1.3 Compatibility of Strain

As seen in the previous section, the displacements can be determined from the strains through integration, to within a rigid body motion. In the two-dimensional case, there are three strain-displacement relations but only two displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**.

1.3.1 The Compatibility Relations

Differentiating the first of 1.2.5 twice with respect to y, the second twice with respect to x and the third once each with respect to x and y yields

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y} \right)$$

It follows that

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}} \qquad \textbf{2-D Compatibility Equation} \qquad (1.3.1)$$

This compatibility condition is an equation which must be satisfied by the strains at all material particles.

Physical Meaning of the Compatibility Condition

When all material particles in a component deform, translate and rotate, they need to meet up again very much like the pieces of a jigsaw puzzle must fit together. Fig. 1.3.1 illustrates possible deformations and rigid body motions for three line elements in a material. Compatibility ensures that they stay together after the deformation.

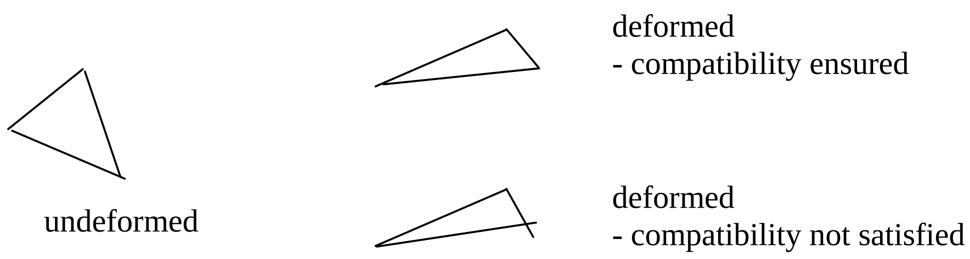

Figure 1.3.1: Deformation and Compatibility

The Three Dimensional Case

There are six compatibility relations to be satisfied in the three dimensional case :

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\
&\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\
&\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right)
\end{aligned} \tag{1.3.2}$$

By inspection, it will be seen that these are satisfied by Eqns. 1.2.19.

1.3.2 Problems

1. The displacement field in a material is given by

$$u_x = Axy, \quad u_y = Ay^2,$$

   where A is a small constant. Determine
   (a) the components of small strain
   (b) the rotation
   (c) the principal strains
   (d) whether the compatibility condition is satisfied