1.2 The Strain-Displacement Relations

The strain was introduced in Book I: §4. The concepts examined there are now extended to the case of strains which vary continuously throughout a material.

1.2.1 The Strain-Displacement Relations

Normal Strain

Consider a line element of length Δx emanating from position (x, y) and lying in the x-direction, denoted by AB in Fig. 1.2.1. After deformation the line element occupies $A'B'$, having undergone a translation, extension and rotation.

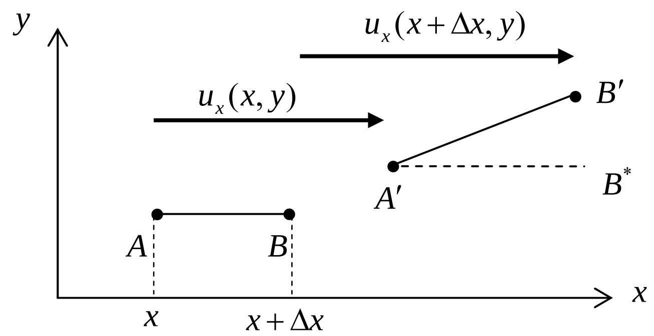

Figure 1.2.1: deformation of a line element

The particle that was originally at x has undergone a displacement $u_x(x, y)$ and the other end of the line element has undergone a displacement $u_x(x+\Delta x, y)$. By the definition of (small) normal strain,

$$\varepsilon_{xx} = \frac{A'B^* - AB}{AB} = \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x} \tag{1.2.1}$$

In the limit $\Delta x \to 0$ one has

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{1.2.2}$$

This partial derivative is a **displacement gradient**, a measure of how rapid the displacement changes through the material, and is the strain *at* (x, y). Physically, it represents the (approximate) unit change in length of a line element, as indicated in Fig. 1.2.2.

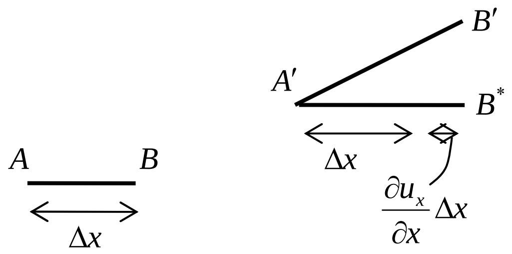

Figure 1.2.2: unit change in length of a line element

Similarly, by considering a line element initially lying in the *y* direction, the strain in the *y* direction can be expressed as

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{1.2.3}$$

Shear Strain

The particles *A* and *B* in Fig. 1.2.1 also undergo displacements in the *y* direction and this is shown in Fig. 1.2.3. In this case, one has

$$B^* B' = \frac{\partial u_y}{\partial x}\Delta x \tag{1.2.4}$$

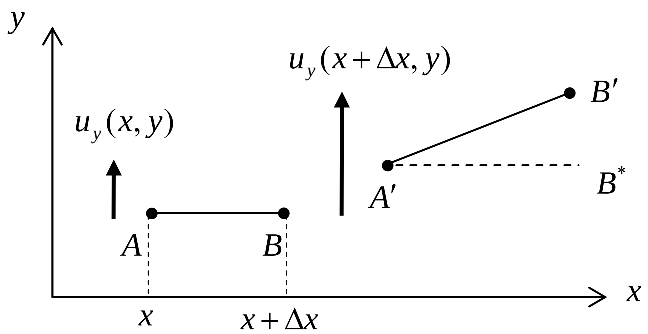

Figure 1.2.3: deformation of a line element

A similar relation can be derived by considering a line element initially lying in the *y* direction. A summary is given in Fig. 1.2.4. From the figure,

$$\theta \approx \tan\theta = \frac{\partial u_y / \partial x}{1 + \partial u_x / \partial x} \approx \frac{\partial u_y}{\partial x}$$

provided that (i) θ is small and (ii) the displacement gradient $\partial u_x / \partial x$ is small. A similar expression for the angle λ can be derived, and hence the shear strain can be written in terms of displacement gradients.

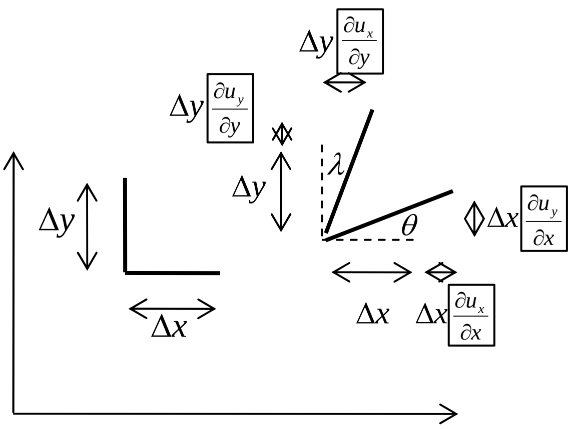

Figure 1.2.4: strains in terms of displacement gradients

The Small-Strain Stress-Strain Relations

In summary, one has

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{aligned}}$$

2-D Strain-Displacement relations (1.2.5)

1.2.2 Geometrical Interpretation of Small Strain

A geometric interpretation of the strain was given in Book I: §4.1.4. This interpretation is repeated here, only now in terms of displacement gradients.

Positive Normal Strain

Fig. 1.2.5a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} > 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial y}{\partial x}\right) = 0 \tag{1.2.6}$$

Negative Normal Strain

Fig 1.2.5b,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} < 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \tag{1.2.7}$$

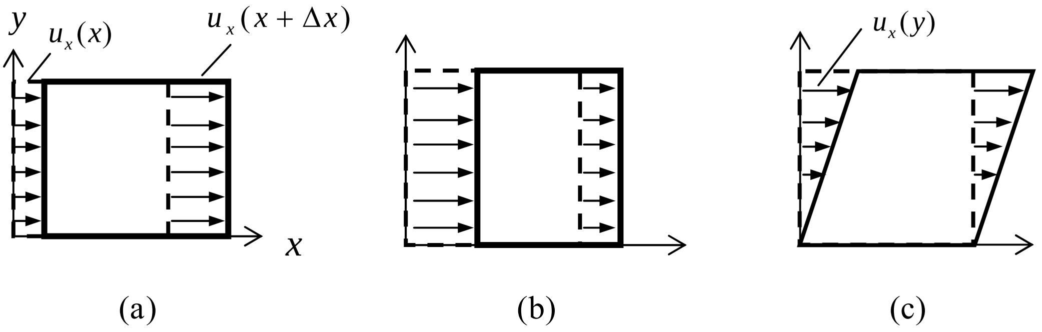

Figure 1.2.5: some simple deformations; (a) positive normal strain, (b) negative normal strain, (c) simple shear

Simple Shear

Fig. 1.2.5c,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\frac{\partial u_x}{\partial y} \tag{1.2.8}$$

Pure Shear

Fig 1.2.6a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} \tag{1.2.9}$$

1.2.3 The Rotation

Consider an arbitrary deformation (omitting normal strains for ease of description), as shown in Fig. 1.2.6. As usual, the angles θ and λ are small, equal to their tangents, and $\theta = \partial u_y / \partial x$, $\lambda = \partial u_x / \partial y$.

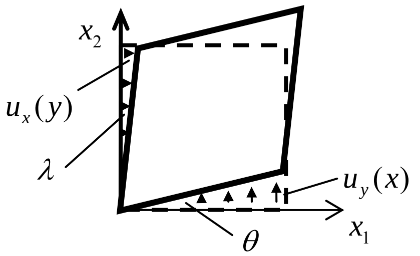

Figure 1.2.6: arbitrary deformation (shear and rotation)

Now this arbitrary deformation can be decomposed into a pure shear and a rigid rotation as depicted in Fig. 1.2.7. In the pure shear, $\theta = \lambda = \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda)$. In the rotation, the angle of rotation is then $\frac{1}{2}(\theta + \lambda)$.

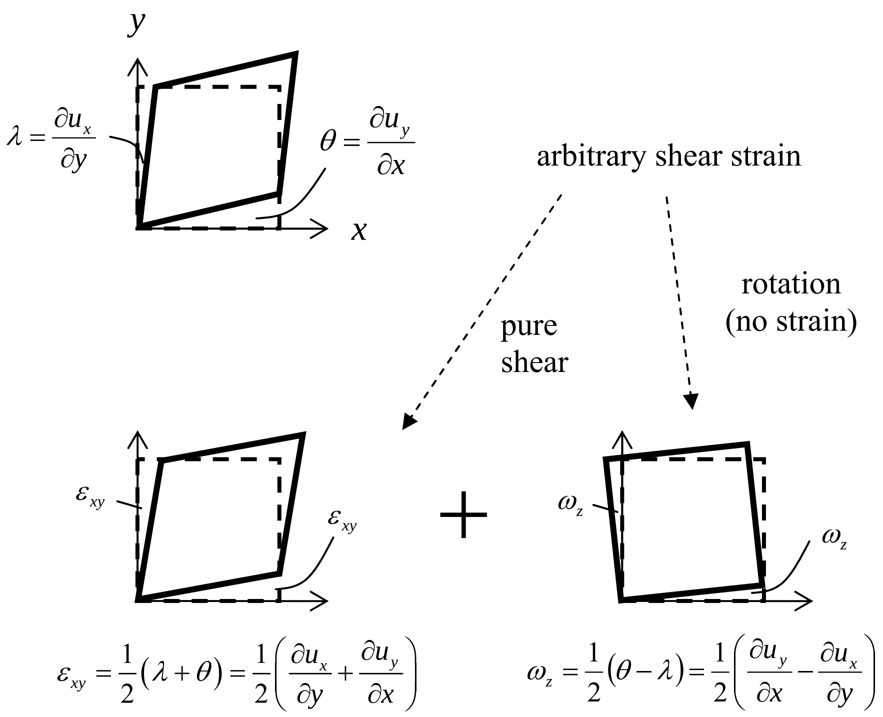

Figure 1.2.7: decomposition of a strain into a pure shear and a rotation

This leads one to define the **rotation** of a material particle, ω_z, the "z" signifying the axis about which the element is rotating:

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{1.2.10}$$

The rotation will in general vary throughout a material. When the rotation is everywhere zero, the material is said to be **irrotational**.

For a pure rotation, note that

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0, \qquad \frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} \tag{1.2.11}$$

1.2.4 Fixing Displacements

The strains give information about the deformation of material particles but, since they do not encompass translations and rotations, they do not give information about the precise location in space of particles. To determine this, one must specify *three* displacement components (in two-dimensional problems). Mathematically, this is equivalent to saying that one cannot uniquely determine the displacements from the strain-displacement relations 1.2.5.

Example

Consider the strain field $\varepsilon_{xx} = 0.01, \varepsilon_{yy} = \varepsilon_{xy} = 0$. The displacements can be obtained by integrating the strain-displacement relations:

$$\begin{aligned} u_x &= \int \varepsilon_{xx} dx = 0.01x + f(y) \\ u_y &= \int \varepsilon_{yy} dy = g(x) \end{aligned} \tag{1.2.12}$$

where *f* and *g* are unknown functions of *y* and *x* respectively. Substituting the displacement expressions into the shear strain relation gives

$$f'(y) = -g'(x). \tag{1.2.13}$$

Any expression of the form $F(x) = G(y)$ which holds for all *x* and *y* implies that *F* and *G* are constant[1]. Since f', g' are constant, one can integrate to get $f(y) = A + Dy, \;\; g(x) = B + Cx$. From 1.2.13, $C = -D$, and

$$\begin{aligned} u_x &= 0.01x + A - Cy \\ u_y &= B + Cx \end{aligned} \tag{1.2.14}$$

There are three arbitrary constants of integration, which can be determined by specifying three displacement components. For example, suppose that it is known that

$$u_x(0,0) = 0, u_y(0,0) = 0, u_x(0,a) = b. \tag{1.2.15}$$

In that case, $A = 0, B = 0, C = -b/a$, and, finally,

$$\begin{aligned} u_x &= 0.01x + (b/a)y \\ u_y &= -(b/a)x \end{aligned} \tag{1.2.16}$$

which corresponds to Fig. 1.2.8, with (b/a) being the (tan of the small) angle by which the element has rotated.

[1] since, if this was not so, a change in *x* would change the left hand side of this expression but would not change the right hand side and so the equality cannot hold

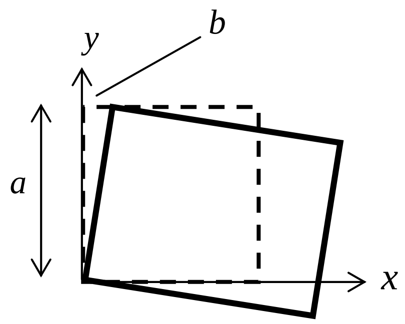

Figure 1.2.8: an element undergoing a normal strain and a rotation

■

In general, the displacement field will be of the form

$$\begin{aligned} u_x &= \;\cdots\cdots\; + A - Cy \\ u_y &= \;\cdots\cdots\; + B + Cx \end{aligned} \tag{1.2.17}$$

and indeed Eqn. 1.2.16 is of this form. Physically, *A*, *B* and *C* represent the possible rigid body motions *of the material as a whole*, since they are the same for all material particles. *A* corresponds to a translation in the x direction, *B* corresponds to a translation in the x direction, and *C* corresponds to a positive (counterclockwise) rotation.

1.2.5 Three Dimensional Strain

The three-dimensional stress-strain relations analogous to Eqns. 1.2.5 are

$$\boxed{\begin{array}{l} \varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z} \\ \varepsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right), \quad \varepsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right), \quad \varepsilon_{yz} = \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}\right) \end{array}}$$

3-D Stress-Strain relations (1.2.18)

The rotations are

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \tag{1.2.19}$$

1.2.6 Problems

1. The displacement field in a material is given by

   $$u_x = A(3x - y), \quad u_y = Axy^2$$

   where *A* is a small constant.

(a) Evaluate the strains. What is the rotation ω_z? Sketch the deformation and any rigid body motions of a differential element at the point $(1,1)$
(b) Sketch the deformation and rigid body motions at the point $(0,2)$, by using a pure shear strain superimposed on the rotation.

2. The strains in a material are given by
$$\varepsilon_{xx} = \alpha x, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = \alpha$$
Evaluate the displacements in terms of three arbitrary constants of integration, in the form of Eqn. 1.2.17,
$$\begin{aligned} u_x &= \cdots\cdots \quad + A - Cy \\ u_y &= \cdots\cdots \quad + B + Cx \end{aligned}$$
What is the rotation?

3. The strains in a material are given by
$$\varepsilon_{xx} = Axy, \quad \varepsilon_{yy} = Ay^2, \quad \varepsilon_{xy} = Ax$$
where A is a small constant. Evaluate the displacements in terms of three arbitrary constants of integration. What is the rotation?

4. Show that, in a state of plane strain ($\varepsilon_{zz} = 0$) with zero body force,
$$\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}$$
where e is the volumetric strain (dilatation), the sum of the normal strains: $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ (see Book I, §4.3).