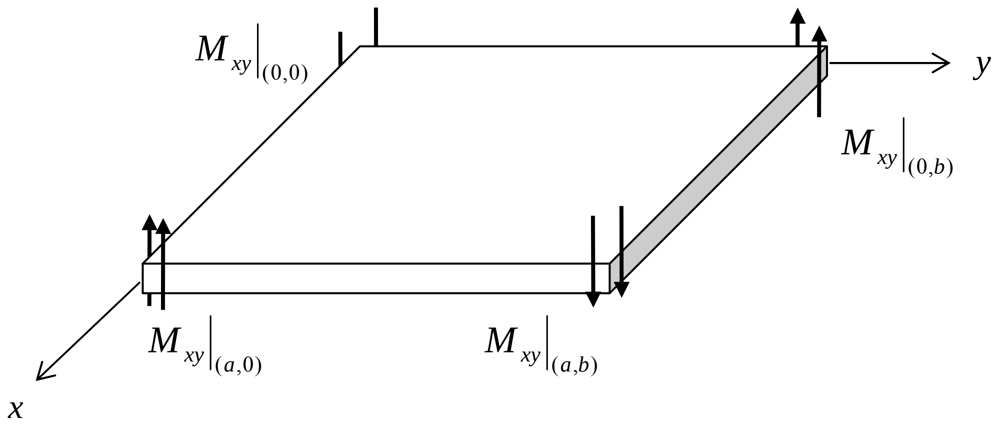

Figure 6.5.8: corner forces in the simply supported plate

The ratio of the resultant downward corner force to the downward force due to the applied pressure, F_{down}, is

$$2(1-\nu)\frac{a^2 b^2}{(a^2+b^2)^2} \tag{6.5.36}$$

For a square plate, this is $(1-\nu)/2$; with $\nu = 0.3$, this is 35%.

6.5.5 A Rectangular Plate Simply Supported at the Edges

The above solution can be used to solve the problem of a simply supported plate loaded by any arbitrary pressure distribution, through the use of Fourier series.

Consider again this plate, whose displacement boundary conditions are

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$$
$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{(0,y)} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{(a,y)} = 0, \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_{(x,0)} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{(x,b)} = 0 \tag{6.5.37}$$

Assume the deflection to be of the form

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.5.38}$$

with A_{mn} coefficients to be determined. It can be seen that this function satisfies the boundary conditions. Taking the derivatives of this function,

$$\frac{\partial^2 w}{\partial x^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-\frac{m^2\pi^2}{a^2}\right) A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.5.39}$$

etc., and substituting into the differential equation 6.4.9, gives

$$\pi^4 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = -q(x,y) \tag{6.5.40}$$

This can be written compactly in the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = -q(x,y) \tag{6.5.41}$$

where

$$C_{mn} = \pi^4 D A_{mn}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \tag{6.5.42}$$

It remains to choose the coefficients of the series so as to satisfy the equation identically over the whole area of the plate.

One can evaluate the coefficients as one does for ordinary Fourier series, although here one has a double series and so one proceeds as follows: first, multiply both sides of (6.5.40) by $\sin(k\pi y / b)$ where k is an integer, and integrate over y between the limits $[0,b]$, so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \int_0^b \sin\frac{n\pi y}{b} \sin\frac{k\pi y}{b} dy = -\int_0^b q(x,y) \sin\frac{k\pi y}{b} dy \tag{6.5.43}$$

Using the orthogonality condition

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy = \begin{cases} 0, & n \neq k \\ b/2, & n = k \end{cases}, \tag{6.5.44}$$

leads to

$$\frac{b}{2}\sum_{m=1}^{\infty} C_{mk}\sin\frac{m\pi x}{a} = -\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy \tag{6.5.45}$$

Now there are functions of *x* only so, multiplying both sides by $\sin(j\pi x/a)$ and following the same procedure, one has

$$\frac{a}{2}\frac{b}{2}C_{jk} = -\int_0^a\left[\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy\right]\sin\frac{j\pi x}{a}dx \tag{6.5.46}$$

and hence the coefficients C_{mn} are (replacing the dummy subscripts j,k with m,n)

$$C_{mn} = -\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5.47}$$

Thus the coefficients A_{mn} of the original expression for the deflection $w(x,y)$, 6.5.38, are

$$A_{mn} = -\frac{1}{\pi^4 D}\frac{4}{ab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5.48}$$

It is now possible to solve for the coefficients given any loading $q(x,y)$ over the plate, and hence evaluate the deflection, moments and stresses in the plate, by taking the derivatives of the infinite series for w.

This solution is due to Navier and is called **Navier's solution** to the rectangular plate problem. A similar solution method has been used by Lévy to solve a more general problem – that of a rectangular plate simply supported on two opposite sides, and any one of the conditions free, simply-supported, or clamped, along the other two opposite sides. For example, considering a square plate, this involves using a trial function for the deflection of the form (compare with 6.5.38)

$$w(x,y) = \sum_{n=1}^{\infty} F_n(y)\sin\frac{n\pi x}{a} \tag{6.5.49}$$

and then attempting to determine the functions $F_n(y)$.

A Uniform Load

In the case of a uniform load $q(x,y) = q$, one has

$$
\begin{aligned}
A_{mn} &= -\frac{4q}{\pi^4 Dab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\int_0^a \sin\frac{m\pi x}{a}dx\int_0^b \sin\frac{n\pi y}{b}dy \\
&= -\frac{4q}{\pi^4 Dab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\left[\frac{a}{m\pi}(1-\cos(m\pi))\right]\left[\frac{b}{n\pi}(1-\cos(n\pi))\right] \\
&= \begin{cases} -\dfrac{16q}{\pi^6 Dmn}\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^{-2} & (m, n = 1, 3, 5, \cdots) \\ 0 & (m, n = 0, 2, 4, \cdots) \end{cases}
\end{aligned}
\tag{6.5.50}
$$

The resulting series in 6.5.50 converges rapidly.

The deflection at the centre of the plate is then

$$
\begin{aligned}
w &= \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} A_{mn}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2} \\
&= -\frac{16qb^4}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn}\left(\frac{m^2}{(a/b)^2}+n^2\right)^{-2}(-1)^{(m+n)/2-1}
\end{aligned}
\tag{6.5.51}
$$

For a square plate,

$$
\begin{aligned}
w &= -\frac{16qa^4}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn}\left(m^2+n^2\right)^{-2}(-1)^{(m+n)/2-1} \\
&= -\frac{qa^4}{D}\times 0.0040624
\end{aligned}
\tag{6.5.52}
$$

Denoting the area a^2 by *A*, this is $\omega = -0.0041qA^2/D$. This can be compared with the clamped circular plate; denoting the area there, πa^2, by *A*, the maximum deflection, Eqn. 6.5.11, gives $\omega = -0.0016qA^2/D$.

Corner Forces

The twisting moment is

$$
M_{xy}(x,y) = -D(1-\nu)\frac{\partial^2\omega}{\partial x\partial y} = -D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\tag{6.5.53}
$$

and the four corner forces required to hold the plate down are now

$$
\begin{aligned}
P_{00} &= +2M_{xy}(0,0) = -2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn} \\
P_{a0} &= -2M_{xy}(a,0) = +2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos m\pi \\
P_{0b} &= -2M_{xy}(0,b) = +2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos n\pi \\
P_{ab} &= +2M_{xy}(a,b) = -2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos m\pi\cos n\pi
\end{aligned}
\tag{6.5.54}
$$

For a uniform load over a square plate, using 6.5.50, the corner forces reduce to

$$
\begin{aligned}
4P &= 4\frac{32q(1-\nu)a^2}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{\left(m^2+n^2\right)^2} \\
&= 4\frac{32q(1-\nu)a^2}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left((2m-1)^2+(2n-1)^2\right)^2} \\
&\approx 4\frac{32q(1-\nu)a^2}{\pi^4}\times 0.2825 \\
&\approx 0.26F_0
\end{aligned}
\tag{6.5.55}
$$

(for $\nu = 0.3$) where $F_0 = qa^2$ is the resultant applied force.

6.5.6 Problems

1. Derive the expressions for the stress components in polar form, for the clamped circular plate under uniform lateral load, Eqn. 6.5.18.