4.2 Differential Equations in Polar Coordinates

Here, the two-dimensional Cartesian relations of Chapter 1 are re-cast in polar coordinates.

4.2.1 Equilibrium equations in Polar Coordinates

One way of expressing the equations of equilibrium in polar coordinates is to apply a change of coordinates directly to the 2D Cartesian version, Eqns. 1.1.8, as outlined in the Appendix to this section, §4.2.6. Alternatively, the equations can be derived from first principles by considering an element of material subjected to stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, as shown in Fig. 4.2.1. The dimensions of the element are Δr in the radial direction, and $r\Delta\theta$ (inner surface) and $(r+\Delta r)\Delta\theta$ (outer surface) in the tangential direction.

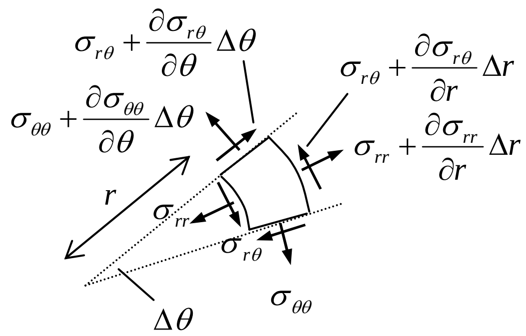

Figure 4.2.1: an element of material

Summing the forces in the radial direction leads to

$$\begin{aligned}\sum F_r &= \left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r}\Delta r\right)(r+\Delta r)\Delta\theta - \sigma_{rr} r\Delta\theta \\ &\quad - \sin\frac{\Delta\theta}{2}\left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta\theta\right)\Delta r - \sin\frac{\Delta\theta}{2}(\sigma_{\theta\theta})\Delta r \\ &\quad + \cos\frac{\Delta\theta}{2}\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\Delta\theta\right)\Delta r - \cos\frac{\Delta\theta}{2}(\sigma_{r\theta})\Delta r \equiv 0\end{aligned} \tag{4.2.1}$$

For a small element, $\sin\theta \approx \theta, \cos\theta \approx 1$ and so, dividing through by $\Delta r\Delta\theta$,

$$\frac{\partial \sigma_{rr}}{\partial r}(r+\Delta r) + \sigma_{rr} - \sigma_{\theta\theta} - \frac{\Delta\theta}{2}\left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right) + \frac{\partial \sigma_{r\theta}}{\partial \theta} \equiv 0 \tag{4.2.2}$$

A similar calculation can be carried out for forces in the tangential direction {▲Problem 1}. In the limit as $\Delta r, \Delta\theta \to 0$, one then has the two-dimensional equilibrium equations in polar coordinates:

$$\boxed{\begin{aligned}&\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)=0\\&\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\sigma_{r\theta}}{r}=0\end{aligned}}$$

Equilibrium Equations (4.2.3)

4.2.2 Strain Displacement Relations and Hooke's Law

The two-dimensional strain-displacement relations can be derived from first principles by considering line elements initially lying in the r and θ directions. Alternatively, as detailed in the Appendix to this section, §4.2.6, they can be derived directly from the Cartesian version, Eqns. 1.2.5,

$$\boxed{\begin{aligned}\varepsilon_{rr}&=\frac{\partial u_r}{\partial r}\\\varepsilon_{\theta\theta}&=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\\\varepsilon_{r\theta}&=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\end{aligned}}$$

2-D Strain-Displacement Expressions (4.2.4)

The stress-strain relations in polar coordinates are completely analogous to those in Cartesian coordinates – the axes through a small material element are simply labelled with different letters. Thus Hooke's law is now

$$\varepsilon_{rr}=\frac{1}{E}\left[\sigma_{rr}-\nu\sigma_{\theta\theta}\right],\quad\varepsilon_{\theta\theta}=\frac{1}{E}\left[\sigma_{\theta\theta}-\nu\sigma_{rr}\right],\quad\varepsilon_{r\theta}=\frac{1+\nu}{E}\sigma_{r\theta}$$
$$\varepsilon_{zz}=-\frac{\nu}{E}\left(\sigma_{rr}+\sigma_{\theta\theta}\right)$$

Hooke's Law (Plane Stress) (4.2.5a)

$$\varepsilon_{rr}=\frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr}-\nu\sigma_{\theta\theta}\right],\quad\varepsilon_{\theta\theta}=\frac{1+\nu}{E}\left[-\nu\sigma_{rr}+(1-\nu)\sigma_{\theta\theta}\right],\quad\varepsilon_{r\theta}=\frac{1+\nu}{E}\sigma_{r\theta}$$

Hooke's Law (Plane Strain) (4.2.5b)

4.2.3 Stress Function Relations

In order to solve problems in polar coordinates using the stress function method, Eqns. 3.2.1 relating the stress components to the Airy stress function can be transformed using the relations in the Appendix to this section, §4.2.6:

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2},\quad\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2},\quad\sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\qquad(4.2.6)$$

It can be verified that these equations automatically satisfy the equilibrium equations 4.2.3 {▲Problem 2}.

The biharmonic equation 3.2.3 becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0 \tag{4.2.7}$$

4.2.4 The Compatibility Relation

The compatibility relation expressed in polar coordinates is (see the Appendix to this section, §4.2.6)

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} - \frac{2}{r}\frac{\partial^2 \varepsilon_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{2}{r^2}\frac{\partial \varepsilon_{r\theta}}{\partial \theta} = 0 \tag{4.2.8}$$

4.2.5 Problems

1. Derive the equilibrium equation 4.2.3b

2. Verify that the stress function relations 4.2.6 satisfy the equilibrium equations 4.2.3.

3. Verify that the strains as given by 4.2.4 satisfy the compatibility relations 4.2.8.

4.2.6 Appendix to §4.2

From Cartesian Coordinates to Polar Coordinates

To transform equations from Cartesian to polar coordinates, first note the relations

$$\begin{gathered} x = r\cos\theta, \quad y = r\sin\theta \\ r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x) \end{gathered} \tag{4.2.9}$$

Then the Cartesian partial derivatives become

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned} \tag{4.2.10}$$

The second partial derivatives are then

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$
$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r}\right) - \cos\theta\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right) + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$
$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \sin^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) + \sin 2\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) \tag{4.2.11}$$

Similarly,

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) - \sin 2\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$
$$\frac{\partial^2}{\partial x\partial y} = -\sin\theta\cos\theta\left(-\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) - \cos 2\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) \tag{4.2.12}$$

Equilibrium Equations

The Cartesian stress components can be expressed in terms of polar components using the stress transformation formulae, Part I, Eqns. 3.4.7. Using a *negative* rotation (see Fig. 4.2.2), one has

$$\begin{aligned}\sigma_{xx} &= \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta\\ \sigma_{yy} &= \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta\\ \sigma_{xy} &= \sin\theta\cos\theta(\sigma_{rr} - \sigma_{\theta\theta}) + \sigma_{r\theta}\cos 2\theta\end{aligned} \tag{4.2.13}$$

Applying these and 4.2.10 to the 2D Cartesian equilibrium equations 3.1.3a-b lead to

$$\begin{aligned}&\cos\theta\left[\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta})\right] - \sin\theta\left[\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r}\right] = 0\\ &\sin\theta\left[\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta})\right] + \cos\theta\left[\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r}\right] = 0\end{aligned} \tag{4.2.14}$$

which then give Eqns. 4.2.3.

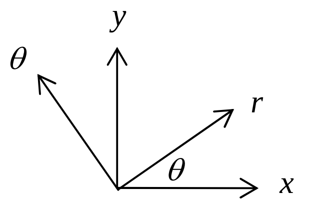

Figure 4.2.2: rotation of axes

The Strain-Displacement Relations

Noting that

$$\begin{aligned} u_x &= u_r \cos\theta - u_\theta \sin\theta \\ u_y &= u_r \sin\theta + u_\theta \cos\theta \end{aligned}, \tag{4.2.15}$$

the strains in polar coordinates can be obtained directly from Eqns. 1.2.5:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ &= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) \left(u_r \cos\theta - u_\theta \sin\theta \right) \\ &= \cos^2\theta \frac{\partial u_r}{\partial r} + \sin^2\theta \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \sin 2\theta \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \end{aligned} \tag{4.2.16}$$

One obtains similar expressions for the strains ε_{yy} and ε_{xy}. Substituting the results into the strain transformation equations Part I, Eqns. 3.8.1,

$$\begin{aligned} \varepsilon_{rr} &= \varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{\theta\theta} &= \varepsilon_{xx} \sin^2\theta + \varepsilon_{yy} \cos^2\theta - \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{r\theta} &= \sin\theta \cos\theta \left(\varepsilon_{yy} - \varepsilon_{xx} \right) + \varepsilon_{xy} \cos 2\theta \end{aligned} \tag{4.2.17}$$

then leads to the equations given above, Eqns. 4.2.4.

The Stress – Stress Function Relations

The stresses in polar coordinates are related to the stresses in Cartesian coordinates through the stress transformation equations (this time a positive rotation; compare with Eqns. 4.2.13 and Fig. 4.2.2)

$$\begin{aligned} \sigma_{rr} &= \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + \sigma_{xy} \sin 2\theta \\ \sigma_{\theta\theta} &= \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - \sigma_{xy} \sin 2\theta \\ \sigma_{r\theta} &= \sin\theta \cos\theta \left(\sigma_{yy} - \sigma_{xx} \right) + \sigma_{xy} \cos 2\theta \end{aligned} \tag{4.2.18}$$

Using the Cartesian stress – stress function relations 3.2.1, one has

$$\sigma_{rr} = \frac{\partial^2 \phi}{\partial y^2} \cos^2\theta + \frac{\partial^2 \phi}{\partial x^2} \sin^2\theta - \frac{\partial^2 \phi}{\partial x \partial y} \sin 2\theta \tag{4.2.19}$$

and similarly for $\sigma_{\theta\theta}, \sigma_{r\theta}$. Using 4.2.11-12 then leads to 4.2.6.

The Compatibility Relation

Beginning with the Cartesian relation 1.3.1, each term can be transformed using 4.2.11-12 and the strain transformation relations, for example

$$\frac{\partial^2 \varepsilon_{xx}}{\partial x^2} = \left(\cos^2\theta \frac{\partial^2}{\partial r^2} + \sin^2\theta \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) + \sin 2\theta \left(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} \right) \right) \times \left(\varepsilon_{rr}\cos^2\theta + \varepsilon_{\theta\theta}\sin^2\theta - \varepsilon_{r\theta}\sin 2\theta \right) \quad (4.2.20)$$

After some lengthy calculations, one arrives at 4.2.8.