1 Differential Equations for Solid Mechanics

Simple problems involving homogeneous stress states have been considered so far, wherein the stress is the same throughout the component under study. An exception to this was the varying stress field in the loaded beam, but there a simplified set of elasticity equations was used. Here the question of varying stress and strain fields in materials is considered. In order to solve such problems, a differential formulation is required. In this Chapter, a number of differential equations will be derived, relating the stresses and body forces (**equations of motion**), the strains and displacements (**strain-displacement relations**) and the strains with each other (**compatibility relations**). These equations are derived from physical principles and so apply to any type of material, although the latter two are derived under the assumption of small strain.

1.1 The Equations of Motion

In Book I, balance of forces and moments acting on any component was enforced in order to ensure that the component was in equilibrium. Here, allowance is made for stresses which vary continuously throughout a material, and force equilibrium of any portion of material is enforced.

One-Dimensional Equation

Consider a one-dimensional differential element of length Δx and cross sectional area *A*, Fig. 1.1.1. Let the *average* body force per unit volume acting on the element be *b* and the *average* acceleration and density of the element be *a* and ρ. Stresses σ act on the element.

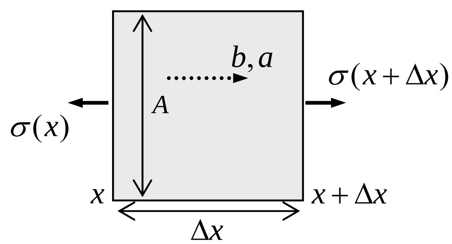

Figure 1.1.1: a differential element under the action of surface and body forces

The net surface force acting is $\sigma(x+\Delta x)A - \sigma(x)A$. If the element is small, then the body force and velocity can be assumed to vary linearly over the element and the average will act at the centre of the element. Then the body force acting on the element is $Ab\Delta x$ and the inertial force is $\rho A \Delta x a$. Applying Newton's second law leads to

$$\begin{aligned} &\sigma(x+\Delta x)A - \sigma(x)A + b\Delta x A = \rho a \Delta x A \\ &\rightarrow \frac{\sigma(x+\Delta x) - \sigma(x)}{\Delta x} + b = \rho a \end{aligned} \tag{1.1.1}$$

so that, by the definition of the derivative, in the limit as $\Delta x \rightarrow 0$,

$$\boxed{\frac{d\sigma}{dx} + b = \rho a} \quad \textbf{1-d Equation of Motion} \tag{1.1.2}$$

which is the one-dimensional **equation of motion**. Note that this equation was derived on the basis of a physical law and must therefore be satisfied for all materials, whatever they be composed of.

The derivative $d\sigma / dx$ is the **stress gradient** – physically, it is a measure of how rapidly the stresses are changing.

Example

Consider a bar of length *l* which hangs from a ceiling, as shown in Fig. 1.1.2.

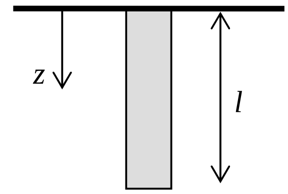

Figure 1.1.2: a hanging bar

The gravitational force is $F = mg$ downward and the body force per unit volume is thus $b = \rho g$. There are no accelerating material particles. Taking the *z* axis positive down, an integration of the equation of motion gives

$$\frac{d\sigma}{dz} + \rho g = 0 \quad \rightarrow \quad \sigma = -\rho g z + c \tag{1.1.3}$$

where *c* is an arbitrary constant. The lower end of the bar is free and so the stress there is zero, and so

$$\sigma = \rho g(l - z) \tag{1.1.4}$$

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Two-Dimensional Equations

Consider now a two dimensional infinitesimal element of width and height Δx and Δy and unit depth (into the page).

Looking at the normal stress components acting in the x –direction, and allowing for variations in stress over the element surfaces, the stresses are as shown in Fig. 1.1.3.

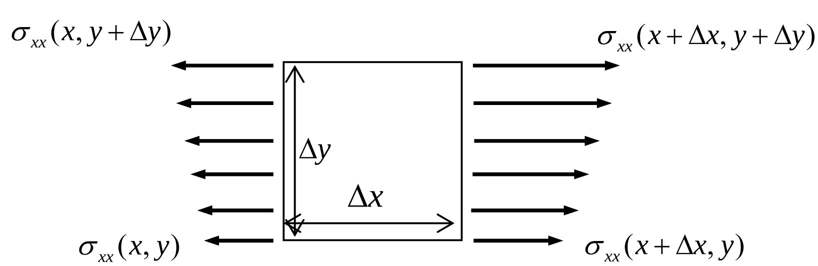

Figure 1.1.3: varying stresses acting on a differential element

Using a (two dimensional) Taylor series and dropping higher order terms then leads to the linearly varying stresses illustrated in Fig. 1.1.4. (where $\sigma_{xx} \equiv \sigma_{xx}(x, y)$ and the partial derivatives are evaluated at (x, y)), which is a reasonable approximation when the element is small.

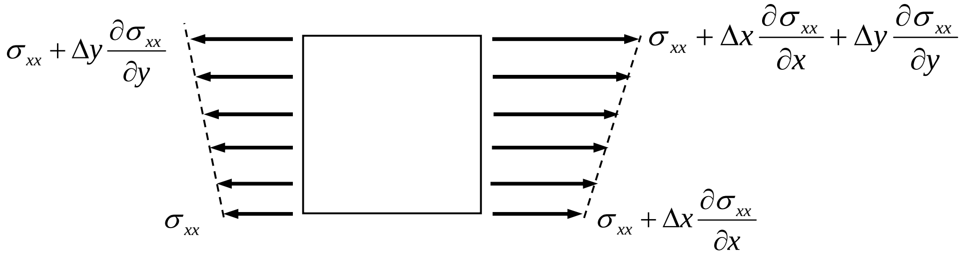

Figure 1.1.4: linearly varying stresses acting on a differential element

The effect (resultant force) of this linear variation of stress on the plane can be replicated by a *constant* stress acting over the whole plane, the size of which is the *average* stress. For the left and right sides, one has, respectively,

$$\sigma_{xx} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y}, \quad \sigma_{xx} + \Delta x \frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y} \tag{1.1.5}$$

One can take away the stress $(1/2)\Delta y \partial \sigma_{xx} / \partial y$ from both sides without affecting the net force acting on the element so one finally has the representation shown in Fig. 1.1.5.

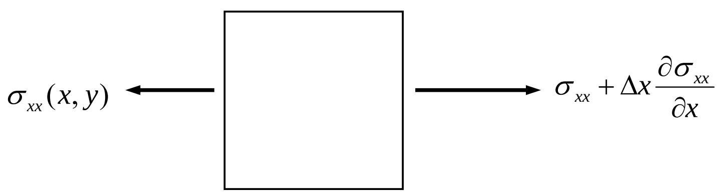

Figure 1.1.5: net stresses acting on a differential element

Carrying out the same procedure for the shear stresses contributing to a force in the x −direction leads to the stresses shown in Fig. 1.1.6.

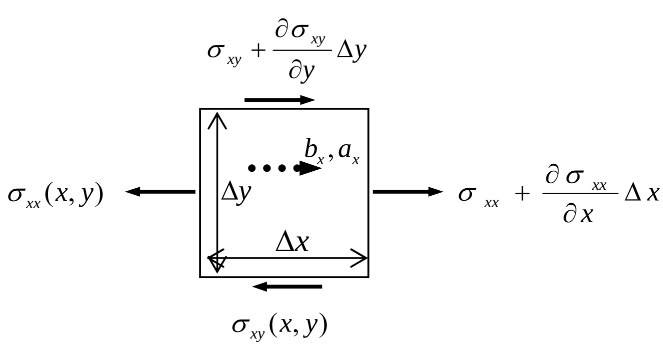

Figure 1.1.6: normal and shear stresses acting on a differential element

Take a_x, b_x to be the average acceleration and body force, and ρ to be the average density. Newton's law then yields

$$-\sigma_{xx}\Delta y+\left(\sigma_{xx}+\Delta x\frac{\partial\sigma_{xx}}{\partial x}\right)\Delta y-\sigma_{xy}\Delta x+\left(\sigma_{xy}+\Delta y\frac{\partial\sigma_{xy}}{\partial y}\right)\Delta x+b_x\Delta x\Delta y=\rho a_x\Delta x\Delta y \tag{1.1.6}$$

which, dividing through by $\Delta x\Delta y$ and taking the limit, gives

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x=\rho a_x \tag{1.1.7}$$

A similar analysis for force components in the $y-$direction yields another equation and one then has the two-dimensional equations of motion:

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x&=\rho a_x\\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+b_y&=\rho a_y\end{aligned}} \quad \textbf{2-D Equations of Motion} \tag{1.1.8}$$

Three-Dimensional Equations

Similarly, one can consider a three-dimensional element, and one finds that

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x&=\rho a_x\\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y&=\rho a_y\\ \frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z&=\rho a_z\end{aligned}} \quad \textbf{3-D Equations of Motion} \tag{1.1.9}$$

These three equations express force-balance in, respectively, the x, y, z directions.

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signant par $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ les projections algébriques de la force accélératrice qui serait capable de produire à elle seule le mouvement effectif d'une particule, et prenant x, y, z, t pour variables indépendantes, on obtiendra, à la place des équations (1), celles qui suivent

$$
(2)\quad \left\{
\begin{aligned}
&\frac{dA}{dx}+\frac{dF}{dy}+\frac{dE}{dz}+\rho X=\rho\mathfrak{X},\\
&\frac{dF}{dx}+\frac{dB}{dy}+\frac{dD}{dz}+\rho Y=\rho\mathfrak{Y},\\
&\frac{dE}{dx}+\frac{dD}{dy}+\frac{dC}{dz}+\rho Z=\rho\mathfrak{Z}.
\end{aligned}
\right.
$$

Enfin, si l'on nomme ξ, η, ζ les déplacements de la particule qui, au bout d'un temps t, coïncide avec le point (x, y, z), mesurés parallèlement aux axes coordonnés, on trouvera, en supposant ces déplacements très-petits,

Figure 1.1.7: from Cauchy's Exercices de Mathematiques (1829)

The Equations of Equlibrium

If the material is not moving (or is moving at constant velocity) and is in static equilibrium, then the equations of motion reduce to the **equations of equilibrium**,

$$
\boxed{
\begin{aligned}
\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x&=0\\
\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y&=0\\
\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z&=0
\end{aligned}
}
\qquad \textbf{3-D Equations of Equilibrium} \qquad (1.1.10)
$$

These equations express the force balance between surface forces and body forces in a material. The equations of equilibrium may also be used as a good approximation in the analysis of materials which have relatively small accelerations.

1.1.2 Problems

1. What does the one-dimensional equation of motion say about the stresses in a bar in the absence of any body force or acceleration?

2. Does equilibrium exist for the following two dimensional stress distribution in the absence of body forces?

$$
\begin{aligned}
\sigma_{xx}&=3x^2+4xy-8y^2\\
\sigma_{xy}&=\sigma_{yx}=x^2/2-6xy-2y^2\\
\sigma_{yy}&=2x^2-xy+3y^2\\
\sigma_{zz}&=\sigma_{zx}=\sigma_{xz}=\sigma_{zy}=\sigma_{yz}=0
\end{aligned}
$$

3. The elementary beam theory predicts that the stresses in a circular beam due to bending are

$$\sigma_{xx} = My/I, \quad \sigma_{xy} = \sigma_{yx} = V(R^2 - y^2)/3I \qquad (I = \pi R^4/4)$$

and all the other stress components are zero. Do these equations satisfy the equations of equilibrium?

4. With respect to axes $0xyz$ the stress state is given in terms of the coordinates by the matrix

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} xy & y^2 & 0 \\ y^2 & yz & z^2 \\ 0 & z^2 & xz \end{bmatrix}$$

Determine the body force acting on the material if it is at rest.

5. What is the acceleration of a material particle of density $\rho = 0.3\mathrm{kgm}^{-3}$, subjected to the stress

$$[\sigma_{ij}] = \begin{bmatrix} 2x^2 - x^4 & 2xy & 2xz \\ 2xy & 2y^2 - y^4 & 2yz \\ 2xz & 2yz & 2z^2 - z^4 \end{bmatrix}$$

and gravity (the z axis is directed vertically upwards from the ground).

6. A fluid at rest is subjected to a hydrostatic pressure p and the force of gravity only.
   (a) Write out the equations of motion for this case.
   (b) A very basic formula of hydrostatics, to be found in any elementary book on fluid mechanics, is that giving the pressure variation in a static fluid,

$$\Delta p = \rho g h$$

   where ρ is the density of the fluid, g is the acceleration due to gravity, and h is the vertical distance between the two points in the fluid (the relative depth). Show that this formula is but a special case of the equations of motion.

1.2 The Strain-Displacement Relations

The strain was introduced in Book I: §4. The concepts examined there are now extended to the case of strains which vary continuously throughout a material.

1.2.1 The Strain-Displacement Relations

Normal Strain

Consider a line element of length Δx emanating from position (x, y) and lying in the x-direction, denoted by AB in Fig. 1.2.1. After deformation the line element occupies $A'B'$, having undergone a translation, extension and rotation.

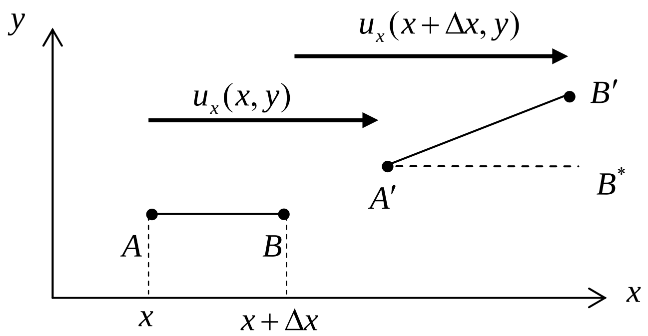

Figure 1.2.1: deformation of a line element

The particle that was originally at x has undergone a displacement $u_x(x, y)$ and the other end of the line element has undergone a displacement $u_x(x+\Delta x, y)$. By the definition of (small) normal strain,

$$\varepsilon_{xx} = \frac{A'B^* - AB}{AB} = \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x} \tag{1.2.1}$$

In the limit $\Delta x \to 0$ one has

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{1.2.2}$$

This partial derivative is a **displacement gradient**, a measure of how rapid the displacement changes through the material, and is the strain *at* (x, y). Physically, it represents the (approximate) unit change in length of a line element, as indicated in Fig. 1.2.2.

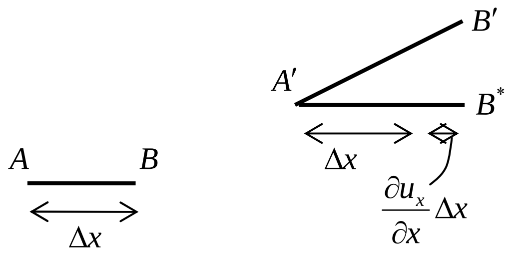

Figure 1.2.2: unit change in length of a line element

Similarly, by considering a line element initially lying in the *y* direction, the strain in the *y* direction can be expressed as

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{1.2.3}$$

Shear Strain

The particles *A* and *B* in Fig. 1.2.1 also undergo displacements in the *y* direction and this is shown in Fig. 1.2.3. In this case, one has

$$B^* B' = \frac{\partial u_y}{\partial x}\Delta x \tag{1.2.4}$$

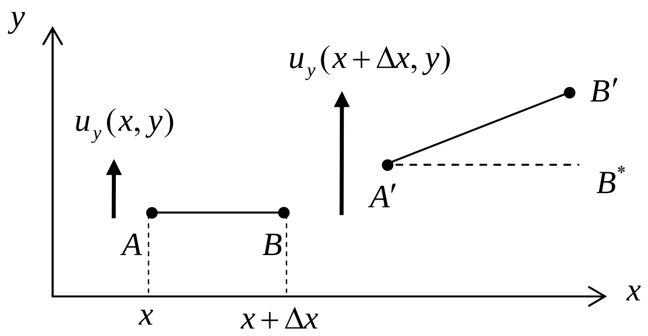

Figure 1.2.3: deformation of a line element

A similar relation can be derived by considering a line element initially lying in the *y* direction. A summary is given in Fig. 1.2.4. From the figure,

$$\theta \approx \tan\theta = \frac{\partial u_y / \partial x}{1 + \partial u_x / \partial x} \approx \frac{\partial u_y}{\partial x}$$

provided that (i) θ is small and (ii) the displacement gradient $\partial u_x / \partial x$ is small. A similar expression for the angle λ can be derived, and hence the shear strain can be written in terms of displacement gradients.

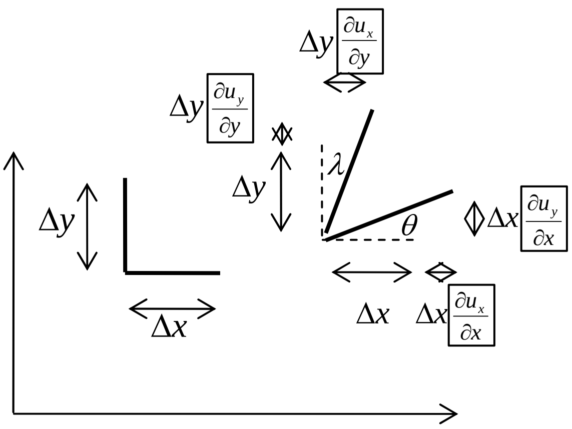

Figure 1.2.4: strains in terms of displacement gradients

The Small-Strain Stress-Strain Relations

In summary, one has

$$\boxed{\begin{aligned}\varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\end{aligned}} \quad \textbf{2-D Strain-Displacement relations} \qquad (1.2.5)$$

1.2.2 Geometrical Interpretation of Small Strain

A geometric interpretation of the strain was given in Book I: §4.1.4. This interpretation is repeated here, only now in terms of displacement gradients.

Positive Normal Strain

Fig. 1.2.5a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} > 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial y}{\partial x}\right) = 0 \qquad (1.2.6)$$

Negative Normal Strain

Fig 1.2.5b,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} < 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \qquad (1.2.7)$$

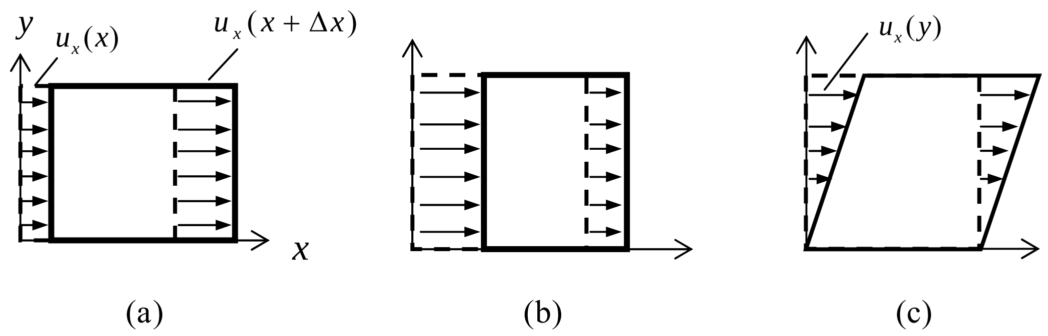

Figure 1.2.5: some simple deformations; (a) positive normal strain, (b) negative normal strain, (c) simple shear

Simple Shear

Fig. 1.2.5c,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\frac{\partial u_x}{\partial y} \qquad (1.2.8)$$

Pure Shear

Fig 1.2.6a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} \qquad (1.2.9)$$

1.2.3 The Rotation

Consider an arbitrary deformation (omitting normal strains for ease of description), as shown in Fig. 1.2.6. As usual, the angles θ and λ are small, equal to their tangents, and $\theta = \partial u_y / \partial x$, $\lambda = \partial u_x / \partial y$.

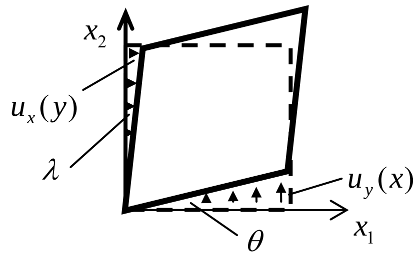

Figure 1.2.6: arbitrary deformation (shear and rotation)

Now this arbitrary deformation can be decomposed into a pure shear and a rigid rotation as depicted in Fig. 1.2.7. In the pure shear, $\theta = \lambda = \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda)$. In the rotation, the angle of rotation is then $\frac{1}{2}(\theta + \lambda)$.

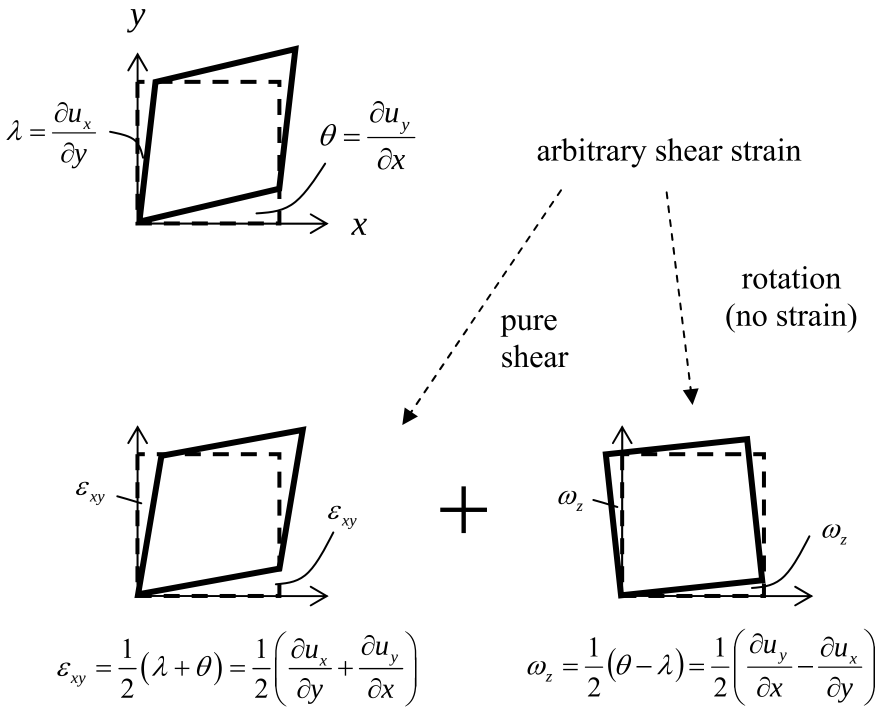

Figure 1.2.7: decomposition of a strain into a pure shear and a rotation

This leads one to define the **rotation** of a material particle, ω_z, the "z" signifying the axis about which the element is rotating:

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{1.2.10}$$

The rotation will in general vary throughout a material. When the rotation is everywhere zero, the material is said to be **irrotational**.

For a pure rotation, note that

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0, \qquad \frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} \tag{1.2.11}$$

1.2.4 Fixing Displacements

The strains give information about the deformation of material particles but, since they do not encompass translations and rotations, they do not give information about the precise location in space of particles. To determine this, one must specify *three* displacement components (in two-dimensional problems). Mathematically, this is equivalent to saying that one cannot uniquely determine the displacements from the strain-displacement relations 1.2.5.

Example

Consider the strain field $\varepsilon_{xx} = 0.01, \varepsilon_{yy} = \varepsilon_{xy} = 0$. The displacements can be obtained by integrating the strain-displacement relations:

$$\begin{aligned} u_x &= \int \varepsilon_{xx} dx = 0.01x + f(y) \\ u_y &= \int \varepsilon_{yy} dy = g(x) \end{aligned} \tag{1.2.12}$$

where *f* and *g* are unknown functions of *y* and *x* respectively. Substituting the displacement expressions into the shear strain relation gives

$$f'(y) = -g'(x). \tag{1.2.13}$$

Any expression of the form $F(x) = G(y)$ which holds for all *x* and *y* implies that *F* and *G* are constant[1]. Since f', g' are constant, one can integrate to get $f(y) = A + Dy,\ g(x) = B + Cx$. From 1.2.13, $C = -D$, and

$$\begin{aligned} u_x &= 0.01x + A - Cy \\ u_y &= B + Cx \end{aligned} \tag{1.2.14}$$

There are three arbitrary constants of integration, which can be determined by specifying three displacement components. For example, suppose that it is known that

$$u_x(0,0) = 0, u_y(0,0) = 0, u_x(0,a) = b. \tag{1.2.15}$$

In that case, $A = 0, B = 0, C = -b/a$, and, finally,

$$\begin{aligned} u_x &= 0.01x + (b/a)y \\ u_y &= -(b/a)x \end{aligned} \tag{1.2.16}$$

which corresponds to Fig. 1.2.8, with (b/a) being the (tan of the small) angle by which the element has rotated.

[1] since, if this was not so, a change in *x* would change the left hand side of this expression but would not change the right hand side and so the equality cannot hold

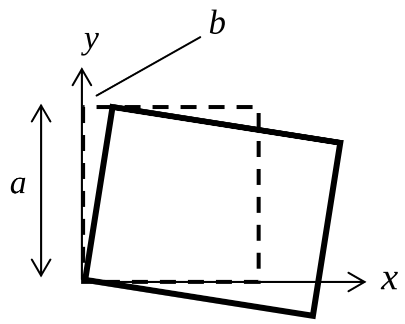

Figure 1.2.8: an element undergoing a normal strain and a rotation

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In general, the displacement field will be of the form

$$\begin{aligned} u_x &= \cdots\cdots \quad + A - Cy \\ u_y &= \cdots\cdots \quad + B + Cx \end{aligned} \tag{1.2.17}$$

and indeed Eqn. 1.2.16 is of this form. Physically, *A*, *B* and *C* represent the possible rigid body motions *of the material as a whole*, since they are the same for all material particles. *A* corresponds to a translation in the x direction, *B* corresponds to a translation in the x direction, and *C* corresponds to a positive (counterclockwise) rotation.

1.2.5 Three Dimensional Strain

The three-dimensional stress-strain relations analogous to Eqns. 1.2.5 are

$$\boxed{\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\ &\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned}}$$

3-D Stress-Strain relations (1.2.18)

The rotations are

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \tag{1.2.19}$$

1.2.6 Problems

1. The displacement field in a material is given by

$$u_x = A(3x - y), \quad u_y = Axy^2$$

where *A* is a small constant.

(a) Evaluate the strains. What is the rotation ω_z? Sketch the deformation and any rigid body motions of a differential element at the point $(1,1)$

(b) Sketch the deformation and rigid body motions at the point $(0,2)$, by using a pure shear strain superimposed on the rotation.

2. The strains in a material are given by

$$\varepsilon_{xx} = \alpha x, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = \alpha$$

Evaluate the displacements in terms of three arbitrary constants of integration, in the form of Eqn. 1.2.17,

$$\begin{aligned} u_x &= \quad \cdots\cdots \quad + A - Cy \\ u_y &= \quad \cdots\cdots \quad + B + Cx \end{aligned}$$

What is the rotation?

3. The strains in a material are given by

$$\varepsilon_{xx} = Axy, \quad \varepsilon_{yy} = Ay^2, \quad \varepsilon_{xy} = Ax$$

where A is a small constant. Evaluate the displacements in terms of three arbitrary constants of integration. What is the rotation?

4. Show that, in a state of plane strain ($\varepsilon_{zz} = 0$) with zero body force,

$$\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}$$

where e is the volumetric strain (dilatation), the sum of the normal strains: $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ (see Book I, §4.3).

1.3 Compatibility of Strain

As seen in the previous section, the displacements can be determined from the strains through integration, to within a rigid body motion. In the two-dimensional case, there are three strain-displacement relations but only two displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**.

1.3.1 The Compatibility Relations

Differentiating the first of 1.2.5 twice with respect to y, the second twice with respect to x and the third once each with respect to x and y yields

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y} \right).$$

It follows that

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}} \qquad \textbf{2-D Compatibility Equation} \qquad (1.3.1)$$

This compatibility condition is an equation which must be satisfied by the strains at all material particles.

Physical Meaning of the Compatibility Condition

When all material particles in a component deform, translate and rotate, they need to meet up again very much like the pieces of a jigsaw puzzle must fit together. Fig. 1.3.1 illustrates possible deformations and rigid body motions for three line elements in a material. Compatibility ensures that they stay together after the deformation.

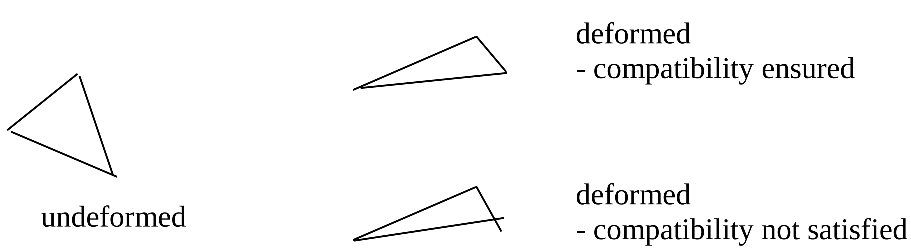

Figure 1.3.1: Deformation and Compatibility

The Three Dimensional Case

There are six compatibility relations to be satisfied in the three dimensional case:

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\
&\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right). \qquad (1.3.2) \\
&\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right)
\end{aligned}$$

By inspection, it will be seen that these are satisfied by Eqns. 1.2.19.

1.3.2 Problems

1. The displacement field in a material is given by

$$u_x = Axy, \quad u_y = Ay^2,$$

   where A is a small constant. Determine
   (a) the components of small strain
   (b) the rotation
   (c) the principal strains
   (d) whether the compatibility condition is satisfied