3 2D Elastostatic Problems in Cartesian Coordinates

Two dimensional elastostatic problems are discussed in this Chapter, that is, static problems of either plane stress or plane strain. Cartesian coordinates are used, which are appropriate for geometries which are have straight boundaries. The two-dimensional Navier equations are derived and the Airy stress function technique is used to solved exactly some important problems.

3.1 Plane Problems

What follows is to be applicable to any two dimensional problem, so it is taken that $\sigma_{yz} = \sigma_{xz} = 0$, which is true of both plane stress and plane strain.

3.1.1 Governing Equations for Plane Problems

To recall, the equations governing the elastostatic problem are the *elastic stress-strain law* (Part I, Eqns. 4.2.11-14), the *strain-displacement relations* (Eqns. 1.2.5) and the *equations of equilibrium* (1.1.10)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right], \quad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right], \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$
$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

Plane Stress (3.1.1a)

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right], \quad \varepsilon_{yy} = \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right], \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$
$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

Plane Strain (3.1.1b)

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Strain-displacement relations (3.1.2)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$
$$\frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

Equations of Equilibrium (3.1.3)

One way of solving these equations is to re-write the stresses in 3.1.3 in terms of strains by using 3.1.1, and then using 3.1.2 to re-write the resulting equations in terms of displacements only. For example in the case of plane strain one arrives at

$$\frac{E}{2(1+\nu)(1-2\nu)}\left\{2(1-\nu)\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x \partial y}+(1-2\nu)\frac{\partial^2 u_x}{\partial y^2}\right\}+b_x=0$$
$$\frac{E}{2(1+\nu)(1-2\nu)}\left\{2(1-\nu)\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_x}{\partial x \partial y}+(1-2\nu)\frac{\partial^2 u_y}{\partial x^2}\right\}+b_y=0 \qquad (3.1.4)$$

These are the 2D Navier's equations, analogous to the 1D version, Eqn. 2.1.2. This set of partial differential equations can be solved subject to boundary conditions on the displacement. Obviously, in the absence of body forces, any linear displacement field satisfies 3.1.4, for example the field

$$u_x=\frac{\sigma_o}{E}x+A-Cy, \qquad u_y=-\frac{\nu\sigma_o}{E}y+B+Cy \qquad (3.1.5)$$

with A, B, C representing the possible rigid body motions; this corresponds to a simple tension $\sigma_{xx}=\sigma_o$.

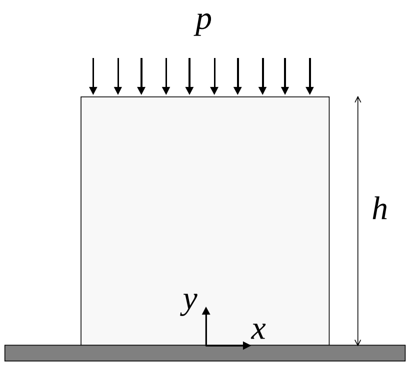
Solving Eqns. 3.1.4 directly for more complex cases is not an easy task. An alternative solution strategy for the plane elastostatic problem is the Airy stress function method described in the next section.

3.1.2 Problems

1. Derive the plane stress Navier equations analogous to 3.1.4.

2. Show that the displacement field $u_x = Ax^2$, $u_y = -4Axy/(1+\nu)$, in the absence of body forces, satisfies the plane stress governing equations derived in Problem 1 (this solution does not satisfy 3.1.4). Determine the corresponding stress field and verify that it satisfies the equilibrium equations.

3. Consider the *thin* plate shown below subjected to a uniform pressure *p* on the top and its own body weight. The plate is *perfectly bonded* to the base plate.
   (a) Does the stress distribution
   $$\sigma_{yy}(x,y)=-p+\rho g(y-h), \quad \sigma_{xx}=\sigma_{xy}=0$$
   satisfy the equations of equilibrium?
   (b) Does it satisfy the boundary conditions at the upper surface, and at the two free surfaces?
   (c) Suppose now that the plate was made out of elastic material. Show that, in that case, the stresses given above are actually not a correct solution to the problem.


p
h
y
x

3.2 The Stress Function Method

An effective way of dealing with many two dimensional problems is to introduce a new "unknown", the **Airy stress function** ϕ, an idea brought to us by George Airy in 1862. The stresses are written in terms of this new function and a new differential equation is obtained, one which can be solved more easily than Navier's equations.

3.2.1 The Airy Stress Function

The stress components are written in the form

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \qquad (3.2.1)$$
$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Note that, unlike stress and displacement, the Airy stress function has no obvious physical meaning.

The reason for writing the stresses in the form 3.2.1 is that, *provided the body forces are zero*, the equilibrium equations are automatically satisfied, which can be seen by substituting Eqns. 3.2.1 into Eqns. 2.2.3 {▲Problem 1}. On this point, the body forces, for example gravitational forces, are generally very small compared to the effect of typical surface forces in elastic materials and may be safely ignored (see Problem 2 of §2.1). When body forces are significant, Eqns. 3.2.1 can be amended and a solution obtained using the Airy stress function, but this approach will not be followed here. A number of examples including non-zero body forces are examined later on, using a different solution method.

3.2.2 The Biharmonic Equation

The Compatability Condition and Stress-Strain Law

In the previous section, it was shown how one needs to solve the equilibrium equations, the stress-strain constitutive law, and the strain-displacement relations, resulting in the differential equation for displacements, Eqn. 3.1.4. An alternative approach is to ignore the displacements and attempt to solve for the *stresses and strains only*. In other words, the strain-displacement equations 3.1.2 are ignored. However, if one is solving for the strains but not the displacements, one must ensure that the compatibility equation 1.3.1 is satisfied.

Eqns. 3.2.1 already ensures that the equilibrium equations are satisfied, so combine now the two dimensional compatibility relation and the stress-strain relations 3.1.1 to get {▲Problem 2}

$$\text{plane stress:} \quad \left(\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) = 0$$

$$\text{plane strain:} \quad \left(\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right)(1-\nu) = 0 \qquad (3.2.2)$$

Thus one has what is known as the biharmonic equation:

$$\boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0} \quad \textbf{The biharmonic equation} \qquad (3.2.3)$$

The biharmonic equation is often written using the short-hand notation $\nabla^4 \phi = 0$.

By using the Airy stress function representation, the problem of determining the stresses in an elastic body is reduced to that of finding a solution to the biharmonic partial differential equation 3.2.3 whose derivatives satisfy certain boundary conditions.

Note that the biharmonic equation is independent of elastic constants, Young's modulus E and Poisson's ratio ν. Thus for bodies in a state of plane stress or plane strain, the stress field is independent of the material properties, provided the boundary conditions are expressed in terms of tractions (stress) [1]; boundary conditions on displacement will bring the elastic constants in through the stress-strain law. Further, the plane stress and plane strain stress fields are identical.

3.2.3 Some Simple Solutions

Clearly, any polynomial of degree 3 or less will satisfy the biharmonic equation. Here follow some elementary examples.

(i) $\phi = Ay^2$

one has $\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2} = 2A, \quad \sigma_{yy} = \sigma_{xy} = 0$, a state if uniaxial tension

(ii) $\phi = Bxy$

here, $\sigma_{xx} = \sigma_{yy} = 0, \quad \sigma_{xy} = -B$, a state of pure shear

(iii) $\phi = Ay^2 + Bxy$

here, $\sigma_{xx} = 2A, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -B$, a superposition of (i) and (ii)

[1] technically speaking, this is true only in simply connected bodies, i.e. ones without any "holes", since problems involving bodies with holes have an implied displacement condition (see, for example, Barber (1992), §2.2).

3.2.4 Pure Bending of a Beam

Consider the bending of a rectangular beam by a moment M_0, as shown in Fig. 3.2.1. The elementary beam theory predicts that the stress σ_{xx} varies linearly with *y*, Fig. 3.2.1, with the $y = 0$ axis along the beam-centre, so a good place to start would be to choose, or guess, as a stress function $\phi = Cy^3$, where C is some constant to be determined. Then $\sigma_{xx} = 6Cy,\ \sigma_{yy} = 0,\ \sigma_{xy} = 0$, and the boundary conditions along the top and bottom of the beam are clearly satisfied.

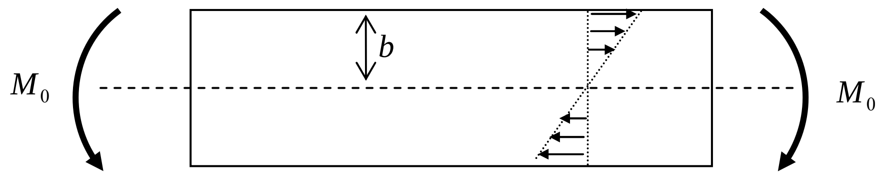


Figure 3.2.1: a beam in pure bending

The moment and stress distribution are related through

$$M_0 = \int_{-b}^{+b} \sigma_{xx} y dy = 6C \int_{-b}^{+b} y^2 dy = 4Cb^3 \qquad (3.2.4)$$

and so $C = M_0 / 4b^3$ and $\sigma_{xx} = 3M_0 y / 2b^3$. The fact that this last expression agrees with the elementary beam theory ($\sigma = -My/I$ with $I = 2b^3 h/3$, where *h* is the depth "into the page") shows that that the beam theory is *exact* in this simple loading case.

Assume now plane strain conditions. In that case, there is another non-zero stress component, acting "perpendicular to the page", $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 3M\nu y / 2b^3$. Using Eqns. 3.1.1b,

$$\begin{aligned} \varepsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] = \left(\frac{1-\nu^2}{E}\frac{3M}{2b^3}\right) y = \alpha y, \text{ say} \\ \varepsilon_{yy} &= \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] = \left(-\frac{\nu(1+\nu)}{E}\frac{3M}{2b^3}\right) y = \beta y, \text{ say} \end{aligned} \qquad (3.2.5)$$

and the other four strains are zero.

As in §1.2.4, once the strains have been found, the displacements can be found by integrating the strain-displacement relations. Thus

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x} = \alpha y \\
&\to u_x = \alpha xy + f(y) \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y} = \beta y \\
&\to u_y = \tfrac{1}{2}\beta y^2 + g(x) \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\big(\alpha x + f'(y) + g'(x)\big) \equiv 0 \\
&\to g'(x) + \alpha x = -f'(y)
\end{aligned} \tag{3.2.6}$$

Therefore $f'(y)$ must be some constant, $-C$ say, so $f(y) = -Cy + A$, and $g(x) = Cx - \frac{1}{2}\alpha x^2 + B$. Finally,

$$\begin{aligned}
u_x &= \alpha xy - Cy + A \\
u_y &= -\tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\beta y^2 + Cx + B
\end{aligned} \tag{3.2.7}$$

which are of the form 1.2.17. For the case when the mid-point of the beam is fixed, so has no translation, $u_x(0,0) = u_y(0,0) = 0$, and if it has no rotation there, $\omega_z(0,0) = 0$, then the three arbitrary constants are zero, and

$$\begin{aligned}
u_x &= \alpha xy \\
u_y &= -\tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\beta y^2
\end{aligned} \tag{3.2.8}$$

3.2.5 A Cantilevered Beam

Consider now the cantilevered beam shown in Fig. 3.2.2. The beam is subjected to a uniform shear stress $\sigma_{xy} = \tau$ over its free end, Fig. 3.2.2a. The boundary conditions are

$$\sigma_{xx}(0,y) = 0, \quad \sigma_{xy}(0,y) = \tau, \quad \sigma_{yy}(x,\pm b) = \sigma_{xy}(x,\pm b) = 0 \tag{3.2.9}$$

It is difficult, if not impossible, to obtain concise expressions for stress and strain for problems even as simple as this[2]. However, a concise solution can be obtained by relaxing one of the above conditions. To this end, consider the similar problem of Fig. 3.2.2b – this beam is subjected to a shear force *F*, the resultant of the shear stresses. The applied force of Fig. 3.2.2b is equivalent to that in Fig. 3.2.2a if

$$\int_{-b}^{+b} \sigma_{xy}(0,y)\,dy = F \tag{3.2.10}$$

[2] an exact solution will usually require an infinite series of terms for the stress and strain

This is known as a **weak boundary condition**, since the stress is not specified in a point-wise sense along the boundary – only the resultant is. However, from Saint-Venant's principle (Part I, §3.3.2), the stress field in both beams will be the same except for in a region close to the applied load.

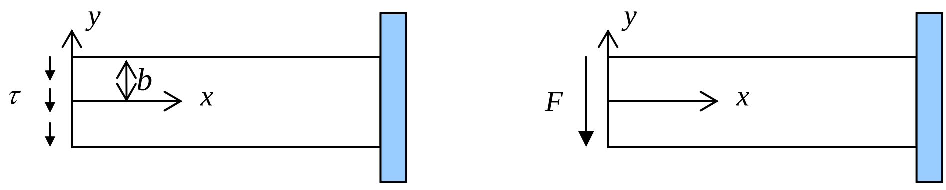


Figure 3.2.2: A cantilevered beam subjected to; (a) a uniform distribution of shear stresses along its free end, (b) a shear force along its free end

The elementary beam theory predicts a stress $\sigma_{xx} = -My/I = Fxy/I$. Thus a good place to start is to choose the stress function $\phi = \alpha xy^3$, where α is a constant to be determined. The stresses are then

$$\sigma_{xx} = 6\alpha xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -3\alpha y^2 \tag{3.2.11}$$

However, it can be seen that $\sigma_{xy}(x, \pm b) = -3\alpha b^2 \neq 0$. To offset this, one can superimpose a constant shear stress $3\alpha b^2$, in other words amend the stress function to

$$\phi = \alpha xy^3 - 3\alpha b^2 xy \tag{3.2.12}$$

The boundary conditions are now satisfied and, from Eqn. 3.2.10,

$$\alpha = \frac{F}{4b^3} \tag{3.2.13}$$

and so

$$\sigma_{xx} = \frac{3F}{2b^3}xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = \frac{3F}{4b^3}\left(b^2 - y^2\right) \tag{3.2.14}$$

3.2.6 Problems

1. Verify that the relations 3.2.1 satisfy the equilibrium equations 2.2.3.

2. Derive Eqn. 3.2.2.

3. A large thin plate is subjected to certain boundary conditions on its thin edges (with its large faces free of stress), leading to the stress function

$$\phi = Ax^3y^2 - Bx^5$$

(i) use the biharmonic equation to express A in terms of B
(ii) calculate all stress components
(iii) calculate all strain components (in terms of B, E, ν)
(iv) derive an expression for the volumetric strain, in terms of B, E, ν, x and y.
(v) check that the compatibility equation is satisfied
(vi) check that the equilibrium equations are satisfied

4. A very thick component has the same boundary conditions on any given cross-section, leading to the following stress function:

$$\phi = x^4y + 4x^2y^3 - y^5$$

(i) is this a valid stress function, i.e. does it satisfy the biharmonic equation?
(ii) calculate all stress components (with $\nu = 1/4$)
(iii) calculate all strain components
(iv) find the displacements
(v) specify any three displacement components which will render the arbitrary constant displacements of (iv) zero

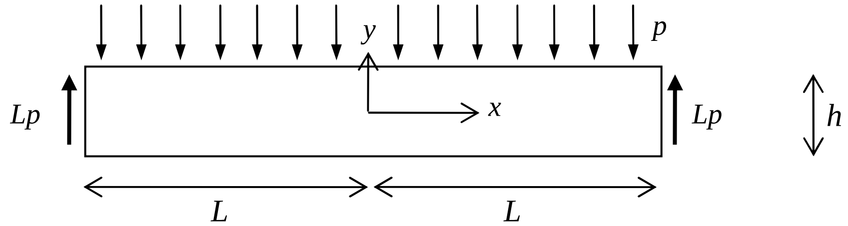
5. For the cantilevered beam discussed in §3.2.5, evaluate the resultant shear force and moment on an arbitrary cross-section $x = \bar{x}$. Are they as you expect? (You will find that the beam is in equilibrium, as expected, since the equilibrium equations have been satisfied.)

6. For the cantilevered beam discussed in §3.2.5, evaluate the strains and displacements, assuming plane stress conditions.
Note: to evaluate the three arbitrary constants of integration, one would be tempted to apply the obvious $u_x = u_y = 0$ all along the built-in end. However, since only weak boundary conditions were imposed, one cannot enforce these strong conditions (try it). Instead, apply the following weaker conditions: (i) the displacement at the built-in end at $y = 0$ is zero ($u_x = u_y = 0$), (ii) the slope there, $\partial u_y / \partial x$, is zero.

7. Show that the stress function

$$\phi = \frac{p}{20h^3}\left[-20y^3\left(L^2 - x^2\right) - 4y^5 - 15h^2x^2y + 2h^2y^3 - 5h^3x^2\right]$$

satisfies the boundary conditions for the simply supported beam subjected to a uniform pressure p shown below. Check the boundary conditions in the *weak* (Saint-Venant) sense on the shorter left and right hand sides (for both normal and shear stress). Since the normal stress σ_{xx} is not zero at the ends, but only its resultant, check also that the moment is zero at each end.



Note that the elementary beam theory predicts an approximate flexural stress but an exact shear stress:

$$\sigma_{xx} = -\frac{6p}{h^3} y\left(L^2 - x^2\right), \quad \sigma_{xy} = \frac{6p}{h^3} x\left(\frac{h^2}{4} - y^2\right)$$

8. Consider the dam shown in the figure below. Assume first a general cubic stress function

$$\phi = \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 y + \frac{1}{2}C_3 x y^2 + \frac{1}{6}C_4 y^3$$

Apply the boundary conditions to determine the constants and hence the stresses in the dam, in terms of ρ, the density of water. (Use the stress transformation equations for the sloped boundary and ignore the weight of the dam.)
[Just consider the effect of the water; to these must be added the stresses resulting from the weight of the dam itself, which are given by

$$\sigma_{xx} = 0, \quad \sigma_{yy} = \rho_s g\left[\frac{1}{\tan\beta} x - y\right], \quad \sigma_{xy} = 0$$

where ρ_s is the density of the dam material.]