2.2 One-dimensional Elastodynamics

In rigid body dynamics, it is assumed that when a force is applied to one point of an object, every other point in the object is set in motion simultaneously. On the other hand, in static elasticity, it is assumed that the object is at rest and is in equilibrium under the action of the applied forces; the material may well have undergone considerable changes in deformation when first struck, but one is only concerned with the final static equilibrium state of the object.

Elastostatics and rigid body dynamics are sufficiently accurate for many problems but when one is considering the effects of forces which are applied *rapidly*, or for very short periods of time, the effects must be considered in terms of the propagation of stress waves.

The analysis presented below is for one-dimensional deformations. Inherent are the assumptions that (1) material properties are uniform over a plane perpendicular to the longitudinal direction, (2) plane sections remain plane and perpendicular to the longitudinal direction and (3) there is no transverse displacement.

2.2.1 The Wave Equation

Consider now the dynamic problem. In this case $u = u(x,t)$ and one considers the governing equations:

$$\frac{\partial \sigma}{\partial x} + b = \rho a \qquad \textbf{Equation of Motion} \qquad (2.2.1a)$$

$$\varepsilon = \frac{\partial u}{\partial x} \qquad \textbf{Strain-Displacement Relation} \qquad (2.2.1b)$$

$$\sigma = E\varepsilon \qquad \textbf{Constitutive Equation} \qquad (2.2.1c)$$

where a is the acceleration. Expressing the acceleration in terms of the displacement, one then obtains the dynamic version of Navier's equation,

$$\boxed{E\frac{\partial^2 u}{\partial x^2} + b = \rho \frac{\partial^2 u}{\partial t^2}} \qquad \textbf{1-D Navier's Equation} \qquad (2.2.2)$$

In most situations, the body forces will be negligible, and so consider the partial differential equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} \qquad \textbf{1-D Wave Equation} \qquad (2.2.3)$$

where

$$c = \sqrt{\frac{E}{\rho}} \qquad (2.2.4)$$

Equation 2.2.3 is the standard one-dimensional **wave equation** with wave speed *c*; note from 2.2.4 that *c* has dimensions of velocity.

The solution to 2.2.3 (see below) shows that a **stress wave** travels at speed *c* through the material from the point of disturbance, e.g. applied load. When the stress wave reaches a given material particle, the particle vibrates about an equilibrium position, Fig. 2.2.1. Since the material is elastic, no energy is lost, and the solution predicts that the particle will vibrate indefinitely, without damping or decay, unless that energy is transferred to a neighbouring particle.

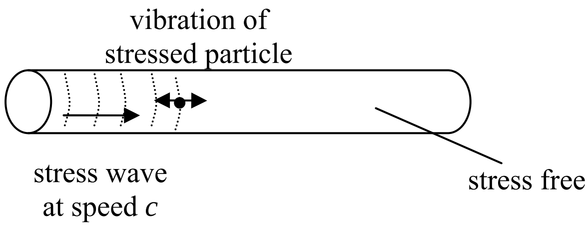

Figure 2.2.1: stress wave travelling at speed *c* through an elastic rod

This type of wave, where the disturbance (particle vibration) is in the same direction as the direction of wave propagation, is called a **longitudinal wave**.

The wave equation is solved subject to the **initial conditions** and **boundary conditions**. The initial conditions are that the displacement *u* and the particle velocity $\partial u / \partial t$ are specified at $t = 0$ (for all *x*). The boundary conditions are that the displacement *u* and the first derivative $\partial u / \partial x$ are specified (for all *t*). This latter derivative is the strain, which is proportional to the stress (see Eqn. 2.2.1b). In problems where there is no boundary (an infinite medium), no boundary conditions are explicitly applied. A semi-infinite medium will have one boundary. For a rod of finite length, there will be two boundaries and a boundary condition will be applied to each boundary.

2.2.2 Particle Velocities and Wave Speed

Before examining the wave equation 2.2.3 directly, first re-express it as

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial v}{\partial t} \tag{2.2.5}$$

where *v* is the velocity. Consider an element of material which has just been reached by the stress wave, Fig. 2.2.2. The length of material passed by the stress wave in a time interval Δt is $c\Delta t$. During this time interval, the stressed material at the left-hand side of the element moves at (average) velocity *v* and so moves an amount $v\Delta t$. The strain of the element is then the change in length divided by the original length:

$$\varepsilon = \frac{v}{c} \tag{2.2.6}$$

Under the small strain assumption, this implies that $v << c$ [1].

Let the stress acting on the element be $\Delta\sigma$; the stress on the free side of the element is zero. Then 2.2.5 leads to

$$\frac{\Delta\sigma}{c\Delta t} = \rho \frac{v}{\Delta t} \tag{2.2.7}$$

and so

$$\Delta\sigma = \rho c v \tag{2.2.8}$$

This is the discontinuity in stress across the wave front.

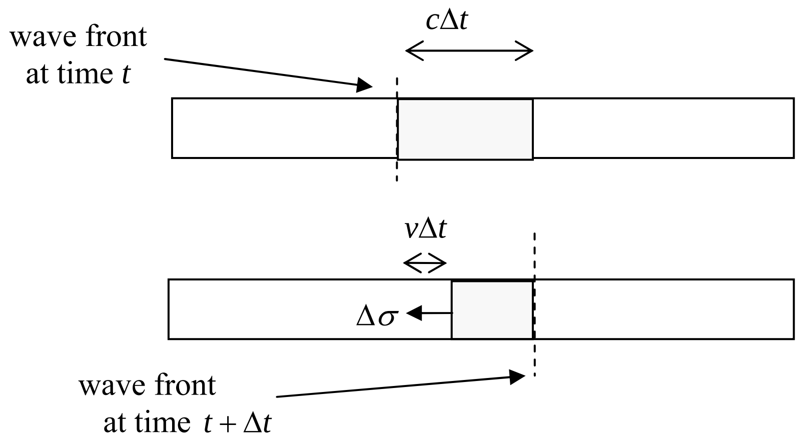

Figure 2.2.2: stress wave passing through a material element

Since $\Delta\sigma = E\varepsilon$, one has $c = \sqrt{E/\rho}$, as in 2.2.4. The wave speeds for some materials are given in Table 2.2.1. As can be seen, the wave speeds for typical engineering materials are of the order km/s and so particle velocities will be in the range $0 - 50\text{m/s}$.

Material	$\rho\ (\text{kg/m}^3)$	$E\ (\text{GPa})$	$c\ (\text{m/s})$
Aluminium Alloy	2700	70	5092
Brass	8300	95	3383
Copper	8500	114	3662
Lead	11300	17.5	1244
Steel	7800	210	5189
Glass	1870	55	5300
Granite	2700	26	3120
Limestone	2600	63	4920
Perspex	490	2.5	2260

Table 2.2.1: Elastic Wave Speeds for Several Materials

[1] note also that the density of the element will change as it is compressed, but again this change in density is small and can be neglected in the linear elastic theory

Consider steel: the velocity at which the material ceases to behave linearly elastic (taking the yield stress to be 400MPa) is $v = Y / \rho c \approx 10\text{m/s}$.

2.2.3 Waves

Before proceeding, it will be helpful to review and summarise the important facts and terminology regarding waves.

Suppose that there is a displacement *u* which is propagated along the *x* axis at velocity *c*. At time $t = 0$ say, the disturbance will have some **wave profile** $u = f(x)$. If the disturbance propagates *without change of shape*, then at some later time *t* the profile will look identical but it will have moved a distance *ct* in the positive direction. If we take a new origin at the point $x = ct$ and let the distance measured from this origin be $\overline{x}$, then the equation of the new wave profile referred to this new origin would be $u = f(\overline{x})$. Referred to the original fixed origin, then,

$$u = f\left(x - ct\right). \tag{2.2.9}$$

This is the most general expression for a wave travelling at constant velocity *c* and without change of shape, along the positive *x* axis. If the wave is travelling in the negative direction, then its form would be $u = f\left(x + ct\right)$.

The simplest type of wave of this kind is the **harmonic wave**, in which the wave profile is a sine or cosine curve. If the wave profile at time $t = 0$ is $u = a\cos\left(kx\right)$, then at time *t* the profile is

$$u = a\cos\left[k\left(x - ct\right)\right]. \tag{2.2.10}$$

The maximum value of the disturbance, *a*, is called the **amplitude**. The wave profile repeats itself at regular distances $2\pi / k$, which is called the **wavelength** λ. The parameter *k* is called the **wave number**[2]; since there is one wave in λ units of distance, it is the number of waves in 2π units of distance:

$$k = \frac{2\pi}{\lambda}. \tag{2.2.11}$$

The distance travelled by the wave in time *t* is ct. The time taken for one complete wave to pass any point is called the **period** *T*, which is the time taken to travel one wavelength:

$$T = \frac{\lambda}{c}. \tag{2.2.12}$$

The **frequency** *f* is the number of waves passing a fixed point in unit time, so

[2] more specifically, this is the **angular wavenumber**, to distinguish it from the (**spectroscopic**) **wavenumber** $1/\lambda$

$$f = \frac{1}{T} = \frac{c}{\lambda} \tag{2.2.13}$$

The **angular frequency** is $\omega = 2\pi f = kc$.

As the wave travels along, the particle at any fixed point displaces back and forth about some equilibrium position; the particle is said to **vibrate**. The period and frequency were defined above in terms of the time taken for a wave to travel along the *x* axis. It can be seen that the period *T* is also equivalent to the time taken for a particle to displace away and then back to its original position, then off in the other direction and back again; the frequency *f* can also be seen to be equivalent to the number of times the particle vibrates about its equilibrium position in unit time.

The wave 2.2.10 can be expressed in the equivalent forms:

$$\begin{aligned}
u &= a\cos\left[k\left(x - ct\right)\right] \\
u &= a\cos\left[\frac{2\pi}{\lambda}\left(x - ct\right)\right] \\
u &= a\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \\
u &= a\cos\left(2\pi\frac{x}{\lambda} - \omega t\right) \\
u &= a\cos\left(kx - \omega t\right)
\end{aligned} \tag{2.2.14}$$

If one has two waves, $u_1 = a\cos\left(kx - \omega t\right)$ and $u_2 = a\cos\left(kx - \omega t + \phi\right)$, then the waves are the same except they are displaced relative to each other by an amount $\phi / k = \phi\lambda / 2\pi$; ϕ is called the **phase** of u_2 relative to u_1. If ϕ is a multiple of 2π, then the displaced distance is a multiple of the wavelength, and the waves are said to be **in phase**.

It can be verified by substitution that the wave 2.2.14 is a solution of the wave equation 2.2.3.

Example

Fig. 2.2.3 shows a wave travelling through steel and vibrating at frequency $f = 1\,\text{kHz}$. Using the data in Table 2.2.1, the wave number is $k = 2\pi f / c \approx 1.21$ and the wavelength is $\lambda = c / f \approx 5.2$. The period is $T = 1/1000$ sec. For unit amplitude, $a = 1$, the wave profiles are shown for $t = 0$ (blue) and $t = 1/1500$ sec $(= \frac{2}{3}T)$ (red). The dashed arrows show the movement of one particle as the wave passes.

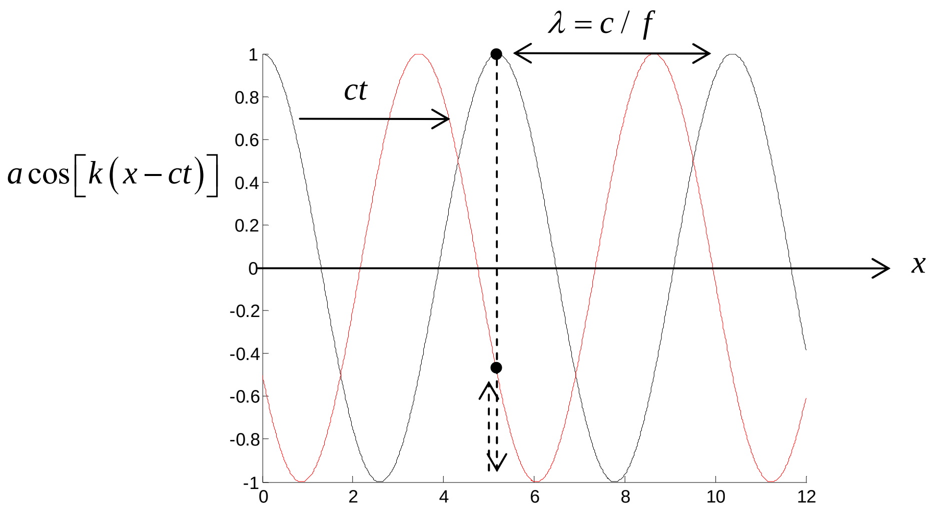

Figure 2.2.3: harmonic wave (Eqn. 2.2.10) travelling through steel at 1 kHz; $a = 1$ with $t = 0$ **(blue) and** $t = 1/1500$ **(red)**

Standing Waves

Because the wave equation is linear, any linear combination of waves is also a solution. In particular, consider two waves which are similar, only travelling in opposite directions; the superposition of these waves is the new wave

$$\begin{aligned} u &= a\cos\left(kx - \omega t\right) + a\cos\left(kx + \omega t\right) \\ &= 2a\cos\left(kx\right)\cos\left(\omega t\right) \end{aligned} \tag{2.2.15}$$

It will be seen that this wave profile does not move forward, and is therefore called a **standing wave** (to distinguish it from the **progressive waves** considered earlier). An example is shown in Fig. 2.2.4 (same parameters as for Fig. 2.2.3); at any fixed point, the wave moves up and down over time. The period is again $T = 1/1000$ sec. Shown is the wave at five instants, from $t = 0$ up to just short of the half-period.

Note that $u = 0$ for $x = \pm(2n+1)\pi / 2k$, $n = 0,1,2,\ldots$; these are called the **nodes** of the wave. The intermediate points, where the amplitude is greatest, are called **antinodes**. The distance between successive nodes (or antinodes) is half the wavelength.

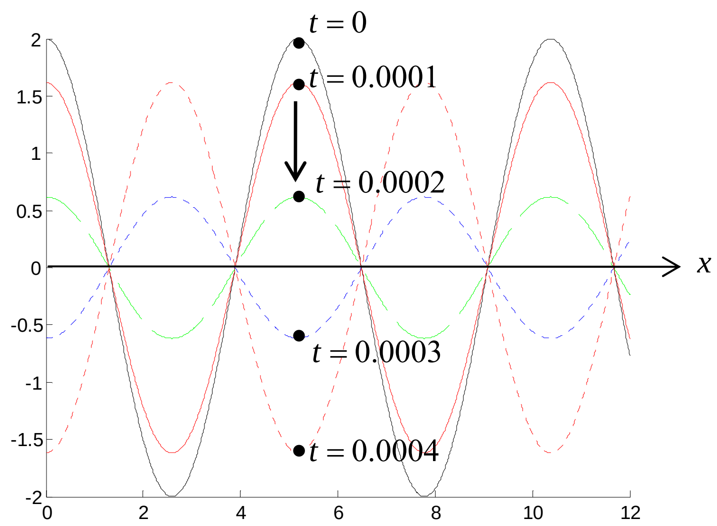

Figure 2.2.4: standing wave (Eqn. 2.2.15) in steel at 1 kHz; with $a = 1$ **at** $t = 0$ **(black),** $t = 0.0001$ **(red),** $t = 0.0002$ **(green dashed),** $t = 0.0003$ **(blue dotted) and** $t = 0.0004$ **(red dotted)**

If the wave is not harmonic, one can use a Fourier analysis (see below) to construct the wave out of a sum of individual harmonic waves; if the profile consists of a regularly repeating pattern, the definitions of wavelength, period, frequency and wave number, and the relations between them, Eqns. 2.2.11-13, still apply.

Complex Exponential Representation

When dealing with progressive waves of harmonic type, it is usually best to represent the wave using a complex exponential function. The reason for this is that exponentials are algebraically simpler than harmonic functions, and also the amplitude and phase are represented by one complex quantity rather than by two separate terms (as will be seen below).

The general wave of the form

$$u = a\cos(kx - \omega t + \phi) \tag{2.2.16}$$

is the real part of the complex exponential

$$ae^{i(kx-\omega t+\phi)} = a\left[\cos(kx - \omega t + \phi) + i\sin(kx - \omega t + \phi)\right] \tag{2.2.17}$$

The phase shift and amplitude can be absorbed into a new constant *A*:

$$u = Ae^{i(kx-\omega t)}, \quad A = ae^{i\phi} \tag{2.2.18}$$

It can be verified that this complex quantity is itself a solution of the wave equation, Eqn. 2.2.3 (and if a complex quantity is a solution, so are its real and imaginary parts). One can carry out analyses using the complex expression 2.2.18, keeping in mind that the "real" solution, Eqn. 2.2.16, is the real part of this expression. Since $\left|e^{i(kx-\omega t)}\right| = 1$, the true amplitude is $|A|$. The true phase shift ϕ is the argument of A, $\arg A$.

Eqn. 2.2.16 is a wave travelling to the "right". It has been seen how a wave travelling to the right is of the form $u = a\cos(kx+\omega t)$, suggesting a complex representation $u = Ae^{i(kx+\omega t)}$. However, this is not an ideal representation, because the difference between a wave travelling left or right, i.e. the difference between this expression and the one in Eqn. 2.2.17, is given by the sign of the frequency. This can make it difficult to solve problems involving reflecting waves[3] (see below), and therefore it is best to use the following representations when adding and subtracting waves:

$$\text{Travelling right: } Ae^{+i(kx-\omega t)} \tag{2.2.19a}$$

$$\text{Travelling left: } Ae^{-i(kx+\omega t)} \tag{2.2.19b}$$

(Note: another popular convention is to use $Ae^{-i(kx-\omega t)}$ for right and $Ae^{+i(kx+\omega t)}$ for left.)

2.2.4 Solution of the Wave Equation (D'Alembert's Solution)

The one-dimensional wave equation 2.2.3 has the very general solution (this is D'Alembert's solution – see the Appendix to this section for its derivation)

$$\boxed{u(x,t) = f(x-ct) + g(x+ct)} \tag{2.2.20}$$

where *f* and *g* are *any* functions[4]; for example, one solution is $f = e^{x-ct}$, $g = \sin(x+ct)$, which can be verified by substitution and carrying out the differentiation. The harmonic waves considered above are special cases of this solution, in which *f* and *g* are cosine functions. The actual forms of the functions *f* and *g* can be determined from the **initial conditions** of the problem, which are the initial displacement profile $u(x,0)$ and the initial velocity $v(x,0) = \partial u/\partial t|_{(x,0)}$. Consider the arbitrary initial conditions

$$\begin{aligned} u(x,0) &= U(x) \\ v(x,0) &= V(x) \end{aligned} \tag{2.2.21}$$

Then, as shown in the Appendix to this section, the solution is

$$u(x,t) = \frac{1}{2}\left[U(x+ct) + U(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\alpha)\,d\alpha \tag{2.2.22}$$

---

[3] for example, when a wave hits a boundary and gets reflected, this representation would force the incident and reflected waves to have different frequencies, when in fact a solution in which the frequencies are the same is often sought

[4] provided they possess second derivatives

Example

Suppose for example that the initial displacement profile was triangular, with maximum displacement $u = \bar{u}$ at $x = 0$, extending to $x = \pm L$, Fig. 2.2.5.

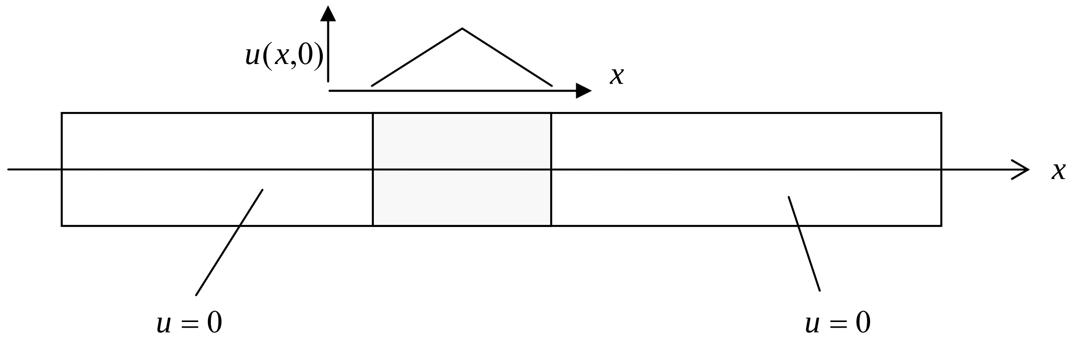

Figure 2.2.5: an initial triangular displacement

The initial conditions are

$$U(x) = u(x,0) = \begin{cases} 0, & |x| \geq L \\ \bar{u}(1 + x/L), & -L \leq x \leq 0 \\ \bar{u}(1 - x/L), & 0 \leq x \leq +L \end{cases}$$

and $V(x) = 0$. D'Alembert's solution is then

$$u(x,t) = \tfrac{1}{2}\left[U(x - ct) + U(x + ct)\right]$$

The solution predicts that at time $2L/c$ there are two triangular displacement profiles of half the magnitude of the original profile; one is to the left and the other is to the right of the original profile, Fig. 2.2.6.

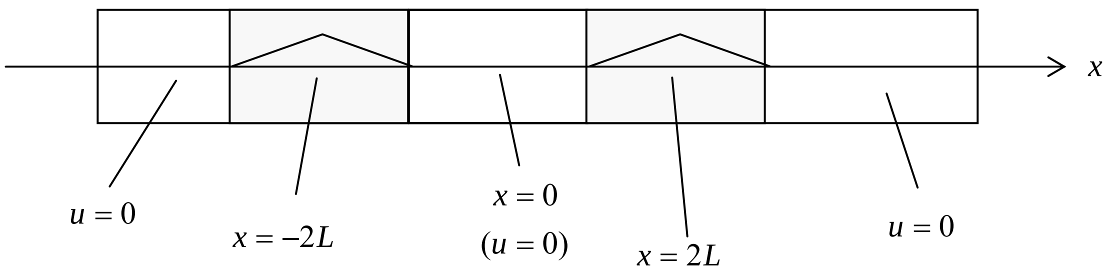

Figure 2.2.6: displacements at time 2*L*/*c*

As the wave passes, particles displace from their equilibrium point, up to the maximum position and then back again. It can be seen that the solution corresponds to a wave of disturbed material propagating through the material from the source, half in one direction and half in the other.

2.2.5 Reflection and Transmission

Let a train of harmonic waves travel from the negative *x* direction in a material with material properties E_1, ρ_1. The waves then meet a second material with different material properties E_2, ρ_2, at the origin $x = 0$. Let the displacements in the first material be u_1 and those in the second, u_2. As will be seen, the incident wave upon the second material will suffer partial reflection and partial transmission. Using the complex exponential representation, Eqn. 2.2.19, and superscripts "*i*" for incident, "*r*" for reflected and "*t*" for transmitted:

$$u_1 = u^{(i)} + u^{(r)}, \qquad u_2 = u^{(t)} \tag{2.2.23}$$

with

$$u^{(i)} = A_i e^{i(k_1 x - \omega t)}, \quad u^{(r)} = A_r e^{-i(k_1 x + \omega t)}, \quad u^{(t)} = A_t e^{i(k_2 x - \omega t)} \tag{2.2.24}$$

A_1 is real, but in general B_1, A_2 could be complex. The wave speeds *c* in each material will be different (if the material properties are different). The frequencies of all three waves are the same – since the material is connected to adjacent material, it must all be vibrating at the same frequency. It follows that the wavenumbers *k* differ also:

$$k_1 c_1 = k_2 c_2 \qquad \text{or} \qquad \frac{k_1}{k_2} = \sqrt{\frac{E_2 \rho_1}{E_1 \rho_2}} \tag{2.2.25}$$

The boundary conditions at the material interface are that

$$\begin{aligned} u_1(0,t) &= u_2(0,t) \\ E_1 \left.\frac{\partial u_1}{\partial x}\right|_{(0,t)} &= E_2 \left.\frac{\partial u_2}{\partial x}\right|_{(0,t)} \end{aligned} \tag{2.2.26}$$

The first of these says that the material remains continuous at the interface. The second says that the stress is also continuous there (see Eqns. 2.2.1b-c). Applying these to Eqn. 2.2.23 gives

$$\begin{aligned} A_i + A_r &= A_t \\ A_i E_1 k_1 - A_r E_1 k_1 &= A_t E_2 k_2 \end{aligned} \tag{2.2.27}$$

so that

$$A_r = A_i \frac{1-\varphi}{1+\varphi}, \qquad A_t = A_i \frac{2}{1+\varphi} \tag{2.2.28}$$

where

$$\varphi = \frac{E_2 k_2}{E_1 k_1} = \frac{c_2 \rho_2}{c_1 \rho_1} = \sqrt{\frac{E_2 \rho_2}{E_1 \rho_1}} \tag{2.2.29}$$

Note that, since A_i is real, so also are A_r, A_t.

The stresses are given by

$$\sigma^{(r)} = -\frac{A_r}{A_i}\sigma^{(i)} = \frac{\varphi - 1}{\varphi + 1}\sigma^{(i)}, \qquad \sigma^{(t)} = \frac{A_t}{A_i}\frac{E_2 k_2}{E_1 k_1}\sigma^{(i)} = \frac{2\varphi}{\varphi + 1}\sigma^{(i)} \tag{2.2.30}$$

The parameter φ determines the nature of the reflected and transmitted waves, and is the ratio of the quantities ρc of each material; this quantity ρc is often referred to as the **mechanical impedance** of the rod. Note that the stiffness E and density ρ are independent, so if $E_2 > E_1$, this does not imply that $\rho_2 > \rho_1$ or that $\varphi > 1$ (see Table 2.2.1).

When $\varphi > 1$, the reflected wave has opposite sign to that of the incident wave and has a smaller amplitude. The transmitted wave is of the same sign and is also smaller. In the limit as $\varphi \to \infty$, which would represent a perfectly rigid material 2 ($E_2 \to \infty$), there is no transmitted wave and the reflected wave has amplitude $A_r = -A_i$. The stress at the boundary is twice the stress due to the incident wave alone.

When $\varphi < 1$, the reflected wave has the same sign to that of the incident wave and has a smaller amplitude. The transmitted wave is of the same sign and is larger. In the limit as $\varphi \to 0$, which would represent "empty" material 2, the reflected wave is equal to the incident wave. The stress at the boundary is zero – this is called a "free boundary" (see below).

Examples of harmonic waves travelling through steel and granite are shown in Fig. 2.2.7. The frequency of vibration is taken to be $f = 1\,\text{kHz}$. Using the data in Table 2.2.1, the wave numbers are $k_s = 2\pi f / c_s \approx 1.21$ and $k_g = 2\pi f / c_g \approx 2.01$. The wavelengths of the waves are $\lambda_s = c_s / f \approx 5.2$ and $\lambda_g = c_g / f \approx 3.1$. The incident wave is taken to have unit amplitude. When the wave travels from steel into granite, $\varphi = 0.207$ and when it travels from granite into steel it is the inverse of this, $\varphi = 4.83$. The interference between the incident and reflected waves produce a new wave in material "1" (denoted by the green plots in Fig. 2.2.7):

$$u_1 = a\left\{\cos(kx - \omega t) + \frac{1-\varphi}{1+\varphi}\cos(kx + \omega t)\right\} \tag{2.2.31}$$

Note that $A_t = A_i$ at time $t = 0$ (full reflected and transmitted wave profiles are plotted at time zero, even though there is no actual wave present right through the material yet at this time).

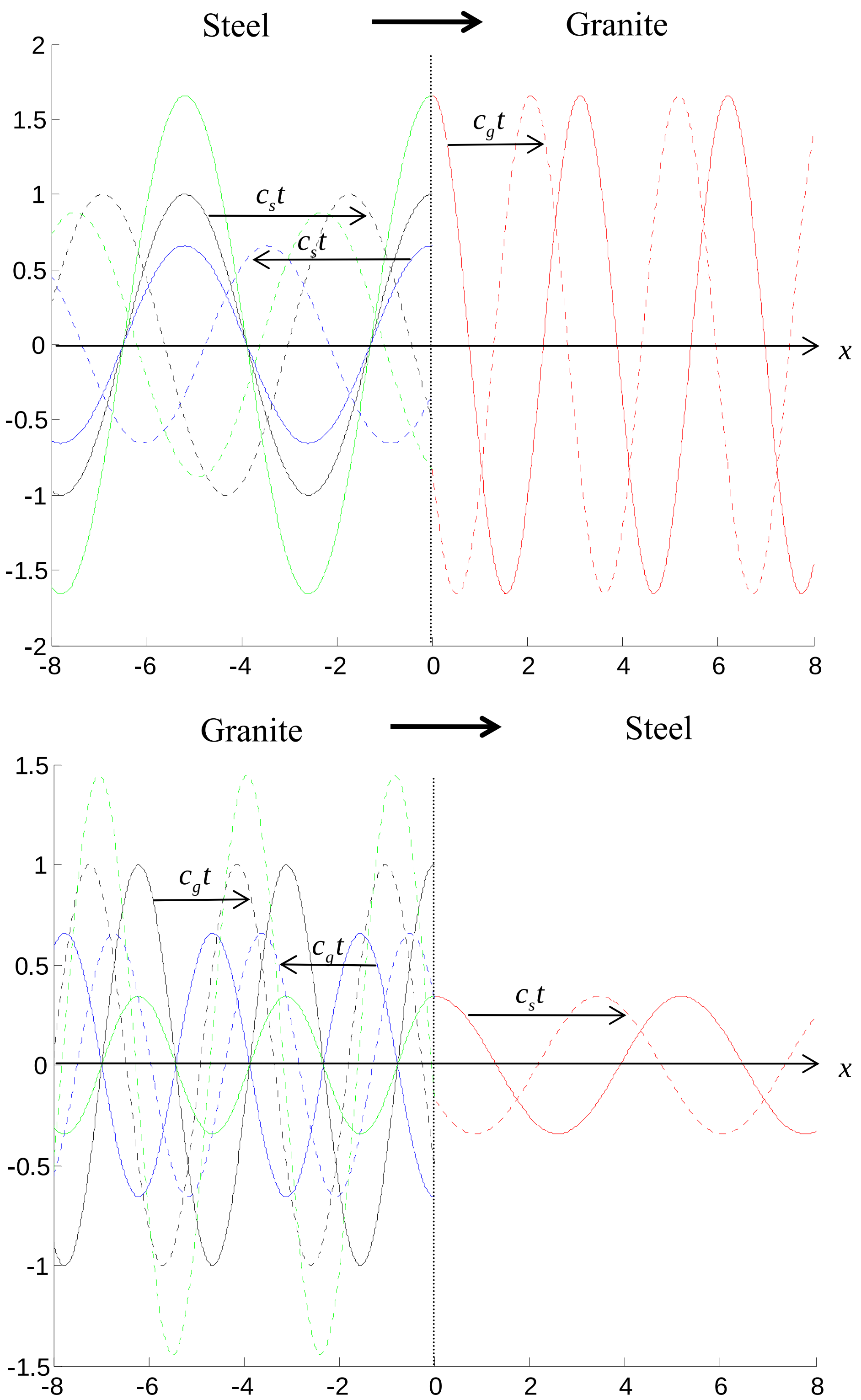

Figure 2.2.7: reflection and transmission of harmonic waves at the boundary between steel and granite; at time $t = 0$ (solid) and time $t = 1/1500$ (dashed); incident (black), reflected (blue), transmitted (red) and composite wave in material "1" (green)

2.2.6 Energy in Vibrating Bars

The kinetic energy in an element of length dx of the bar is $dK = \frac{1}{2} A\rho \left(\partial u / \partial t\right)^2 dx$, where A is the cross-sectional area. The total kinetic energy in a bar of length L is then

$$K = \frac{1}{2}\rho A \int_0^L \left(\partial u / \partial t\right)^2 dx, \tag{2.2.32}$$

The potential energy is the elastic strain energy; for a small element of length dx this is $dW = \frac{1}{2}\sigma\varepsilon A dx$, so

$$W = \frac{1}{2} AE \int_0^L \left(\partial u / \partial x\right)^2 dx, \tag{2.2.33}$$

2.2.7 Solution of the Wave Equation (Standing Waves)

D'Alembert's solution gives results for progressive waves travelling in an infinitely extended medium. Standing waves in an infinite medium can also be a solution. For example if one has the initial profile $U(x) = a\cos(kx)$ and zero initial velocity, $V(x) = 0$, one gets from Eqn. 2.2.22 the standing wave 2.2.15.

Standing waves can be generated more generally by using a **separation of variables** solution procedure for Eqn. 2.2.3. Using this method, detailed in the Appendix to this section, one has the general solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(\bar{A}_n \cos k_n x + \bar{B}_n \sin k_n x\right)\left(\bar{C}_n \cos ck_n t + \bar{D}_n \sin ck_n t\right) \tag{2.2.34}$$

The (infinite number of) constants $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ and **eigenvalues**[5] k can be obtained from the initial and boundary conditions (see later). What are termed "eigenvalues" in this context can be seen to be the wave number.

The terms $\cos k_n x$ and $\sin k_n x$ are called **modes** or **mode shapes**. At any given time *t*, the displacement is a linear combination of these modes. Example modes are shown in Fig. 2.2.8. Some modes will dominate over others, for example perhaps only the first few modes (terms in the series 2.2.34) are significant and need be considered.

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[5] note that some authors use the term "eigenvalue" to mean the quantity (ck_n) in this expression

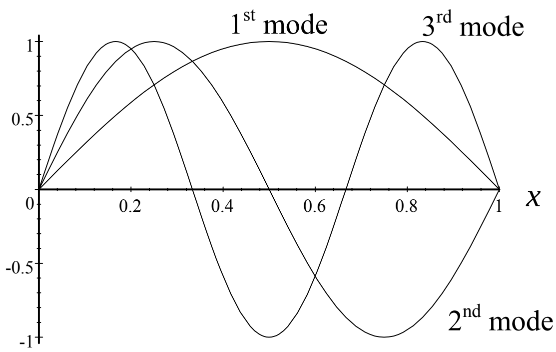

Figure 2.2.8: mode shapes for a vibrating elastic rod

Natural Frequencies

The eigenvalues (or, equivalently, the **natural frequencies** $\omega = ck$) depend on the boundary conditions. There are four possible cases for the one-dimensional rod. Taking the bar to have end-points $x = 0, L$, the boundary conditions are (these are the same as for the static elasticity problem):

1. **fixed-fixed** - $u(0,t) = 0$, $u(L,t) = 0$
2. **free-free** - $\partial u / \partial x\big|_{(0,t)} = 0$, $\partial u / \partial x\big|_{(L,t)} = 0$
3. **fixed-free** - $u(0,t) = 0$, $\partial u / \partial x\big|_{(L,t)} = 0$ (2.2.35)
4. **free-fixed** - $\partial u / \partial x\big|_{(0,t)} = 0$, $u(L,t) = 0$

The natural frequencies and modes for each of these boundary conditions are solved for and given in the Appendix to this section (in the boxes). For example, considering the "fixed-fixed" case, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos(k_n ct) + B_n \sin(k_n ct) \right] \sin(k_n x) \tag{2.2.36}$$

with

$$\begin{aligned} &\text{Frequencies: } \omega_n = k_n c = \frac{n\pi c}{L}, \quad n = 0,1,\ldots \\ &\text{Modes: } \sin\left(k_n x\right), \quad n = 0,1,\ldots \end{aligned} \tag{2.2.37}$$

One can plot these sine functions over $[0, L]$ to see the displacement profile of each mode (the first three are those plotted in Fig. 2.2.8 – it can be seen that the higher the mode, the higher the frequency).

The complete solution and precise profile is then obtained by applying the initial conditions of the problem to determine the coefficients A_n, B_n in Eqn. 2.2.36. Some examples of this complete calculation are given in the Appendix.

Vibration Analysis

A **vibration analysis** is one in which the eigenvalues (natural frequencies) and modes are evaluated without regard to which of them might be important in an application. The boundary conditions *alone* determine the modes and natural frequencies. Thus a vibration analysis is carried out *without regard to how the vibration is initiated*. The exact combination of the modes for a particular problem is determined from the initial conditions; the initial conditions will determine the arbitrary constants in the above equations and hence the actual amplitude of vibration.

The vibration is termed **free** if the load is zero or constant; **forced vibration** occurs when the load itself oscillates.

Even though a vibration analysis does not completely solve the problem of a material model loaded in a certain way, for example solving for the propagation paths of stress waves, the amplitudes of vibration, and so on, the natural frequencies and modes are very useful information in themselves, for design and other purposes.

Dynamic response analysis or **transient response analysis** is the calculation of the *complete* response to any arbitrary boundary and initial conditions. This is more difficult than the vibration analysis, since it is a time-dependent problem.

Non-Homogeneous Boundary Conditions

The boundary conditions in 2.2.35 are all **homogeneous** (i.e. $u = 0$ or $\partial u / \partial x = 0$). In practice, the boundary conditions will not be homogeneous, but the natural frequencies do not depend on whether the boundary conditions are homogeneous or non-homogeneous. In other words, if one wants to determine the natural frequencies, *one needs only consider the case of homogeneous boundary conditions*, as will be seen now.

Consider the following **non-homogeneous** boundary conditions:

$$\text{BC's:} \quad u(0,t) = \hat{u}\,, \; u(L,t) = 0 \tag{2.2.38}$$

Since the wave equation is linear, the solution can be written as the superposition of two separate solutions,

$$u(x,t) = u_p(x,t) + u_h(x,t) \tag{2.2.39}$$

The u_h is the **homogeneous** solution, and is chosen to satisfy the wave equation with homogeneous boundary conditions; u_p is some **particular** solution and accounts for the non-homogeneous boundary condition:

$$\begin{aligned} \text{BC's:} \qquad & u_h(0,t) = 0\,, \quad u_h(L,t) = 0 \\ & u_p(0,t) = \hat{u}\,, \quad u_p(L,t) = 0 \end{aligned} \tag{2.2.40}$$

Substituting 2.2.39 into the wave equation 2.2.3 gives

$$\frac{\partial^2 u_h}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_h}{\partial t^2} = -\left(\frac{\partial^2 u_p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_p}{\partial t^2}\right) \tag{2.2.41}$$

The left hand side is zero. The right hand side can be made zero by choosing u_p to be *any* particular solution of the wave equation. For a simple constant displacement boundary condition, one can choose the linear function

$$u_p(x) = \hat{u}\left(1 - \frac{x}{L}\right) \tag{2.2.42}$$

which can be seen to satisfy 2.2.40b. The complete solution u is illustrated in Fig. 2.2.9.

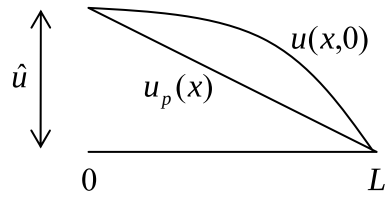

Figure 2.2.9: displacements as a superposition of two separate solutions

Suppose now that the initial conditions are

$$\text{IC's:} \quad \begin{aligned} u(x,0) &= \bar{u}(x) \\ v(x,0) &= \bar{v}(x) \end{aligned} \tag{2.2.43}$$

The initial conditions can be split between u_h and u_p according to

$$\text{IC's:} \quad \begin{aligned} u_h(x,0) &= \bar{u}(x) - u_p(x), \quad u_p(x,0) = u_p(x) \\ v_h(x,0) &= \bar{v}(x) - v_p(x), \quad v_p(x,0) = v_p(x) \end{aligned} \tag{2.2.44}$$

Thus, the complete solution is obtained by adding together:

(i) the function u_h which satisfies the wave equation with homogeneous boundary conditions on displacement, and initial conditions

$$\text{IC's:} \quad \begin{aligned} u_h(x,0) &= \bar{u}(x) - u_p(x) \\ v_h(x,0) &= \bar{v}(x) - v_p(x) \end{aligned}$$

(ii) the function

$$u_p(x) = \hat{u}\left(1 - \frac{x}{L}\right)$$

Thus, using the "fixed-fixed" homogeneous solution from the Appendix,

$$u(x,t) = \hat{u}\left(1 - \frac{x}{L}\right) + \sum_{n=0}^{\infty}\left[A_n \cos(k_n ct) + B_n \sin(k_n ct)\right]\sin(k_n x) \tag{2.2.45}$$

and the natural frequencies are given by 2.2.37. The constants A_n, B_n can be obtained from the initial conditions, as outlined in the Appendix.

The important point to be made here is that the modes and natural frequencies are determined from (i), i.e. the problem involving the homogeneous boundary conditions, and so, as stated above, the non-homogeneous boundary condition does not affect the modes and natural frequencies.

Forced Vibration

Suppose now that the boundary conditions and initial conditions are given by

$$\text{BC's:} \quad \begin{array}{l} u(0,t) = \alpha \cos(\Omega t) \\ u(L,t) = 0 \end{array}, \quad \text{IC's:} \quad \begin{array}{l} u(x,0) = \hat{u}\cos(x\pi / 2L) \\ v(x,0) = 0 \end{array} \tag{2.2.46}$$

Again, let $u(x,t) = u_p(x,t) + u_h(x,t)$ and substitute into the wave equation. In this case, the particular solution will be of the general form 2.2.34,

$$u_p = \left(A\cos kx + B\sin kx\right)\left(C\cos ckt + D\sin ckt\right) \tag{2.2.47}$$

Applying the boundary conditions, one finds that {▲Problem 1}

$$u_p(x,t) = \alpha\left\{\cos\left(\frac{\Omega x}{c}\right) - \cot\left(\frac{\Omega L}{c}\right)\sin\left(\frac{\Omega x}{c}\right)\right\}\cos(\Omega t) \tag{2.2.48}$$

As with the constant non-homogeneous boundary condition, the initial conditions can now be split appropriately between the homogeneous and particular solutions. Again, the complete solution is obtained by adding together:

(i) the function u_h which satisfies the wave equation with homogeneous boundary conditions on displacement, and initial conditions

$$\text{IC's:} \quad \begin{array}{l} u_h(x,0) = \hat{u}\cos\left(\dfrac{x\pi}{2L}\right) - \alpha\left\{\cos\left(\dfrac{\Omega x}{c}\right) - \cot\left(\dfrac{\Omega L}{c}\right)\sin\left(\dfrac{\Omega x}{c}\right)\right\} \\ v_h(x,0) = 0 \end{array}$$

(ii) the function 2.2.48

The complete solution is

$$u(x,t) = \alpha\left\{\cos\left(\frac{\Omega x}{c}\right) - \cot\left(\frac{\Omega l}{c}\right)\sin\left(\frac{\Omega x}{c}\right)\right\}\cos(\Omega t)$$
$$+\sum_{n=0}^{\infty}\left[A_n\cos(k_n ct) + B_n\sin(k_n ct)\right]\sin(k_n x) \tag{2.2.49}$$

Resonance occurs when the displacements become "infinite", which from 2.2.49 occurs when

$$\sin\frac{\Omega L}{c} = 0 \rightarrow \Omega = n\frac{\pi c}{L}.$$

These are precisely the natural frequencies of the system, i.e. the natural frequencies of (i). Thus the problem of resonance becomes more prominent when the forcing frequency Ω approaches any of the natural frequencies k_n.

2.2.8 Problems

1. Consider the case of forced vibration. Use the boundary conditions 2.2.46 to evaluate the constants in the particular solution 2.2.47 and hence derive the particular solution 2.2.48.

2. Consider a fixed-free problem, with the end $x = 0$ subjected to a forced displacement $u = \alpha\sin\Omega t$ and the end $x = L$ free.
   (a) Find the vibration of the material. What are the natural frequencies?
   (b) When does resonance occur?
   [note: the appropriate homogeneous solution and natural frequencies are given in the Appendix to this section]

3. Consider a vibrating bar with an oscillatory stress applied to one end, $\sigma(0) = \alpha\cos\Omega t$. The end $x = L$ is fixed, $u(L) = 0$.
   (a) Find the vibration of the material. What are the natural frequencies?
   (b) When does resonance occur?
   [note: the appropriate homogeneous solution and natural frequencies are given in the Appendix to this section]

2.2.9 Appendix to Section 2.2

1. D'Alembert's Solution of the Wave Equation

In the wave equation 2.2.3, change variables through

$$\xi = x - ct, \qquad \eta = x + ct \tag{2.2.50}$$

Then $u = u\left(\xi(x,t), \eta(x,t)\right)$ and the chain rule gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \tag{2.2.51}$$

and similarly for the variable *t*. Another differentiation gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \tag{2.2.52}$$

and similarly for the variable *t*. Substituting these expression into the wave equation 2.2.3 leads to

$$4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{2.2.53}$$

Integrating with respect to ξ gives $\partial u / \partial \eta = \gamma(\eta)$ where $\gamma(\eta)$ is some arbitrary function. A further integration then gives $u = \int \gamma(\eta) d\eta + f(\xi) = f(\xi) + g(\eta)$, which is D'Alembert's solution, Eqn. 2.2.20:

$$u(x,t) = f(x - ct) + g(x + ct) \tag{2.2.54}$$

Let the initial conditions be

$$\begin{aligned} u(x,0) &= U(x) \\ \left.\frac{\partial u}{\partial t}\right|_{(x,0)} &= V(x) \end{aligned} \tag{2.2.55}$$

Thus, from 2.2.54,

$$U(x) = f(x) + g(x). \tag{2.2.56}$$

Now

$$\frac{\partial u}{\partial t}=\frac{\partial f\left(\xi(x,t)\right)}{\partial t}+\frac{\partial g\left(\eta(x,t)\right)}{\partial t}=\frac{df}{d\xi}\frac{\partial \xi}{\partial t}+\frac{dg}{d\eta}\frac{\partial \eta}{\partial t}=-c\frac{df}{d\xi}+c\frac{dg}{d\eta} \tag{2.2.57}$$

At $t=0$, $f(\xi)=f(x)$ and $g(\eta)=g(x)$, so

$$V(x)=\left.\frac{\partial u}{\partial t}\right|_{(x,0)}=-c\frac{df(x)}{dx}+c\frac{dg(x)}{dx} \tag{2.2.58}$$

Integrating then gives

$$\frac{1}{c}\int_{x_0}^{x}V(\alpha)d\alpha=-\int_{x_0}^{x}\frac{df(\alpha)}{d\alpha}d\alpha+\int_{x_0}^{x}\frac{dg(\alpha)}{d\alpha}d\alpha=g(x)-f(x)+\vartheta(x_0),\quad \vartheta(x_0)=f(x_0)-g(x_0) \tag{2.2.59}$$

Subtracting this from Eqn. 2.2.56, and also adding it to Eqn. 2.2.56, gives

$$\begin{aligned}f(x)&=\frac{1}{2}U(x)-\frac{1}{2c}\int_{x_0}^{x}V(\alpha)d\alpha+\frac{1}{2}\vartheta(x_0)\\ g(x)&=\frac{1}{2}U(x)+\frac{1}{2c}\int_{x_0}^{x}V(\alpha)d\alpha-\frac{1}{2}\vartheta(x_0)\end{aligned} \tag{2.2.60}$$

If one now replaces x with $x-ct$ in the first of these, and with $x+ct$ in the latter, addition of the two expressions leads to Eqn. 2.2.22:

$$u(x,t)=\frac{1}{2}\left[U\left(x+ct\right)+U\left(x-ct\right)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}V(\alpha)d\alpha \tag{2.2.61}$$

2. Method of Separation of Variables Solution to the Wave Equation

Assuming a separable solution, write $u(x,t)=X(x)T(t)$ so that $\partial^2 u/\partial t^2=X(x)\ddot{T}(t)$ and $\partial^2 u/\partial x^2=X''(x)T(t)$. Inserting these into the wave equation gives

$$\begin{aligned}&X\frac{d^2T}{dt^2}=c^2\frac{d^2X}{dX}T\\ &\to\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{X}\frac{d^2X}{dX}\end{aligned} \tag{2.2.62}$$

This relation states that a function of t equals a function of x and it must hold for all t and x. It follows that both sides of this expression must be equal to a constant, say k (if the left hand side were not constant it would change in value as t is changed, but then the equality would no longer hold because the right hand side does not change when t is changed – it is a function of x only). Thus there are two second order ordinary differential equations:

$$\frac{d^2X}{dx^2} - kX = 0, \qquad \frac{d^2T}{dt^2} - c^2kT = 0 \tag{2.2.63}$$

which have solutions

$$X = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} = 0, \qquad T = Ce^{c\sqrt{k}t} + De^{c\sqrt{k}t} \tag{2.2.64}$$

Modes and Natural Frequencies for Homogeneous Boundary Conditions

Suppose first that k is positive. Consider homogeneous boundary conditions, that is, $u = 0$ and/or $\partial u / \partial x = 0$ at the end points $x = 0, L$. Suppose first that $u(0,t) = 0$. Then $u(0,t) = X(0)T(t) = 0 \to X(0) = 0$ and so $A + B = 0$. If also $u(L,t) = 0$, then $Ae^{\sqrt{k}L} + Be^{-\sqrt{k}L} = 0$ which implies that $A = B = 0$, and $u(x,t) = 0$. Similarly, if one uses the conditions $\partial u / \partial x(0,t) = 0$ or $\partial u / \partial x(L,t) = 0$, or a combination of zero u and first derivative, one arrives at the same conclusion: a trivial zero solution. Therefore, to obtain a non-zero solution, one must have k negative, and

$$X(x) = \overline{A}\cos(\lambda x) + \overline{B}\sin(\lambda x), \qquad k = -\lambda^2 \tag{2.2.65}$$

The solution for $T(t)$ must then be

$$T(t) = \overline{C}\cos(\lambda ct) + \overline{D}\sin(\lambda ct) \tag{2.2.66}$$

and the full solution is

$$u(x,t) = \left(\overline{A}\cos(\lambda x) + \overline{B}\sin(\lambda x)\right)\left(\overline{C}\cos(\lambda ct) + \overline{D}\sin(\lambda ct)\right) \tag{2.2.67}$$

There are four possible combinations of boundary conditions.

1. Fixed-Fixed

Here, $u(0,t) = u(L,t) = 0$. Thus $X(0) = \overline{A} = 0$ and $X(L) = \overline{B}\sin(\lambda L) = 0$. For non-zero $\overline{B}$ one must have $\sin(\lambda L) = 0 \to \lambda = \pm n\pi / L,\ n = 0, 1, \ldots$. Thus one has the infinite number of solutions $X_n(x) = \overline{B}_n \sin(\lambda_n x)$, and the complete general solution is ($A = \overline{B}\overline{C}$, $B = \overline{B}\overline{D}$)[6]

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right]\sin(\lambda_n x) \tag{2.2.68}$$

with

[6] the solutions corresponding to negative values of n, i.e. $\lambda = -n\pi / L$, $n = 1, 2, \ldots$, can be subsumed into 2.2.68 through the constants A_n, B_n; the solution for $n = 0$ is zero

Frequencies: $\boxed{\omega_n = \lambda_n c = \frac{n\pi c}{L}, \quad n = 1,2,\dots}$ Modes: $\boxed{\sin(\lambda_n x), \quad n = 1,2,\dots}$ (2.2.69)

It can be proved that the series 2.2.68 converges and that it is indeed a solution of the wave equation, provided some fairly weak conditions are fulfilled (see a text on Advanced Calculus).

The first three modes are plotted in Fig. 2.2.10.

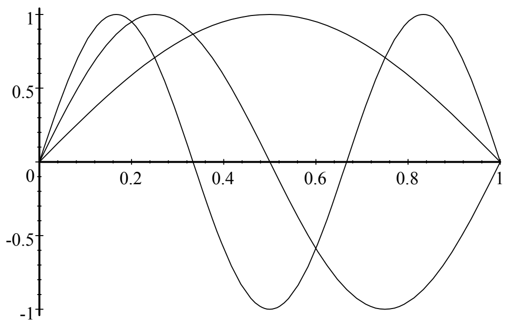

Figure 2.2.10: first three mode shapes for fixed-fixed

Case 2. Free-Free

Here, $\partial u / \partial x(0,t) = \partial u / \partial x(L,t) = 0$. Thus $X'(0) = \lambda \overline{B} = 0$ and $X'(L) = -\lambda \overline{A} \sin(\lambda L) = 0$. Thus the general solution is ($A = \overline{A}\overline{C}$, $B = \overline{A}\overline{D}$)

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct) \right] \cos(\lambda_n x) \tag{2.2.70}$$

with the λ_n as for fixed-fixed.

Frequencies: $\boxed{\omega_n = \lambda_n c = \frac{n\pi c}{L}, \quad n = 1,2,\dots}$ Modes: $\boxed{\cos(\lambda_n x), \quad n = 1,2,\dots}$ (2.2.71)

The displacement profiles of the first three modes are shown in Fig. 2.2.11.

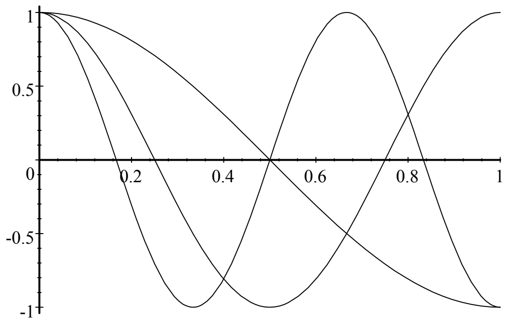

Figure 2.2.11: first three mode shapes for free-free

Case 3. Fixed-Free

Here, $u(0,t) = \partial u / \partial x(L,t) = 0$. Thus $X(0) = \overline{A} = 0$ and $X(L) = \lambda \overline{B} \cos(\lambda L) = 0$. For non-zero $\overline{B}$ one must have $\cos(\lambda L) = 0 \to \lambda = (2n-1)\pi / 2L,\ n = \ldots -2, -1, 0, 1, 2, \ldots$. The solution is again given by 2.2.68, which is repeated here,

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct) \right] \sin(\lambda_n x) \qquad (2.2.72)$$

only now

$$\text{Frequencies: } \boxed{\omega_n = \lambda_n c = \frac{(2n-1)\pi c}{2L}, \quad n = 1, 2, \ldots} \qquad \text{Modes: } \boxed{\sin(\lambda_n x), \quad n = 1, 2, \ldots} \qquad (2.2.73)$$

The displacement profiles of the first three modes are shown in Fig. 2.2.12.

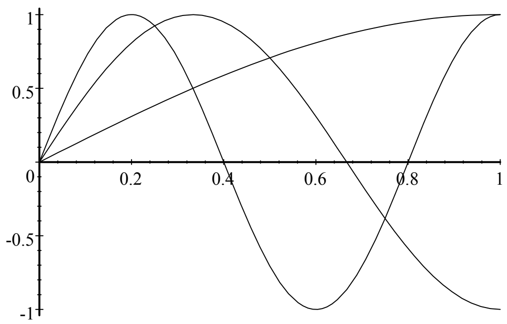

Figure 2.2.12: first three mode shapes for fixed-free

Case 4. Free-Fixed

Here, $\partial u/\partial x(0,t) = u(L,t) = 0$. Thus $X'(0) = \lambda\overline{B} = 0$ and $X(L) = \overline{A}\cos(\lambda L) = 0$. For non-zero $\overline{A}$ one must have $\cos(\lambda L) = 0$ so the general solution is as for free-free, Eqn. 2.2.70, but with $A_0 = 0$:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right]\cos(\lambda_n x) \tag{2.2.74}$$

with the λ_n as for fixed-free.

$$\text{Frequencies: } \boxed{\omega_n = \lambda_n c = \frac{(2n-1)\pi c}{2L}, \quad n = 1,2,\ldots} \qquad \text{Modes: } \boxed{\cos(\lambda_n x), \quad n = 1,2,\ldots} \tag{2.2.75}$$

The displacement profiles of the first three modes are shown in Fig. 2.2.13.

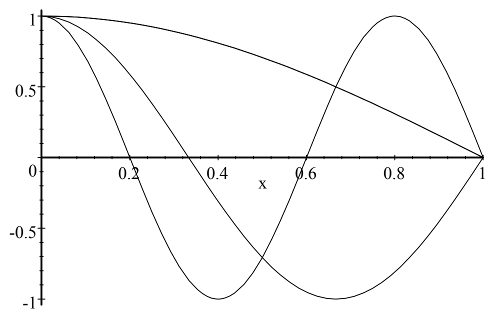

Figure 2.2.13: first three mode shapes for free-fixed

Full Solution (incorporating Initial Conditions)

(a) Initial Condition on Displacement

The initial condition on displacement is

$$u(x,0) = u_0(x) \tag{2.2.76}$$

which give, from 2.2.68, 2.2.70, 2.2.72, 2.2.74,

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) = u_0(x) && \text{fixed-fixed/fixed-free} \\ u(x,0) &= A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) = u_0(x) && \text{free-free} \end{aligned} \tag{2.2.77}$$

$$u(x,0)=\sum_{n=1}^{\infty} A_n \cos(\lambda_n x)=u_0(x) \qquad \text{free-fixed}$$

These can be solved by using the **orthogonality condition** of the trigonometric functions:

$$\int_0^L \sin(\lambda_n x)\sin(\lambda_m x)dx=\int_0^L \cos(\lambda_n x)\cos(\lambda_m x)dx=\begin{cases}0, & m\neq n\\ L/2, & m=n\end{cases} \qquad (2.2.78)$$

for either of $\lambda_n = n\pi/L,\ (2n-1)\pi/2L$. Thus multiplying both sides of 2.2.77a by $\sin(\lambda_m x)$ and 2.2.77b-c by $\cos(\lambda_m x)$ and integrating over $[0, L]$ gives

$$\begin{aligned}
&A_n=\frac{2}{L}\int_0^L u_0(x)\sin(\lambda_n x)dx && \text{fixed-fixed/fixed-free}\\
&A_0=\frac{1}{L}\int_0^L u_0(x)dx,\quad A_n=\frac{2}{L}\int_0^L u_0(x)\cos(\lambda_n x)dx,\ n=1,2,\ldots && \text{free-free} \qquad (2.2.79)\\
&A_n=\frac{2}{L}\int_0^L u_0(x)\cos(\lambda_n x)dx && \text{free-fixed}
\end{aligned}$$

(b) Initial Condition on Velocity

The initial condition on velocity, $\dot{u}(x,0)=v_0(x)$, gives

$$\begin{aligned}
&\dot{u}(x,0)=\sum_{n=1}^{\infty}\lambda_n c B_n \sin(\lambda_n x)=v_0(x) && \text{fixed-fixed/fixed-free}\\
&\dot{u}(x,0)=\sum_{n=1}^{\infty}\lambda_n c B_n \cos(\lambda_n x)=v_0(x) && \text{free-fixed/free-free} \qquad (2.2.80)
\end{aligned}$$

Using the orthogonality conditions again gives

$$\begin{aligned}
&B_n=\frac{2}{Lc\lambda_n}\int_0^L v_0(x)\sin(\lambda_n x)dx && \text{fixed-fixed/fixed-free}\\
&B_n=\frac{2}{Lc\lambda_n}\int_0^L v_0(x)\cos(\lambda_n x)dx && \text{free-fixed/free-free} \qquad (2.2.81)
\end{aligned}$$

Example

Consider the fixed-free case with initial conditions $u_0(x)=0,\ v_0(x)=2x/L$. Thus $A_n=0$ and

$$B_n = \frac{8}{(2n-1)\pi Lc}\int_0^L x\sin\left(\frac{(2n-1)\pi}{2L}x\right)dx = \frac{8}{(2n-1)\pi Lc}\frac{4L^2(-1)^{n+1}}{\pi^2(2n-1)^2}$$
$$= \frac{32(-1)^{n+1}}{(2n-1)^3\pi^3}\frac{L}{c}$$

so that

$$u(x,t) = \frac{32L}{\pi^3 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\sin(\lambda_n x)\sin(\lambda_n ct), \qquad \omega_n = \lambda_n c = \frac{(2n-1)\pi c}{2L}, \qquad n = 1,2,\ldots$$

The period for the first (dominant) mode is $T_1 = 2\pi/\lambda_1 c = 4L/c$. The solution is plotted in Fig. 2.2.14 for $c = 5000\text{m/s}$, $L = 0.1\text{m}$, for the five times $iT_1/16,\ i = 0\ldots4$ (up to the quarter-period). Thereafter, the solution decreases back to zero, down through negative displacements, back to zero and then repeats.

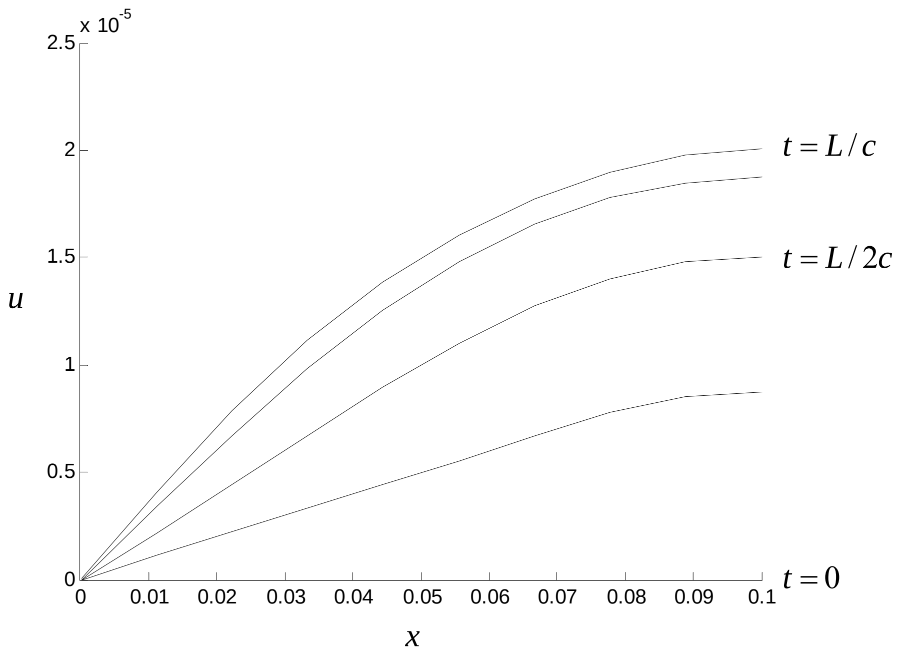

Figure 2.2.14: displacements for fixed-free example

Example

Consider the free-free case with initial conditions $u_0(x) = \bar{u}x,\ v_0(x) = 0$. Thus $B_n = 0$ and

$$A_0 = \frac{\bar{u}}{L}\int_0^L x dx = \frac{\bar{u}L}{2}$$

$$A_n = \frac{2\bar{u}}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2\bar{u}L}{\pi^2 n^2}\left((-1)^n - 1\right), \quad n = 1, 2, \ldots$$

so that

$$u(x,t) = \bar{u}L\left[\frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left((-1)^n - 1\right)}{n^2}\cos(\lambda_n ct)\cos(\lambda_n x)\right], \quad \lambda_n = \frac{n\pi}{L} \tag{2.2.34}$$

The period for the first (dominant) mode is $T_1 = 2\pi/\lambda_1 c = 2L/c$. The solution is plotted in Fig. 2.2.15 again for $c = 5000\text{m/s}$, $L = 0.1\text{m}$, for the nine times $iT_1/16,\ i = 0\ldots 8$ (up to the half-period). Thereafter, the solution returns back to the initial position and then repeats.

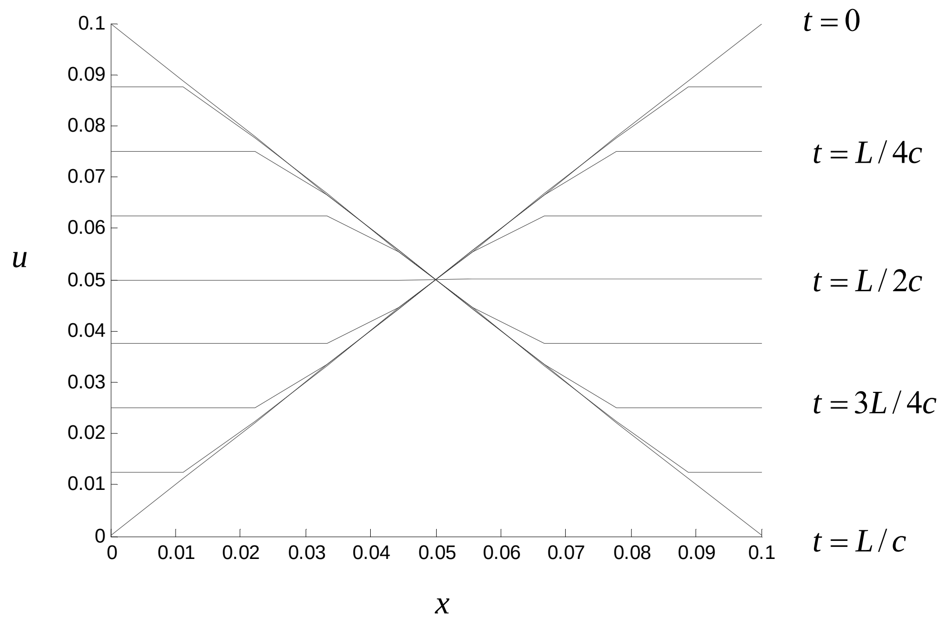

Figure 2.2.15: displacements for free-free example