8 Energy and Virtual Work

Thus far in this book, problems have been solved by using a combination of force-equilibrium and kinematics. Here, another approach is explored, in which expressions for work and energy are derived and utilised.

Two important topics are discussed in this Chapter. The first is Energy Methods, which are techniques for solving problems involving elastic materials. Some of these methods, for example Castigliano's second theorem, apply only to linear elastic materials, but most apply to generally non-linear elastic materials.

The second topic is that of Virtual Work. The virtual work approach leads to powerful methods which can be used to solve static or dynamic problems involving any material model.

8.1 Energy in Deforming Materials

There are many different types of **energy**: mechanical, chemical, nuclear, electrical, magnetic, etc. Energies can be grouped into **kinetic energies** (which are due to movement) and **potential energies** (which are stored energies – energy that a piece of matter has because of its position or because of the arrangement of its parts).

A rubber ball held at some height above the ground has (gravitational) potential energy. When dropped, this energy is progressively converted into kinetic energy as the ball's speed increases until it reaches the ground where all its energy is kinetic. When the ball hits the ground it begins to deform elastically and, in so doing, the kinetic energy is progressively converted into **elastic strain energy**, which is stored *inside* the ball. This elastic energy is due to the re-arrangement of molecules in the ball – one can imagine this to be very like numerous springs being compressed inside the ball. The ball reaches maximum deformation when the kinetic energy has been completely converted into strain energy. The strain energy is then converted back into kinetic energy, "pushing" the ball back up for the rebound.

Elastic strain energy is a potential energy – elastically deforming a material is in many ways similar to raising a weight off the ground; in both cases the potential energy is increased.

Similarly, **work** is done in stretching a rubber band. This work is converted into elastic strain energy within the rubber. If the applied stretching force is then slowly reduced, the rubber band will use this energy to "pull" back. If the rubber band is stretched and then released suddenly, the band will retract quickly; the strain energy in this case is converted into kinetic energy – and sound energy (the "snap").

When a small weight is placed on a large metal slab, the slab will undergo minute strains, too small to be noticed visually. Nevertheless, the metal behaves like the rubber ball and when the weight is removed the slab uses the internally stored strain energy to return to its initial state. On the other hand, a metal bar which is bent considerably, and then laid upon the ground, will not nearly recover its original un-bent shape. It has undergone *permanent* deformation. Most of the energy supplied has been lost; it has been converted into heat energy, which results in a very slight temperature rise in the bar. Permanent deformations of this type are accounted for by **plasticity theory**, which is treated in Chapter 11.

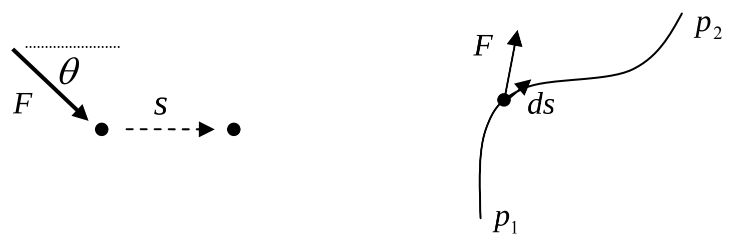
In any real material undergoing deformation, at least some of the supplied energy will be converted into heat. However, with the ideal elastic material under study in this chapter, it is assumed that *all* the energy supplied is converted into strain energy. When the loads are removed, the material returns to its precise initial shape and there is no energy loss; for example, a purely elastic ball dropped onto a purely elastic surface would bounce back up to the precise height from which it was released.

As a prelude to a discussion of the energy of elastic materials, some important concepts from elementary particle mechanics are reviewed in the following sections. It is shown that Newton's second law, the **principle of work and kinetic energy** and the **principle of conservation of mechanical energy** are equivalent statements; each can be derived from the other. These concepts are then used to study the energetics of elastic materials.

8.1.1 Work and Energy in Particle Mechanics

Work

Consider a force F which acts on a particle, causing it to move through a displacement *s*, the directions in which they act being represented by the arrows in Fig. 8.1.1a. The work *W* done by *F* is defined to be $Fs\cos\theta$ where θ is the angle formed by positioning the start of the *F* and *s* arrows at the same location with $0 \le \theta \le 180$. Work can be positive or negative: when the force and displacement are in the same direction, then $0 \le \theta \le 90$ and the work done is positive; when the force and displacement are in opposite directions, then $90 \le \theta \le 180$ and the work done is negative.



(a) (b)

Figure 8.1.1: (a) force acting on a particle, which moves through a displacement *s*; (b) a varying force moving a particle along a path

Consider next a particle moving along a certain path between the points p_1, p_2 by the action of some force *F*, Fig. 8.1.1b. The work done is

$$W = \int_{p_1}^{p_2} F\cos\theta\, ds \tag{8.1.1}$$

where s is the displacement. For motion along a straight line, so that $\theta = 0$, the work is $W = \int_{p_1}^{p_2} F ds$; if *F* here is *constant* then the work is simply *F* times the distance between p_1 to p_2 but, in most applications, *the force will vary* and an integral needs to be evaluated.

Conservative Forces

From Eqn. 8.1.1, the work done by a force in moving a particle through a displacement will in general *depend on the path* taken. There are many important practical cases, however, when the work is *independent* of the path taken, and simply depends on the initial and final positions, for example the work done in deforming elastic materials (see later) – these lead to the notion of a **conservative** (or **potential**) **force**. Looking at the one-dimensional case, a conservative force F_{con} is one which can always be written as the derivative of a function *U* (the minus sign will become clearer in what follows),

$$F_{\text{con}} = -\frac{dU}{dx}, \tag{8.1.2}$$

since, in that case,

$$W = \int_{p_1}^{p_2} F_{\text{con}}\, dx = -\int_{p_1}^{p_2} \frac{dU}{dx} dx = -\int_{p_1}^{p_2} dU = -\left(U(p_2) - U(p_1)\right) = -\Delta U \tag{8.1.3}$$

In this context, the function U is called the **potential energy** and ΔU is the change in potential energy of the particle as it moves from p_1 to p_2. If the particle is moved from p_1 to p_2 and then back to p_1, the net work done is zero and the potential energy U of the particle is that with which it started.

Potential Energy

The potential energy of a particle/system can be defined as follows:

Potential Energy:
the work done in moving a system from some standard configuration to the current configuration

Potential energy has the following characteristics:
(1) The existence of a force field
(2) To move something in the force field, work must be done
(3) The force field is conservative
(4) There is some reference configuration
(5) The force field itself does negative work when another force is moving something against it
(6) It is recoverable energy

These six features are evident in the following example: a body attached to the coil of a spring is extended slowly by a force F, overcoming the spring (restoring) force F_{spr} (so that there are no accelerations and $F = -F_{\text{spr}}$ at all times), Fig. 8.1.2.

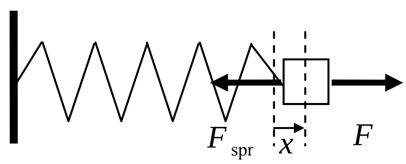


Figure 8.1.2: a force extending an elastic spring

Let the initial position of the block be x_0 (relative to the reference configuration, $x = 0$). Assuming the force to be proportional to deflection, $F = kx$, the work done by F in extending the spring to a distance x is

$$W = \int_{x_0}^{x} F dx = \int_{x_0}^{x} kx dx = \tfrac{1}{2}kx^2 - \tfrac{1}{2}kx_0^2 \equiv U(x) - U(x_0) = \Delta U \qquad (8.1.4)$$

This is the work done to move something in the elastic spring "force field" and by definition is the potential energy (change in the body). The energy supplied in moving the body is said to be **recoverable** because the spring is ready to pull back and do the same amount of work.

The corresponding work done by the conservative spring force F_{spr} is

$$W_{\text{spr}} = -\int_{x_0}^{x} F dx = -\left(\tfrac{1}{2}kx^2 - \tfrac{1}{2}kx_0^2\right) \equiv -\Delta U \qquad (8.1.5)$$

This work can be seen from the area of the triangles in Fig. 8.1.3: the spring force is zero at the equilibrium/reference position ($x = 0$) and increases linearly as x increases.

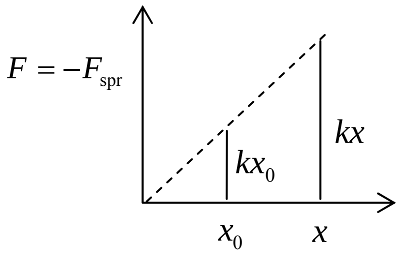


Figure 8.1.3: force-extension curve for a spring

■

The forces in this example depend on the amount by which the spring is stretched. This is similar to the potential energy stored in materials – the potential force will depend in some way on the separation between material particles (see below).

Also, from the example, it can be seen that an alternative definition for the potential energy U of a system is *the negative of the work done by a conservative force in moving the system from some standard configuration to the current configuration*.

In general then, the work done by a conservative force is related to the potential energy through

$$W_{con} = -\Delta U \qquad (8.1.6)$$

Dissipative (Non-Conservative) Forces

When the forces are not conservative, that is, they are **dissipative**, one cannot find a universal function U such that the work done is the difference between the values of U at the beginning and end points – *one has to consider the path* taken by the particle and the work done will be different in each case. A general feature of non-conservative forces is that if one moves a particle and then returns it to its original position the net work done will not be zero. For example, consider a block being dragged across a rough surface,

Fig. 8.1.4. In this case, if the block slides over and back a number of times, the work done by the pulling force F keeps increasing, and the work done is not simply determined by the final position of the block, but by its complete path history. The energy used up in moving the block is dissipated as heat (the energy is **irrecoverable**).

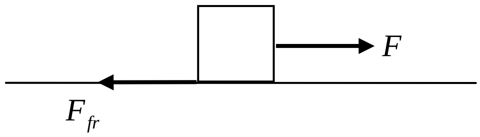


Figure 8.1.4: Dragging a block over a frictional surface

8.1.2 The Principle of Work and Kinetic Energy

In general, a mechanics problem can be solved using either Newton's second law or the principle of work and energy (which is discussed here). These are two different equations which basically say the same thing, but one might be preferable to the other depending on the problem under consideration. Whereas Newton's second law deals with *forces*, the work-energy principle casts problems in terms of *energy*.

The kinetic energy of a particle of mass m and velocity v is defined to be $K = \frac{1}{2}mv^2$. The rate of change of kinetic energy is, using Newton's second law $F = ma$,

$$\dot{K} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dt} = (ma)v = Fv \tag{8.1.7}$$

The change in kinetic energy over a time interval (t_0, t_1) is then

$$\Delta K = K_1 - K_0 = \int_{t_0}^{t_1} \frac{dK}{dt}dt = \int_{t_0}^{t_1} Fv dt \tag{8.1.8}$$

where K_0 and K_1 are the initial and final kinetic energies. The work done over this time interval is

$$W = \int_{W(t_0)}^{W(t_1)} dW = \int_{x(t_0)}^{x(t_1)} F dx = \int_{t_0}^{t_1} Fv dt \tag{8.1.9}$$

and it follows that

$$\boxed{W = \Delta K} \qquad \textbf{Work – Energy Principle} \tag{8.1.10}$$

One has the following:

The principle of work and kinetic energy:
the total work done by the external forces acting on a particle equals the change in kinetic energy of the particle

It is not a new principle of mechanics, rather a rearrangement of Newton's second law of motion (or one could have started with this principle, and derived Newton's second law).

The following example shows how the principle holds for conservative, dissipative and applied forces.

Example

A block of mass m is attached to a spring and dragged along a rough surface. It is dragged from left to right, Fig. 8.1.5. Three forces act on the block, the applied force F_{apl} (taken to be constant), the spring force F_{spr} and the friction force F_{fri} (assumed constant).

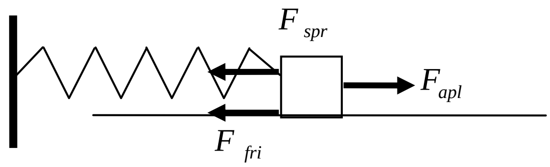


Figure 8.1.5: a block attached to a spring and dragged along a rough surface

Newton's second law, with $F_{spr} = kx$, leads to a standard non-homogeneous second order linear ordinary differential equation with constant coefficients:

$$m\frac{d^2x}{dt^2} = F_{apl} - F_{fri} - F_{spr} \tag{8.1.11}$$

Taking the initial position of the block to be x_0 and the initial velocity to be $\dot{x}_0$, the solution can be found to be

$$x(t) = \frac{F_{apl} - F_{fri}}{k} + \left(x_0 - \frac{F_{apl} - F_{fri}}{k} \right)\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t \tag{8.1.12}$$

where $\omega = \sqrt{k/m}$. The total work done W is the sum of the work done by the applied force W_{apl}, the work done by the spring force W_{spr} and that done by the friction force W_{fri}:

$$W = W_{apl} + W_{spr} + W_{fri} = F_{apl}(x - x_0) - \frac{1}{2}k\left(x^2 - x_0^2\right) - F_{fri}(x - x_0) \tag{8.1.13}$$

The change in kinetic energy of the block is

$$\Delta K = \frac{1}{2} m \left(\dot{x}^2 - \dot{x}_0^2 \right) \tag{8.1.14}$$

Substituting Eqn. 8.1.12 into 8.1.13-14 and carrying out the algebra, one indeed finds that $W = \Delta K$:

$$W = W_{apl} + W_{spr} + W_{fri} = \Delta K \tag{8.1.15}$$

Now the work done by the spring force is equivalent to the negative of the potential energy change, so the work-energy equation (8.1.15) can be written in the alternative form[1]

$$W_{apl} + W_{fri} = \Delta U_{spr} + \Delta K \tag{8.1.16}$$

The friction force is dissipative – it leads to energy loss. In fact, the work done by the friction force is converted into heat which manifests itself as a temperature change in the block. Denoting this energy loss by (see Eqn. 8.1.13) $H_{fri} = F_{fri}(x - x_0)$, one has

$$W_{apl} - H_{fri} = \Delta U_{spr} + \Delta K \tag{8.1.17}$$

■

8.1.3 The Principle of Conservation of Mechanical Energy

In what follows, it is assumed that *there is no energy loss*, so that no dissipative forces act. Define the **total mechanical energy** of a body to be the sum of the kinetic and potential energies of the body. The work-energy principle can then be expressed in two different ways, for this special case:

1. The total work done by the external forces acting on a body equals the change in kinetic energy of the body:

$$W = W_{con} + W_{apl} = \Delta K \tag{8.1.18}$$

2. The total work done by the external forces acting on a body, exclusive of the conservative forces, equals the change in the total mechanical energy of the body

$$W_{\mathrm{apl}} = \Delta U + \Delta K \tag{8.1.19}$$

The special case where there are no external forces, or where all the external forces are conservative/potential, leads to $0 = \Delta U + \Delta K$, so that the mechanical energy is *constant*. This situation occurs, for example, for a body in free-fall {▲Problem 3} and for a freely oscillating spring {▲Problem 4}. Both forms of the work-energy principle can also be seen to apply for a spring subjected to an external force {▲Problem 5}.

[1] it is conventional to keep work terms on the left and energy terms on the right

The Principle of Conservation of Mechanical Energy

The **principle of conservation of energy** states that the total energy of a system remains constant – energy cannot be created or destroyed, it can only be changed from one form of energy to another.

The principle of conservation of energy in the case where there is no energy dissipation is called the **principle of conservation of mechanical energy** and states that, *if a system is subject only to conservative forces, its mechanical energy remains constant*; any system in which non-conservative forces act will inevitable involve non-mechanical energy (heat transfer).

So, when there are only conservative forces acting, one has

$$0 = \Delta U + \Delta K \tag{8.1.20}$$

or, equivalently,

$$K_f + U_f = K_i + U_i \tag{8.1.21}$$

where K_i, K_f are the initial and final kinetic energies and U_i, U_f are the initial and final potential energies.

Note that the principle of mechanical energy conservation is not a new separate law of mechanics, it is merely a re-expression of the work-energy principle (or of Newton's second law).

8.1.4 Deforming Materials

The discussion above which concerned particle mechanics is now generalized to that of a deforming material.

Any material consists of many molecules and particles, all interacting in some complex way. There will be a complex system of **internal forces** acting between the molecules, even when the material is in a natural (undeformed) equilibrium state. If **external forces** are applied, the material will deform and the molecules will move, and hence not only will work be done by the external forces, but *work will be done by the internal forces*. The work-energy principle in this case states that the total work done by the external and internal forces equals the change in kinetic energy,

$$W_{\text{ext}} + W_{\text{int}} = \Delta K \tag{8.1.22}$$

In the special case where no external forces act on the system, one has

$$W_{\text{int}} = \Delta K \tag{8.1.23}$$

which is a situation known as **free vibration**. The case where the kinetic energy is unchanging is

$$W_{\text{ext}} + W_{\text{int}} = 0 \tag{8.1.24}$$

and this situation is known as **quasi-static** (the quantities here can still depend on time).

The force interaction between the molecules can be grouped into:
(1) conservative internal force systems
(2) non-conservative internal force systems (or at least partly non-conservative)

Conservative Internal Forces

First, assuming a conservative internal force system, one can imagine that the molecules interact with each other in the manner of elastic springs. Suppose one could apply an external force to pull two of these molecules apart, as shown in Fig. 8.1.6.

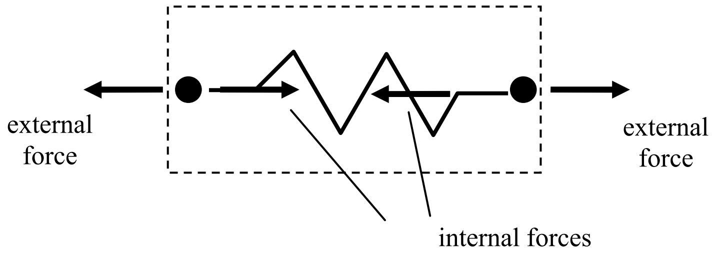


Figure 8.1.6: external force pulling two molecules/particles apart

In this ideal situation one can say that the work done by the external forces equals the change in potential energy plus the change in kinetic energy,

$$W_{\text{ext}} = \Delta U + \Delta K \tag{8.1.25}$$

The energy U in this case of deforming materials is called the **elastic strain energy**, the energy due to the molecular arrangement relative to some equilibrium position.

The free vibration case is now $0 = \Delta U + \Delta K$ and the quasi-static situation is $W_{\text{ext}} = \Delta U$.

Non-Conservative Internal Forces

Consider now another example of internal forces acting within materials, that of a polymer with long-chain molecules. If one could somehow apply an external force to a pair of these molecules, as shown in Fig. 8.1.7, the molecules would slide over each other. Frictional forces would act between the molecules, very much like the frictional force between the block and rough surface of Fig. 8.1.4. This is called **internal friction**. Assuming that the internal forces are dissipative, the external work cannot be written in terms of a potential energy, $W_{\text{ext}} \neq \Delta U + \Delta K$, since *the work done depends on the path taken*. One would have to calculate the work done by evaluating an integral.

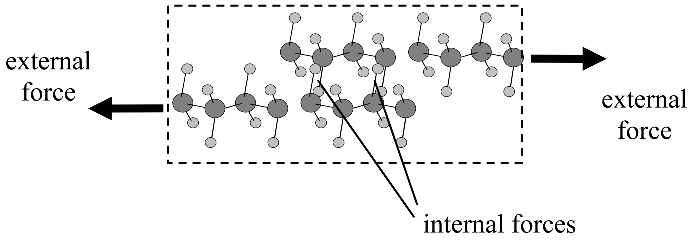


Figure 8.1.7: external force pulling two molecules/particles apart

Similar to Eqn. 8.1.17, however, the energy balance can be written as

$$W_{\text{ext}} - H = \Delta K + \Delta U \tag{8.1.26}$$

where H is the energy dissipated during the deformation *and will depend on the precise deformation process*. This energy is dissipated through heat transfer and is conducted away through the material.

8.1.5 Energy Methods

The work-energy principle provides a method for obtaining solutions to conservative static problems and will be pursued in the next section. The principle is one of a number of tools which can be grouped under the heading **Energy Methods**, such as Castigliano's theorems and the Crotti-Engesser theorem (see later). These methods can be used to solve a wide range of problems involving elastic (linear or non-linear) materials.

Virtual work methods are closely related to energy methods and provide powerful means for solving problems whether they involve elastic materials or not; for the case of elastic materials, they lead naturally to the principle of minimum potential energy discussed in a later section. These virtual work methods will be discussed in sections 8.5-8.6.

8.1.6 Problems

1. Consider the conservative force field
$$F = \frac{1}{x^4} - \frac{1}{x^2}$$
What is the potential energy of a particle at some position $x = x_1$ (define the point at infinity to be the reference point)? What work is done by F as the particle moves from the reference point to $x = x_1$? What is the work done by the applied force which moves the particle from the reference point to $x = x_1$?

2. Consider the gravitational force field mg. Consider a body acted upon by its weight $w = mg$ and by an equal and opposite upward force F (arising, for instance, in a string). Suppose the weight to be moved at slow speed from one position to another one (so that there is no acceleration and $F = -w$). Calculate the work done by F and

show that it is independent of the path taken. What is the potential energy of the body? What is the work done by the gravitational force?

3. Show that both forms of the work-energy principle, Eqns. 8.1.18, 8.1.19, hold for a body in free-fall and that the total mechanical energy is constant. (Use Newton's second law with x positive up, initial height h and zero initial velocity.)

4. Consider a mass m attached to a freely oscillating spring, at initial position x_0 and with initial velocity $\dot{x}_0$. Use Newton's second law to show that
$$x = x_0 \cos\omega t + (\dot{x}_0 / \omega)\sin\omega t$$
$$\dot{x} = \omega\left(-x_0 \sin\omega t + (\dot{x}_0 / \omega)\cos\omega t\right)$$
where $\omega = \sqrt{k/m}$. Show that both forms of the work-energy principle, Eqns. 8.1.18, 8.1.19, hold for the mass and that the total mechanical energy is constant.

5. Consider the case of an oscillating mass m attached to a spring with a *constant* force F applied to the mass. From Newton's second law, one has $m\ddot{x} = -kx + F$ which can be solved to obtain
$$x = \left(x_0 - \frac{F}{k}\right)\cos\omega t + (\dot{x}_0 / \omega)\sin\omega t + \frac{F}{k}$$
$$\dot{x} = \omega\left\{-\left(x_0 - \frac{F}{k}\right)\sin\omega t + (\dot{x}_0 / \omega)\cos\omega t\right\}$$
Evaluate the change in kinetic energy and the total work done by the applied force to show that $W = \Delta K$. Show also that the total work done by the applied force, exclusive of the conservative spring force, is equivalent to $\Delta U + \Delta K$.

6. Consider a body dragged a distance s along a rough horizontal surface by a force F, Fig. 8.1.4. By Newton's second law, $F - F_{fr} = m\ddot{x}$. By directly integrating this equation twice and letting the initial position and velocity of the body be x_0 and $\dot{x}_0$ respectively, show that the work done and the change in kinetic energy of the block are both given by
$$(F - F_{fr})\left\{\frac{1}{2}\left(\frac{F - F_{fr}}{m}\right)t^2 + \dot{x}_0 t\right\}$$
so that the principle of work and kinetic energy holds. How much energy is dissipated?

8.2 Elastic Strain Energy

The strain energy stored in an elastic material upon deformation is calculated below for a number of different geometries and loading conditions. These expressions for stored energy will then be used to solve some elasticity problems using the energy methods mentioned in the previous section.

8.2.1 Strain energy in deformed Components

Bar under axial load

Consider a bar of elastic material fixed at one end and subjected to a steadily increasing force *P*, Fig. 8.2.1. The force is applied slowly so that kinetic energies are negligible. The initial length of the bar is L. The work dW done in extending the bar a small amount $d\Delta$ is[1]

$$dW = Pd\Delta \tag{8.2.1}$$

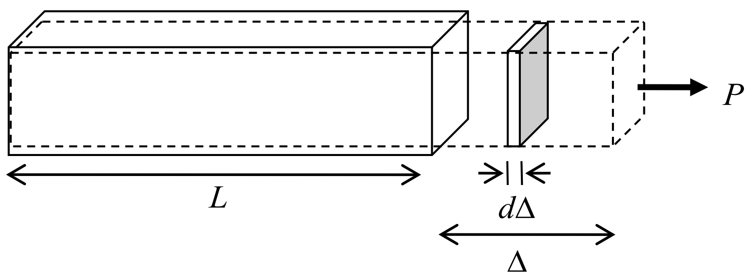


Figure 8.2.1: a bar loaded by a force

It was shown in §7.1.2 that the force and extension Δ are linearly related through $\Delta = PL/EA$, Eqn. 7.1.5, where *E* is the Young's modulus and *A* is the cross sectional area. This linear relationship is plotted in Fig. 8.2.2. The work expressed by Eqn. 8.2.1 is the white region under the force-extension curve (line). The total work done during the complete extension up to a *final* force *P* and *final* extension Δ is the total area beneath the curve.

The work done is stored as elastic strain energy *U* and so

$$U = \frac{1}{2}P\Delta = \frac{P^2 L}{2EA} \tag{8.2.2}$$

If the axial force (and/or the cross-sectional area and Young's modulus) varies along the bar, then the above calculation can be done for a small element of length dx. The energy stored in this element would be $P^2 dx / 2EA$ and the total strain energy stored in the bar would be

[1] the small change in force d*P* which occurs during this small extension may be neglected, since it will result in a smaller-order term of the form d*P*dΔ

$$U = \int_0^L \frac{P^2}{2EA} dx \tag{8.2.3}$$

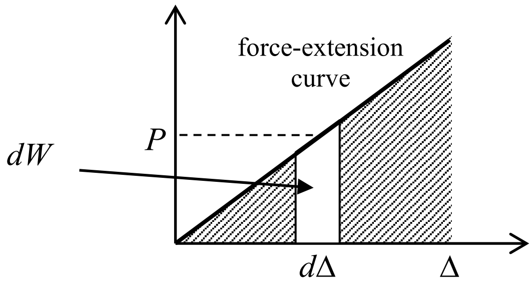


Figure 8.2.2: force-displacement curve for uniaxial load

The strain energy is always positive, due to the square on the force *P*, regardless of whether the bar is being compressed or elongated.

Note the factor of one half in Eqn. 8.2.2. The energy stored is not simply force times displacement because *the force is changing* during the deformation.

Circular Bar in Torsion

Consider a circular bar subjected to a torque *T*. The torque is equivalent to a couple: two forces of magnitude *F* acting in opposite directions and separated by a distance $2r$ as in Fig. 8.2.3; $T = 2Fr$. As the bar twists through a small angle $\Delta\phi$, the forces each move through a distance $\Delta s = r\Delta\phi$. The work done is therefore $\Delta W = 2\left(F\Delta s\right) = T\Delta\phi$.

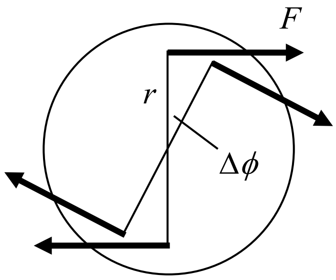


Figure 8.2.3: torque acting on a circular bar

It was shown in §7.2 that the torque and angle of twist are linearly related through Eqn. 7.2.10, $\phi = TL/GJ$, where *L* is the length of the bar, *G* is the shear modulus and *J* is the polar moment of inertia. The angle of twist can be plotted against the torque as in Fig. 8.2.4.

The total strain energy stored in the cylinder during the straining up to a final angle of twist ϕ is the work done, equal to the shaded area in Fig. 8.2.4, leading to

$$U = \frac{1}{2}\phi T = \frac{T^2 L}{2GJ} \tag{8.2.4}$$

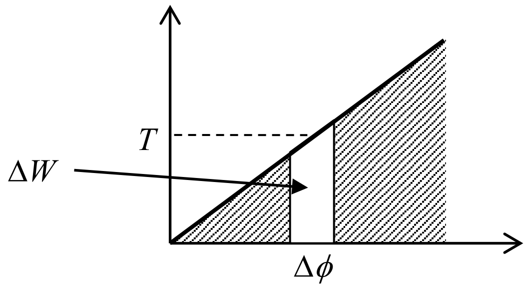


Figure 8.2.4: torque – angle of twist plot for torsion

Again, if the various quantities are varying along the length of the bar, then the total strain energy can be expressed as

$$U = \int_0^L \frac{T^2}{2GJ} dx \tag{8.2.5}$$

Beam subjected to a Pure Moment

As with the bar under torsion, the work done by a moment M as it moves through an angle $d\theta$ is $Md\theta$. The moment is related to the radius of curvature R through Eqns. 7.4.36-37, $M = EI/R$, where E is the Young's modulus and I is the moment of inertia. The length L of a beam and the angle subtended θ are related to R through $L = R\theta$, Fig. 8.2.5, and so moment and angle θ are linearly related through $\theta = ML/EI$.

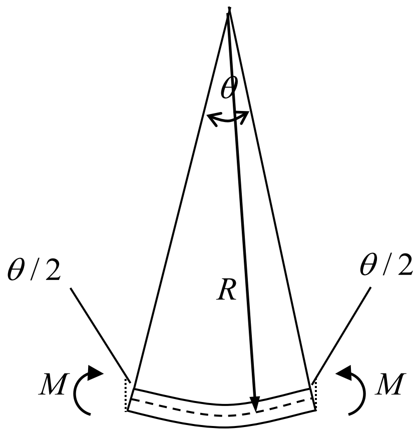


Figure 8.2.5: beam of length L under pure bending

The total strain energy stored in a bending beam is then

$$U = \frac{1}{2}\theta M = \frac{M^2 L}{2EI} \tag{8.2.6}$$

and if the moment and other quantities vary along the beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (8.2.7)$$

This expression is due to the flexural stress σ. A beam can also store energy due to shear stress τ; this latter energy is usually much less than that due to the flexural stresses provided the beam is slender – this is discussed further below.

Example

Consider the bar with varying circular cross-section shown in Fig. 8.2.6. The Young's modulus is 200GPa.

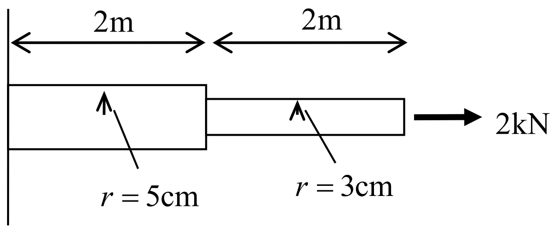


Figure 8.2.6: a loaded bar

The strain energy stored in the bar when a force of 2kN is applied at the free end is

$$U = \int_0^L \frac{P^2}{2EA} dx = \frac{\left(2\times 10^3\right)^2 (2)}{2\left(2\times 10^{11}\right)\pi}\left(\frac{1}{\left(5\times 10^{-2}\right)^2} + \frac{1}{\left(3\times 10^{-2}\right)^2}\right) = 9.62\times 10^{-3} \text{ Nm} \qquad (8.2.8)$$

■

8.2.2 The Work-Energy Principle

The work-energy principle for elastic materials, that is, the fact that the work done by external forces is stored as elastic energy, can be used directly to solve some simple problems. To be precise, it can be used to solve problems involving a single force and for solving for the displacement in the direction of that force. By force and displacement here it is meant **generalised force** and **generalised displacement**, that is, a force/displacement pair, a torque/angle of twist pair or a moment/bending angle pair.

More complex problems need to be solved using more sophisticated energy methods, such as Castigliano's method discussed further below.

Example

Consider the beam of length *L* shown in Fig. 8.2.7, pinned at one end (A) and simply supported at the other (C). A moment M_0 acts at B, a distance L_1 from the left-hand end. The cross-section is rectangular with depth *b* and height *h*. The work-energy principle can be used to calculate the angle θ_B through which the moment at B rotates.

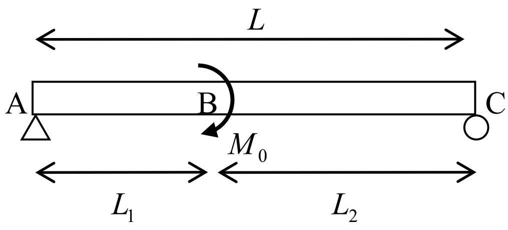


Figure 8.2.7: a beam subjected to a moment at B

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -M_0 x/L, & 0 < x < L_1 \\ M_0(1 - x/L), & L_1 < x < L \end{cases} \tag{8.2.9}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$U = \int_0^L \frac{M^2}{2EI} dx = \frac{6M_0^2}{Ebh^3} \left\{ \int_0^{L_1} \left(\frac{x}{L}\right)^2 dx + \int_{L_1}^{L} \left(1 - \frac{x}{L}\right)^2 dx \right\} = \frac{6M_0^2 L}{Ebh^3} \left[\frac{1}{3} - \left(\frac{L_1}{L}\right) - \left(\frac{L_1}{L}\right)^2 \right] \tag{8.2.10}$$

The work done by the applied moment is $M_0 \theta_B / 2$ and so

$$\theta_B = \frac{12 M_0 L}{Ebh^3} \left[\frac{1}{3} - \left(\frac{L_1}{L}\right) - \left(\frac{L_1}{L}\right)^2 \right] \tag{8.2.11}$$

■

8.2.3 Strain Energy Density

The strain energy will in general vary throughout a body and for this reason it is useful to introduce the concept of **strain energy density**, which is a measure of how much energy is stored in small volume elements throughout a material.

Consider again a bar subjected to a uniaxial force *P*. A small volume element with edges aligned with the x, y, z axes as shown in Fig. 8.2.8 will then be subjected to a stress σ_{xx} only. The volume of the element is $dV = dxdydz$.

From Eqn. 8.2.2, the strain energy in the element is

$$U = \frac{(\sigma_{xx} dydz)^2 dx}{2Edydz} \tag{8.2.12}$$

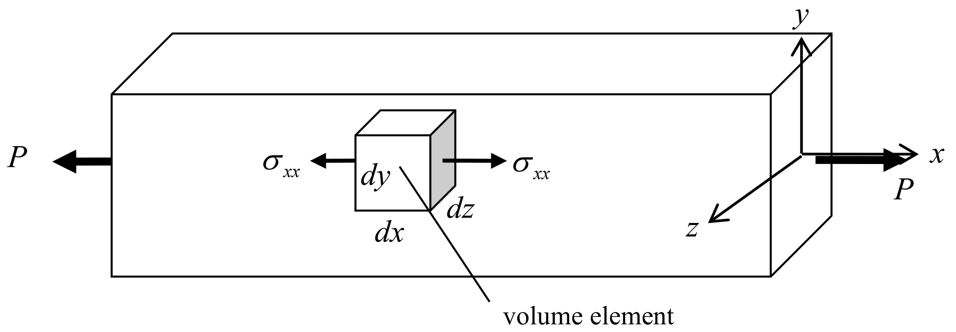


Figure 8.2.8: a volume element under stress

The strain energy density *u* is defined as the strain energy *per unit volume*:

$$u = \frac{\sigma_{xx}^2}{2E} \tag{8.2.13}$$

The total strain energy in the bar may now be expressed as this quantity integrated over the whole volume,

$$U = \int_V u dV , \tag{8.2.14}$$

which, for a constant cross-section *A* and length *L* reads $U = A\int_0^L u dx$. From Hooke's law, the strain energy density of Eqn. 8.2.13 can also be expressed as

$$u = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} \tag{8.2.15}$$

As can be seen from Fig. 8.2.9, this is the area under the uniaxial stress-strain curve.

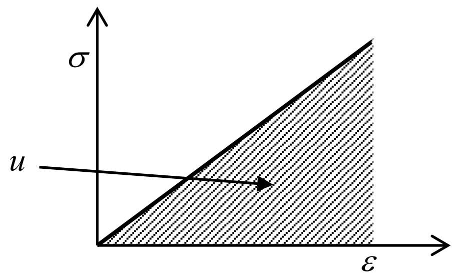


Figure 8.2.9: stress-strain curve for elastic material

Note that the element *does* deform in the *y* and *z* directions but no work is associated with those displacements since there is no force acting in those directions.

The strain energy density for an element subjected to a σ_{yy} stress only is, by the same arguments, $\sigma_{yy}\varepsilon_{yy}/2$, and that due to a σ_{zz} stress is $\sigma_{zz}\varepsilon_{zz}/2$. Consider next a shear stress σ_{xy} acting on the volume element to produce a shear strain ε_{xy} as illustrated in Fig. 8.2.10. The element deforms with small angles θ and λ as illustrated. Only the stresses on the upper and right-hand surfaces are shown, since the stresses on the other two surfaces do no work. The force acting on the upper surface is $\sigma_{xy} dxdz$ and moves through a displacement λdy. The force acting on the right-hand surface is $\sigma_{xy} dydz$ and moves through a displacement θdx. The work done when the element moves through angles $d\theta$ and $d\lambda$ is then, using the definition of shear strain,

$$dW = (\sigma_{xy} dxdz)(d\lambda dy) + (\sigma_{xy} dydz)(d\theta dx) = (dxdydz)\sigma_{xy}(2d\varepsilon_{xy}) \qquad (8.2.16)$$

and, with shear stress proportional to shear strain, the strain energy density is

$$u = 2\int \sigma_{xy} d\varepsilon_{xy} = \sigma_{xy}\varepsilon_{xy} \qquad (8.2.17)$$

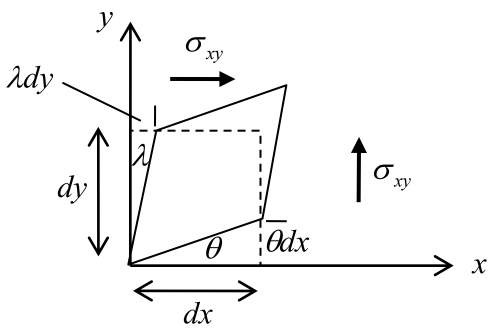


Figure 8.2.10: a volume element under shear stress

The strain energy can be similarly calculated for the other shear stresses and, in summary, the strain energy density for a volume element subjected to arbitrary stresses is

$$u = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz}) + (\sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}) \qquad (8.2.18)$$

Using Hooke's law, Eqns. 6.1.9, and Eqn. 6.1.5, the strain energy density can also be written in the alternative and useful forms {▲Problem 4}

$$\begin{aligned} u &= \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1}{2\mu}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) \\ &= \frac{\mu}{1-2\nu}\left[(1-\nu)(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + 2\nu(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx})\right] + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \\ &= \frac{\nu\mu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + \mu(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \end{aligned} \qquad (8.2.19)$$

Strain Energy in a Beam due to Shear Stress

The shear stresses arising in a beam at location y from the neutral axis are given by Eqn. 7.4.28, $\tau(y) = Q(y)V / Ib(y)$, where Q is the first moment of area of the section of beam from y to the outer surface, V is the shear force, I is the moment of inertia of the complete cross-section and b is the thickness of the beam at y. From Eqns. 8.2.19a and 8.2.14 then, the total strain energy in a beam of length L due to shear stress is

$$U = \int_V \frac{\tau^2}{2\mu} dV = \frac{1}{2}\int_0^L \frac{V^2}{\mu I^2}\left[\int_A \frac{Q^2}{b^2} dA\right] dx \tag{8.2.20}$$

Here V, μ and I are taken to be constant for any given cross-section but may vary along the beam; Q varies and b may vary over any given cross-section. Expression 8.2.20 can be simplified by introducing the **form factor for shear** f_s, defined as

$$f_s(x) = \frac{A}{I^2}\int_A \frac{Q^2}{b^2} dA \tag{8.2.21}$$

so that

$$U = \frac{1}{2}\int_0^L \frac{f_s V^2}{\mu A} dx \tag{8.2.22}$$

The form factor depends only on the shape of the cross-section. For example, for a rectangular cross-section, using Eqn. 7.4.29,

$$f_s(x) = \frac{bh}{\left(bh^3/12\right)^2}\int_{-h/2}^{+h/2} \frac{1}{b^2}\left[\frac{b}{2}\left(\frac{h^2}{4} - y^2\right)\right]^2 dy \int_{-b/2}^{+b/2} dz = \frac{6}{5} \tag{8.2.23}$$

In a similar manner, the form factor for a circular cross-section is found to be $10/9$ and that of a very thin tube is 2.

8.2.4 Castigliano's Second Theorem

The work-energy method is the simplest of energy methods. A more powerful method is that based on **Castigliano's second theorem**[2], which can be used to solve problems involving *linear* elastic materials. As an introduction to Castigliano's second theorem, consider the case of uniaxial tension, where $U = P^2 L / 2EA$. The displacement through which the force moves can be obtained by a differentiation of this expression with respect to that force,

$$\frac{dU}{dP} = \frac{PL}{EA} = \Delta \tag{8.2.24}$$

[2] Casigliano's first theorem will be discussed in a later section

Similarly, for torsion of a circular bar, $U = T^2 L / 2GJ$, and a differentiation gives $dU / dT = TL / GJ = \phi$. Further, for bending of a beam it is also seen that $dU / dM = \theta$.

These are examples of Castigliano's theorem, which states that, provided the body is in equilibrium, *the derivative of the strain energy with respect to the force gives the displacement corresponding to that force, in the direction of that force*. When there is more than one force applied, then one takes the partial derivative. For example, if *n* independent forces $P_1, P_2, \ldots, P_n$ act on a body, the displacement corresponding to the *i*th force is

$$\Delta_i = \frac{\partial U}{\partial P_i} \tag{8.2.25}$$

Before proving this theorem, here follow some examples.

Example

The beam shown in Fig. 8.2.11 is pinned at A, simply supported half-way along the beam at B and loaded at the end C by a force *P* and a moment M_0.

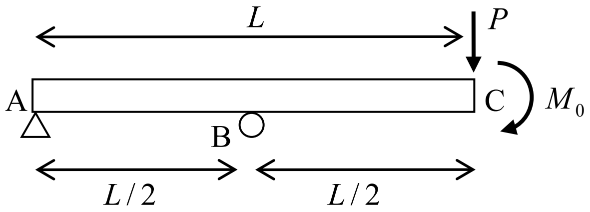


Figure 8.2.11: a beam subjected to a force and moment at C

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -Px - 2M_0 x / L, & 0 < x < L/2 \\ -M_0 - P(L - x), & L/2 < x < L \end{cases} \tag{8.2.26}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$\begin{aligned} U &= \frac{1}{2EI}\left\{\left(P + \frac{2M_0}{L}\right)^2 \int_0^{L/2} x^2 dx + \int_{L/2}^{L} \left(M_0 + P(L - x)\right)^2 dx\right\} \\ &= \frac{P^2 L^3}{24EI} + \frac{5PM_0 L^2}{24EI} + \frac{M_0^2 L}{3EI} \end{aligned} \tag{8.2.27}$$

In order to apply Castigliano's theorem, the strain energy is considered to be a function of the two external loads, $U = U(P, M_0)$. The displacement associated with the force *P* is then

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{PL^3}{12EI} + \frac{5M_0L^2}{24EI} \tag{8.2.28}$$

The rotation associated with the moment is

$$\theta_C = \frac{\partial U}{\partial M_0} = \frac{5PL^2}{24EI} + \frac{2M_0L}{3EI} \tag{8.2.29}$$

■

Example

Consider next the beam of length *L* shown in Fig. 8.2.12, built in at both ends and loaded centrally by a force *P*. This is a statically indeterminate problem. In this case, the strain energy can be written as a function of the applied load and one of the unknown reactions.

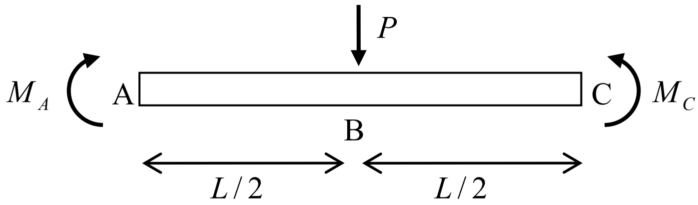


Figure 8.2.12: a statically indeterminate beam

First, the moment in the beam is found from equilibrium considerations to be

$$M = M_A + \frac{P}{2}x, \qquad 0 < x < L/2 \tag{8.2.30}$$

where M_A is the unknown reaction at the left-hand end. Then the strain energy in the left-hand half of the beam is

$$U = \frac{1}{2EI}\int_0^{L/2}\left(M_A + \frac{P}{2}x\right)^2 dx = \frac{P^2L^3}{192EI} + \frac{PM_AL^2}{16EI} + \frac{M_A^2L}{4EI} \tag{8.2.31}$$

The strain energy in the complete beam is double this:

$$U = \frac{P^2L^3}{96EI} + \frac{PM_AL^2}{8EI} + \frac{M_A^2L}{2EI} \tag{8.2.32}$$

Writing the strain energy as $U = U(P, M_A)$, the rotation at A is

$$\theta_A = \frac{\partial U}{\partial M_A} = \frac{PL^2}{8EI} + \frac{M_AL}{EI} \tag{8.2.33}$$

But $\theta_A = 0$ and so Eqn. 8.2.33 can be solved to get $M_A = -PL/8$. Then the displacement at the centre of the beam is

$$\Delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{M_A L^2}{8EI} = \frac{PL^3}{192EI} \tag{8.2.34}$$

This is positive in the direction in which the associated force is acting, and so is downward.

■

Proof of Castigliano's Theorem

A proof of Castiligliano's theorem will be given here for a structure subjected to a single load. The load *P* produces a displacement Δ and the strain energy is $U = P\Delta/2$, Fig. 8.2.13. If an additional force dP is applied giving an additional deformation $d\Delta$, the additional strain energy is

$$dU = Pd\Delta + \frac{1}{2}dPd\Delta \tag{8.2.35}$$

If the load $P + dP$ is applied from zero in one step, the work done is $(P + dP)(\Delta + d\Delta)/2$. Equating this to the strain energy $U + dU$ given by Eqn. 8.2.35 then gives $Pd\Delta = \Delta dP$. Substituting into Eqn. 8.2.35 leads to

$$dU = \Delta dP + \frac{1}{2}dPd\Delta \tag{8.2.36}$$

Dividing through by dP and taking the limit as $d\Delta \to 0$ results in Castigliano's second theorem, $dU/dP = \Delta$.

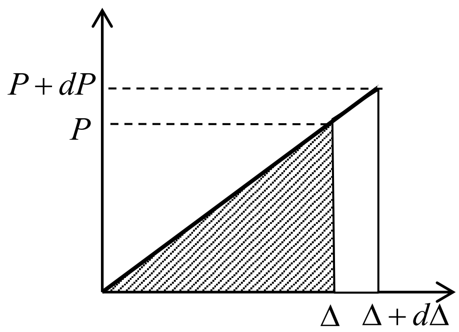


Figure 8.2.13: force-displacement curve

In fact, dividing Eqn. 8.2.35 through by $d\Delta$ and taking the limit as $d\Delta \to 0$ results in **Castigliano's first theorem**, $dU/d\Delta = P$. It will be shown later that this first theorem, unlike the second, in fact holds also for the case when the elastic material is *non-linear*.

8.2.5 Dynamic Elasticiy

Impact and Dynamic Loading

Consider the case of a weight *P* dropped instantaneously onto the end of an elastic bar. If the weight *P* had been applied gradually from zero, the strain energy stored at the final force *P* and final displacement Δ_0 would be $\frac{1}{2}P\Delta_0$. However, the instantaneously applied load is constant throughout the deformation and work done up to a displacement Δ_0 is $P\Delta_0$, Fig. 8.2.14. The difference between the two implies that the bar acquires a kinetic energy (see Eqn. 8.1.19); the material particles accelerate from their equilibrium positions during the compression.

As deformation proceeds beyond Δ_0, it is clear from Fig. 8.2.14 that the strain energy is increasing faster than the work being done by the weight and so there must be a drop in kinetic energy; the particles begin to decelerate. Eventually, at $\Delta_{max} = 2\Delta_0$, the work done by the weight exactly equals the strain energy stored and the material is at rest. However, the material is not in equilibrium – the equilibrium position for a load *P* is Δ_0 – and so the material begins to accelerate back to Δ_0.

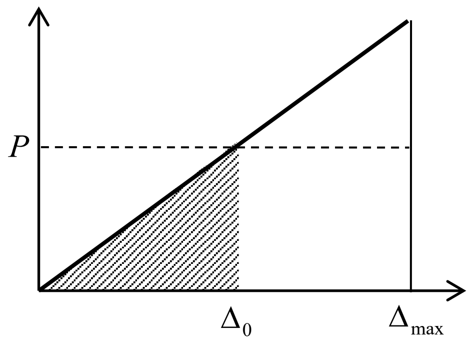


Figure 8.2.14: non-equilibrium loading

The bar and weight will continue to oscillate between 0 and Δ_{max} indefinitely. In a real (inelastic) material, internal friction will cause the vibration to decay.

Thus the maximum compression of a bar under impact loading is twice that of a bar subjected to the same load gradually.

Example

Consider a weight *w* dropped from a height *h*. If one is interested in the final, maximum, displacement of the bar, Δ_{max}, one does not need to know about the detailed and complex transfer of energies during the impact; the energy lost by the weight equals the strain energy stored in the bar:

$$w(h + \Delta_{max}) = \frac{1}{2}P\Delta_{max} \tag{8.2.37}$$

where P is the force acting on the bar at its maximum compression. For an elastic bar, $P = \Delta_{max} EA / L$, or, introducing the **stiffness** k so that $P = k\Delta_{max}$,

$$w(h + \Delta_{max}) = \frac{1}{2} k\Delta_{max}^2, \qquad k = \frac{EA}{L} \tag{8.2.38}$$

which is a quadratic equation in Δ_{max} and can be solved to get

$$\Delta_{max} = \frac{w}{k}\left\{1 + \sqrt{1 + \frac{2hk}{w}}\right\} \tag{8.2.39}$$

If the force w had been applied gradually, then the displacement would have been $\Delta_{st} = w / k$, the "st" standing for "static", and Eqn. 8.2.39 can be re-written as

$$\Delta_{max} = \Delta_{st}\left\{1 + \sqrt{1 + \frac{2h}{\Delta_{st}}}\right\} \tag{8.2.40}$$

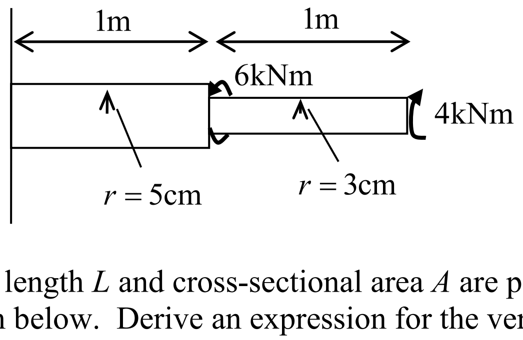
If $h = 0$, so that the weight is just touching the bar when released, then $\Delta_{max} = 2\Delta_{st}$.

■

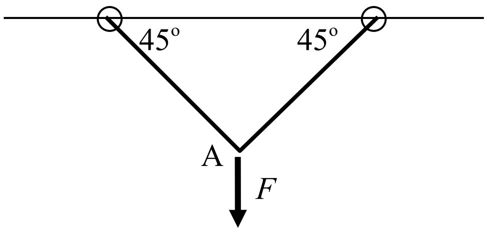
8.2.6 Problems

1. Show that the strain energy in a bar of length L and cross sectional area A hanging from a ceiling and subjected to its own weight is given by (at any section, the force acting is the weight of the material below that section)
$$U = \frac{A\rho^2 g^2 L^3}{6E}$$

2. Consider the circular bar shown below subject to torques at the free end and where the cross-sectional area changes. The shear modulus is $G = 80\text{GPa}$. Calculate the strain energy in the bar(s).



3. Two bars of equal length L and cross-sectional area A are pin-supported and loaded by a force F as shown below. Derive an expression for the vertical displacement at point A using the direct work-energy method, in terms of L, F, A and the Young's modulus E.

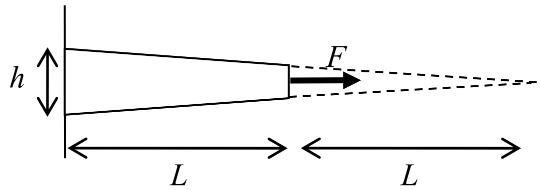


4. Derive the strain energy density equations 8.2.19.

5. For the beam shown in Fig. 8.2.7, use the expression 8.2.22 to calculate the strain energy due to the shear stresses. Take the shear modulus to be $G = 80\text{GPa}$. Compare this with the strain energy due to flexural stress given by Eqn. 8.2.10.

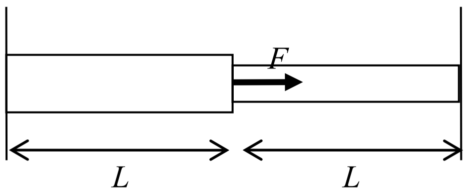
6. Consider a simply supported beam of length L subjected to a uniform load w N/m. Calculate the strain energy due to both flexural stress and shear stress for (a) a rectangular cross-section of depth times height $b \times h$, (b) a circular cross-section with radius r. What is the ratio of the shear-to-flexural strain energies in each case?

7. Consider the tapered bar of length L and square cross-section shown below, built-in at one end and subjected to a uniaxial force F at its free end. The thickness is h at the built-in end. Evaluate the displacement in terms of the (constant) Young's modulus E at the free end using (i) the work-energy theorem, (ii) Castigliano's theorem



8. Consider a cantilevered beam of length L and constant cross-section subjected to a uniform load w N/m. The beam is built-in at $x = 0$ and has a Young's modulus E. Use Castigliano's theorem to calculate the deflection at $x = L$. Consider only the flexural strain energy. [Hint: place a fictitious "dummy" load F at $x = L$ and set to zero once Castigliano's theorem has been applied]

9. Consider the statically indeterminate uniaxial problem shown below, two bars joined at $x = L$, built in at $x = 0$ and $x = 2L$, and subjected to a force F at the join. The cross-sectional area of the bar on the left is $2A$ and that on the right is A. Use (i) the work-energy theorem and (ii) Castigliano's theorem to evaluate the displacement at $x = L$.



8.3 Complementary Energy

The linear elastic solid was considered in the previous section, with the characteristic straight force-deflection curve for axial deformations, Fig.8.2.2. Here, consider the more general case of a bar of *non-linear* elastic material, of length L, fixed at one end and subjected to a steadily increasing force P. The work dW done in extending the bar a small amount $d\Delta$ is

$$dW = Pd\Delta \,. \tag{8.3.1}$$

Force is now no longer proportional to extension Δ, Fig. 8.3.1. However, the total work done during the complete extension up to a final force P and final extension Δ is once again the total area beneath the force-extension curve. The work done is equal to the stored elastic strain energy which must now be expressed as an integral,

$$U = \int_0^\Delta P d\Delta \tag{8.3.2}$$

The strain energy can be calculated if the precise force-deflection relationship is known.

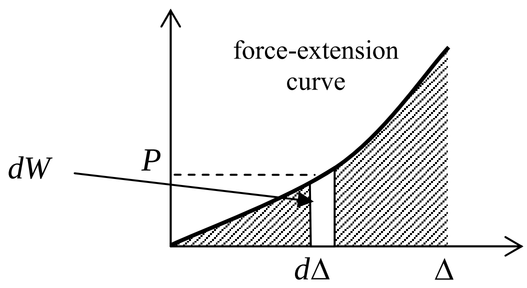


Figure 8.3.1: force-displacement curve for a non-linear material

8.3.1 Complementary Energy

The force-deflection curve is naturally divided into two regions, beneath the curve and above the curve, Fig. 8.3.2. The area of the region under the curve is the strain energy. It is helpful to introduce a new concept, the **complementary energy** C, which is the area above the curve; this can be seen to be given by

$$C = \int_0^P \Delta dP \,. \tag{8.3.3}$$

For a linear elastic material, $C = U$. Although C has units of energy, it has no real physical meaning.

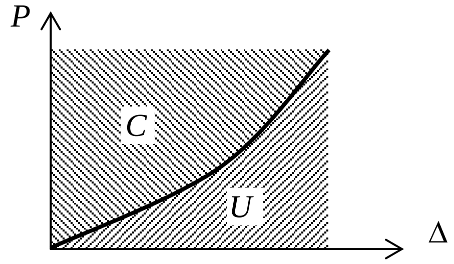


Figure 8.3.2: strain energy and complementary energy for an elastic material

8.3.2 The Crotti-Engesser Theorem

Suppose an elastic body is loaded by *n* independent loads $P_1, P_2, \ldots, P_n$. The strain energy is then the work done by these loads,

$$U = \int_0^{\Delta_1} P_1 d\Delta_1 + \int_0^{\Delta_2} P_2 d\Delta_2 + \cdots + \int_0^{\Delta_n} P_n d\Delta_n \tag{8.3.4}$$

It follows that

$$\frac{\partial U}{\partial \Delta_j} = P_j \tag{8.3.5}$$

which is known as Castigliano's first theorem.

Similarly, the total complementary energy is

$$C = \int_0^{P_1} \Delta_1 dP_1 + \int_0^{P_2} \Delta_2 dP_2 + \cdots + \int_0^{P_n} \Delta_n dP_n \tag{8.3.6}$$

and it follows that

$$\frac{\partial C}{\partial P_j} = \Delta_j \tag{8.3.7}$$

which is known as the **Crotti-Engesser theorem**. For a *linear* elastic material, $C = U$, and the Crotti-Engesser theorem reduces to Castigliano's second theorem, $\Delta_j = \partial U / \partial P_j$, Eqn. 8.2.25.

8.3.3 Problems

1. The force-deflection equation for a non-linear elastic material is given by $P = \alpha \Delta^3$. Find expressions for the strain energy and the complementary energy in terms of (i) P only, (ii) Δ only. Check that $U + C = P\Delta$. What is the ratio C/U?

8.4 Strain Energy Potentials

8.4.1 The Linear Elastic Strain Energy Potential

The strain energy u was introduced in §8.2[1]. From Eqn 8.2.19, the strain energy can be regarded as a function of the strains:

$$\begin{aligned} u &= u(\varepsilon_{ij}) \\ &= \frac{\mu}{1-2\nu}\left[(1-\nu)\left(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2\right)+2\nu\left(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{zz}\varepsilon_{xx}\right)\right]+2\mu\left(\varepsilon_{xy}^2+\varepsilon_{yz}^2+\varepsilon_{zx}^2\right) \end{aligned} \tag{8.4.1}$$

Differentiating with respect to ε_{xx} (holding the other strains constant),

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \frac{2\mu}{1-2\nu}\left[(1-\nu)(\varepsilon_{xx})+\nu\left(\varepsilon_{yy}+\varepsilon_{zz}\right)\right] \tag{8.4.2}$$

From Hooke's law, Eqn 6.1.9, with Eqn 6.1.5, $\mu = E/\left[2(1+\nu)\right]$, the expression on the right is simply σ_{xx}. The strain energy can also be differentiated with respect to the other normal strain components and one has

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz} \tag{8.4.3}$$

The strain energy is a **potential**, meaning that it provides information through a differentiation. Note the similarity between these equations and the equation relating a conservative force and the potential energy seen in §8.1: $dU/dx = F$.

Differentiating Eqn. 8.4.1 with respect to the shear stresses results in

$$\frac{\partial u}{\partial \varepsilon_{xy}} = 2\sigma_{xy}, \quad \frac{\partial u}{\partial \varepsilon_{yz}} = 2\sigma_{yz}, \quad \frac{\partial u}{\partial \varepsilon_{zx}} = 2\sigma_{zx} \tag{8.4.4}$$

The fact that Eqns. 8.4.4 has the factor of 2 on the right hands side but Eqns. 8.4.3 do not is not ideal. There are two common ways of viewing the strain energy potential to overcome this lack of symmetry. First, the strain energy can be taken to be a function of the *six* independent strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$, the latter three being the engineering shear strains, $\gamma_{xy} = 2\varepsilon_{xy}$, etc. Re-writing Eqn. 8.4.1 in terms of the engineering shear strains then leads to the set of equations {▲Problem 1}

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz}, \quad \frac{\partial u}{\partial \gamma_{xy}} = \sigma_{xy} \quad , \quad \frac{\partial u}{\partial \gamma_{yz}} = \sigma_{yz}, \quad \frac{\partial u}{\partial \gamma_{zx}} = \sigma_{zx} \tag{8.4.5}$$

[1] strictly speaking, this is the strain energy *density*, but it should be clear from the context whether it is energy per unit volume or not; the word density will often be omitted henceforth for brevity

The second method is to treat the strain energy as a function of *nine* independent strains, the three normal strains and $\varepsilon_{xy}, \varepsilon_{yx}, \varepsilon_{yz}, \varepsilon_{zy}, \varepsilon_{zx}, \varepsilon_{xz}$. In other words the fact that the strains ε_{xy} and ε_{yx} are the same is ignored and the strain energy is differentiated with respect to these as though they were independent. In order to implement this approach, the strain energy needs to be derived anew treating σ_{xy} and σ_{yx} as independent quantities. This simply means that Fig. 8.2.10 is re-drawn as Fig. 8.4.1 below, and Eqn. 8.2.16 is re-expressed using $\varepsilon_{xy} = (\varepsilon_{xy} + \varepsilon_{yx})/2$ as

$$dW = (\sigma_{yx} dxdz)(d\lambda dy) + (\sigma_{xy} dydz)(d\theta dx) = (dxdydz)\left[\sigma_{xy} d\varepsilon_{xy} + \sigma_{yx} d\varepsilon_{yx}\right] \qquad (8.4.6)$$

so that

$$u = \frac{1}{2}\sigma_{xy}\varepsilon_{xy} + \frac{1}{2}\sigma_{yx}\varepsilon_{yx} \qquad (8.4.7)$$

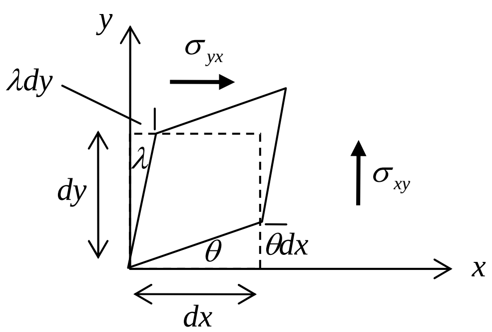


Figure 8.4.1: a volume element under shear stress

Re-writing Eqn. 8.4.1 and differentiation then leads to {▲Problem 2}

$$\begin{gathered}\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz}, \\ \frac{\partial u}{\partial \varepsilon_{xy}} = \sigma_{xy}, \quad \frac{\partial u}{\partial \varepsilon_{yz}} = \sigma_{yz}, \quad \frac{\partial u}{\partial \varepsilon_{zx}} = \sigma_{zx}, \quad \frac{\partial u}{\partial \varepsilon_{yx}} = \sigma_{yx}, \quad \frac{\partial u}{\partial \varepsilon_{zy}} = \sigma_{zy}, \quad \frac{\partial u}{\partial \varepsilon_{xz}} = \sigma_{xz}\end{gathered} \qquad (8.4.8)$$

These equations can be expressed in the succinct form

$$\boxed{\frac{\partial u}{\partial \varepsilon_{ij}} = \sigma_{ij}} \qquad (8.4.9)$$

8.4.2 The Elastic Strain Energy Potential

Eqns. 8.4.9 was derived for an isotropic linear elastic material. In fact these equations are valid very generally, for non-linear and not-necessarily isotropic materials. Generalising the above discussion, recall that the strain energy is the area beneath the stress-strain curve, Fig. 8.4.2, and

$$dW = du = \sigma d\varepsilon \tag{8.4.10}$$

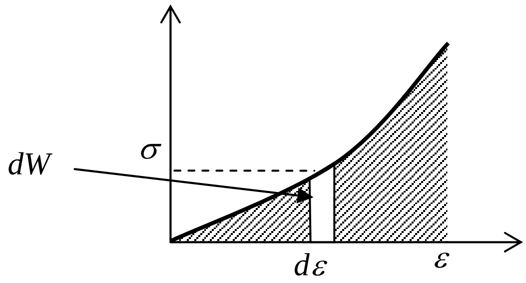


Figure 8.4.2: stress-strain curve for a non-linear material

When the material undergoes increments in strain $d\varepsilon_{xx}$, $d\varepsilon_{xy}$, etc., the increment in strain energy is

$$du = \sigma_{xx} d\varepsilon_{xx} + \sigma_{yy} d\varepsilon_{yy} + \sigma_{xy} d\varepsilon_{xy} + \cdots \tag{8.4.11}$$

If the strain energy is a function of the nine strains ε_{ij}, $u = u(\varepsilon_{ij})$, its increment can also be expressed as

$$du = \frac{\partial u}{\partial \varepsilon_{xx}} d\varepsilon_{xx} + \frac{\partial u}{\partial \varepsilon_{yy}} d\varepsilon_{yy} + \frac{\partial u}{\partial \varepsilon_{xy}} d\varepsilon_{xy} + \cdots \tag{8.4.12}$$

Subtracting Eqns 8.4.12 from 8.4.11 then gives

$$0 = \left(\sigma_{xx} - \frac{\partial u}{\partial \varepsilon_{xx}} \right) d\varepsilon_{xx} + \left(\sigma_{yy} - \frac{\partial u}{\partial \varepsilon_{yy}} \right) d\varepsilon_{yy} + \left(\sigma_{xy} - \frac{\partial u}{\partial \varepsilon_{xy}} \right) d\varepsilon_{xy} + \cdots \tag{8.4.13}$$

Because the strains are independent, that is, any one of them can be adjusted without changing the others, one again arrives at Eqns. 8.4.8-9, only now it has been shown that this result holds very generally.

Note that in the case of an incompressible material, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0$, so that the strains are *not* independent, and Eqns. 8.4.8-9 must be amended.

8.4.3 Symmetry of the Elastic Stiffness Matrix

Consider again the generalised Hooke's Law, Eqn. 6.3.1. Using Eqn. 8.4.9:

$$\frac{\partial}{\partial \varepsilon_{xy}} \left(\frac{\partial u}{\partial \varepsilon_{xx}} \right) = \frac{\partial}{\partial \varepsilon_{xy}} (\sigma_{xx}) = C_{16}, \qquad \frac{\partial}{\partial \varepsilon_{xx}} \left(\frac{\partial u}{\partial \varepsilon_{xy}} \right) = \frac{\partial}{\partial \varepsilon_{xx}} (\sigma_{xy}) = C_{61} \tag{8.4.14}$$

Since the order of partial differentiation for these second partial derivatives should be immaterial, it follows that $C_{16} = C_{61}$. Following the same procedure for the rest of the stresses and strains, it can be seen that the stiffness matrix in Eqn. 6.3.1 is *symmetric* and so there are only 21 independent elastic constants in the most general case of anisotropic elasticity.

8.4.4 The Complementary Energy Potential

Analogous to Eqns. 8.4.10-13, an increment in complementary energy density can be expressed as

$$dc = \varepsilon d\sigma \tag{8.4.15}$$

with

$$dc = \varepsilon_{xx} d\sigma_{xx} + \varepsilon_{yy} d\sigma_{yy} + \varepsilon_{xy} d\sigma_{xy} + \cdots \tag{8.4.16}$$

and

$$dc = \frac{\partial c}{\partial \sigma_{xx}} d\sigma_{xx} + \frac{\partial c}{\partial \sigma_{yy}} d\sigma_{yy} + \frac{\partial c}{\partial \sigma_{xy}} d\sigma_{xy} + \cdots \tag{8.4.17}$$

so that

$$\left(\varepsilon_{xx} - \frac{\partial u}{\partial \sigma_{xx}} \right) d\sigma_{xx} + \left(\varepsilon_{yy} - \frac{\partial u}{\partial \sigma_{yy}} \right) d\sigma_{yy} + \left(\varepsilon_{xy} - \frac{\partial u}{\partial \sigma_{xy}} \right) d\sigma_{xy} + \cdots \tag{8.4.18}$$

With the stresses independent, one has an expression analogous to 8.4.9,

$$\boxed{\frac{\partial c}{\partial \sigma_{ij}} = \varepsilon_{ij}} \tag{8.4.19}$$

8.4.5 Problems

1. Derive equations 8.4.5

2. Derive equations 8.4.8

8.5 Virtual Work

Consider a mass attached to a spring and pulled by an applied force F_{apl}, Fig. 8.5.1a. When the mass is in equilibrium, $F_{spr} + F_{apl} = 0$, where $F_{spr} = -kx$ is the spring force with x the distance from the spring reference position.

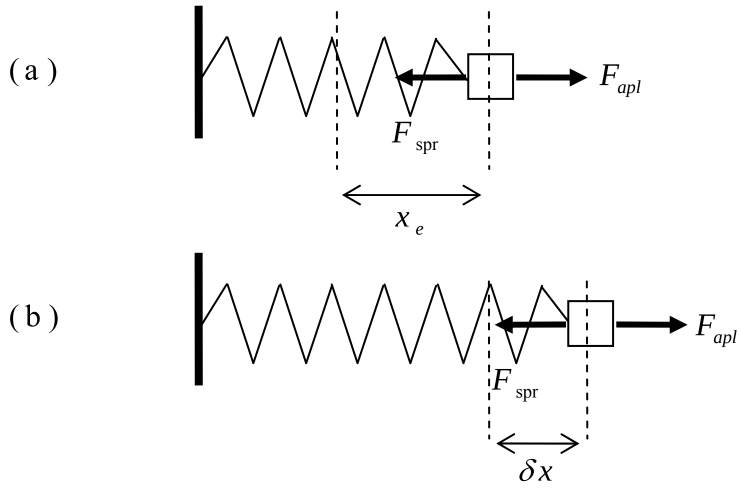


Figure 8.5.1: a force extending an elastic spring; (a) block in equilibrium, (b) block not at its equilibrium position

In order to develop a number of powerful techniques based on a concept known as **virtual work**, imagine that the mass is not in fact at its equilibrium position but at an (incorrect) non-equilibrium position $x + \delta x$, Fig. 8.5.1b. The imaginary displacement δx is called a **virtual displacement**. Define the *virtual work* δW done by a force to be the equilibrium force times this small imaginary displacement δx. It should be emphasized that virtual work is not real work – no work has been performed since δx is not a real displacement which has taken place; this is more like a "thought experiment". The virtual work of the spring force is then $\delta W_{spr} = F_{spr}\delta x = -kx\delta x$. The virtual work of the applied force is $\delta W_{apl} = F_{apl}\delta x$. The total virtual work is

$$\delta W = \delta W_{spr} + \delta W_{apl} = \left(-kx + F_{apl}\right)\delta x \tag{8.5.1}$$

There are two ways of viewing this expression. First, if the system is in equilibrium ($-kx + F_{apl} = 0$) then the virtual work is zero, $\delta W = 0$. Alternatively, if the virtual work is zero then, since δx is arbitrary, the system must be in equilibrium. Thus the virtual work idea gives one an alternative means of determining whether a system is in equilibrium.

The symbol δ is called a **variation** so that, for example, δx is a *variation in the displacement* (from equilibrium).

Virtual work is explored further in the following section.

8.5.1 Principle of Virtual Work: a single particle

A particle of mass *m* is acted upon by a number of forces, $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$, Fig. 8.5.2. Suppose the particle undergoes a virtual displacement $\delta\mathbf{u}$; to reiterate, these impressed forces $\mathbf{f}_i$ do not *cause* the particle to move, one imagines it to be incorrectly positioned a little away from the true equilibrium position.

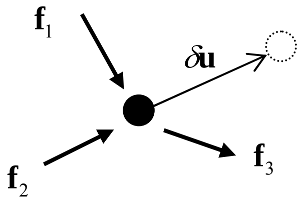


Figure 8.5.2: a particle in equilibrium under the action of a number of forces

If the particle is moving with an acceleration **a**, the quantity $-m\mathbf{a}$ is treated as an inertial force. The total virtual work is then (each term here is the dot product of two vectors)

$$\delta W = \left(\sum_{i=1}^{N} \mathbf{f}_i - m\mathbf{a} \right) \cdot \delta\mathbf{u} \tag{8.5.2}$$

Now *if* the particle is in equilibrium by the action of the effective (impressed plus inertial) force, then

$$\delta W = 0 \tag{8.5.3}$$

This can be expressed as follows:

The principle of virtual work (or **principle of virtual displacements**) **I**:
if a particle is in equilibrium under the action of a number of forces (including the inertial force) the total work done by the forces for a virtual displacement is zero

Alternatively, one can define the external virtual work $\delta W_{\text{ext}} = \sum \mathbf{f}_i \cdot \delta\mathbf{u}$ and the virtual kinetic energy $\delta K = m\mathbf{a} \cdot \delta\mathbf{u}$ in which case the principle takes the form $\delta W_{\text{ext}} = \delta K$ (compare with the work-energy principle, Eqn. 8.1.10).

In the above, the principle of virtual work was derived using Newton's second law. One could just as well regard the principle of virtual work as the fundamental principle and from it derive the conditions for equilibrium. In this case one can say that[1]

[1] note the word *any* here: this must hold for *all* possible virtual displacements, for it will always be possible to find one virtual displacement which is perpendicular to the resultant of the forces, so that $\left(\sum \mathbf{f}\right) \cdot \delta\mathbf{u} = 0$ even though $\sum \mathbf{f}$ is not necessarily zero

The principle of virtual work (or **principle of virtual displacements**) **II**:
a particle is in equilibrium under the action of a system of forces (including the inertial force) if the total work done by the forces is zero for any virtual displacement of the particle

Constraints

In many practical problems, the particle will usually be constrained to move in only certain directions. For example consider a ball rolling over a table, Fig. 8.5.3. If the ball is in equilibrium then all the forces sum to zero, $\mathbf{R} + \sum \mathbf{f} - m\mathbf{a} = \mathbf{0}$, where one distinguishes between the non-reaction forces $\mathbf{f}_i$ and the reaction force **R**. If the virtual displacement $\delta\mathbf{u}$ is such that the constraint is not violated, that is the ball is not allowed to go "through" the table, then $\delta\mathbf{u}$ and **R** are perpendicular, the virtual work done by the reaction force is zero and $\delta W = \left(\sum \mathbf{f} - m\mathbf{a}\right) \cdot \delta\mathbf{u} = 0$. This is one of the benefits of the principle of virtual work; one does not need to calculate the forces of constraint **R** in order to determine the forces $\mathbf{f}_i$ which maintain the particle in equilibrium.

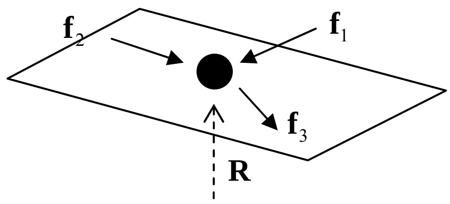


Figure 8.5.3: a particle constrained to move over a surface

The term **kinematically admissible displacement** is used to mean one that does not violate the constraints, and hence one arrives at the version of the principle which is often used in practice:

The principle of virtual work (or **principle of virtual displacements**) **III**:
a particle is in equilibrium under the action of a system of forces (including the inertial force) if the total work done by the forces (excluding reaction forces) is zero for any kinematically admissible virtual displacement of the particle

Whether one uses a kinematically admissible virtual displacement and so disregard reaction forces, or permit a virtual displacement that violates the constraint conditions will usually depend on the problem at hand. In this next example, use is made of a kinematically inadmissible virtual displacement.

Example

Consider a rigid bar of length *L* supported at its ends and loaded by a force *F* a distance *a* from the left hand end, Fig. 8.5.3a. Reaction forces R_A, R_C act at the ends. Let point *C* undergo a virtual displacement δu. From similar triangles, the displacement at *B* is $(a/L)\delta u$. End A does not move and so no virtual work is performed there. The total virtual work is

$$\delta W = R_C \delta u - F \frac{a}{L} \delta u \tag{8.5.4}$$

Note the minus sign here – the displacement at B is in a direction opposite to that of the action of the load and hence the work is negative. The beam is in equilibrium when $\delta W = 0$ and hence $R_C = aF / L$.

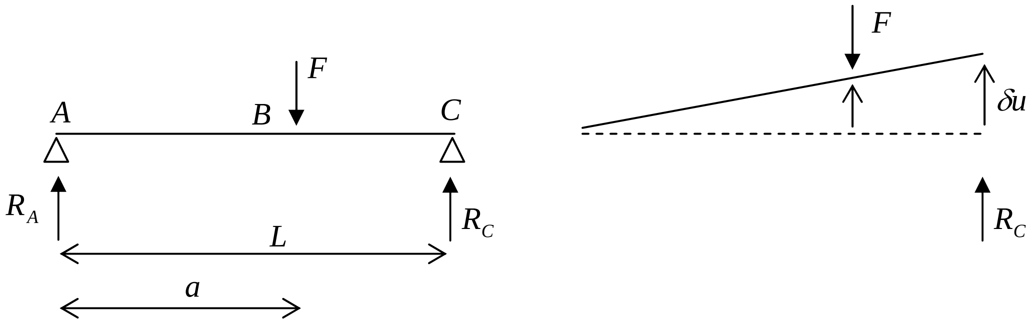


Figure 8.5.3: a loaded rigid bar; (a) bar geometry, (b) a virtual displacement at end C

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8.5.2 Principle of Virtual Work: deformable bodies

A deformable body can be imagined to undergo virtual displacements (not necessarily the same throughout the body). Virtual work is done by the externally applied forces – **external virtual work** – and by the internal forces – **internal virtual work**. Looking again at the spring problem of Fig. 8.5.1, the external virtual work is $\delta W_{ext} = F_{apl} \delta x$ and, considering the spring force to be an "internal" force, the internal virtual work is $\delta W_{\text{int}} = -kx\delta x$. This latter virtual work can be re-written as $\delta W_{\text{int}} = -\delta U$ where δU is the virtual potential energy change which occurs when the spring is moved a distance δx (keeping the spring force constant).

In the same way, the internal virtual work of an elastic body is the (negative of the) virtual strain energy and the principle of virtual work can be expressed as

$$\boxed{\delta W_{ext} = \delta U} \quad \textbf{Principle of Virtual Work for an Elastic Body} \tag{8.5.4}$$

The principle can be extended to accommodate dissipation (energy loss), but only elastic materials will be examined here.

The virtual strain energy for a uniaxial rod is derived next.

8.5.3 Virtual Strain Energy for a Uniaxially Loaded Bar

In what follows, to distinguish between the strain energy and the displacement, the former will now be denoted by *w* and the latter by *u*.

Consider a uniaxial bar which undergoes strains ε. The strain is the unit change in length and, considering an element of length dx, Fig. 8.5.4a, the strain is

$$\varepsilon = \frac{\left[\Delta x + u(x+\Delta x) - u(x)\right] - \Delta x}{\Delta x} = \frac{du}{dx} \tag{8.5.5}$$

in the limit as $\Delta x \to 0$. With $dw = \sigma d\varepsilon$, the strain energy density is

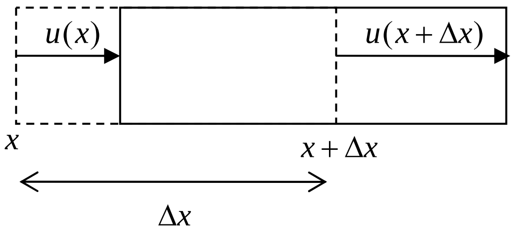
$$w = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{du}{dx}\right)^2 \tag{8.5.6}$$

and the strain energy is

$$U = \int_v \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx \tag{8.5.7}$$

This is the actual strain energy change when the bar undergoes actual strains ε. For the simple case of constant *A* and *L* and constant strain $du/dx = \Delta / L$ where Δ is the elongation of the bar, Eqn. 8.5.7 reduces to $U = AE\Delta^2 / 2L$ (equivalent to Eqn. 8.2.2).

(a)



(b)

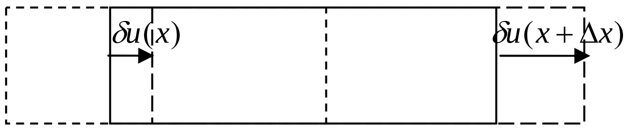


Figure 8.5.4: element undergoing actual and virtual displacements; (a) actual displacements, (b) virtual displacements

It will now be shown that the internal virtual work done as material particles undergo virtual displacements δu is given by δU, with *U* given by Eqn. 8.5.7.

Consider an element to "undergo" virtual displacements δu, Fig. 8.5.4b, which are, by definition, *measured from the actual displacements*. The virtual displacements give rise to **virtual strains**:

$$\delta\varepsilon = \frac{\delta u(x+\Delta x) - \delta u(x)}{\Delta x} = \frac{d(\delta u)}{dx} \tag{8.5.8}$$

again in the limit as $\Delta x \to 0$. Since $\delta\varepsilon = \delta(du/dx)$, it follows that

$$\delta\left(\frac{du}{dx}\right) = \frac{d(\delta u)}{dx} \tag{8.5.9}$$

In other words, the variation of the derivative is equal to the derivative of the variation[2].

One other result is needed before calculating the internal virtual work. Consider a function of the displacement, $f(u)$. The variation of f when u undergoes a virtual displacement is, by definition,

$$\delta f \equiv f(u+\delta u) - f(u) = \frac{f(u+\delta u) - f(u)}{\delta u}\delta u = \frac{df}{du}\delta u \tag{8.5.10}$$

now in the limit as the virtual displacement $\delta u \to 0$. From this one can write

$$\delta\left[\left(\frac{du}{dx}\right)^2\right] = 2\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right) \tag{8.5.11}$$

The stress σ applied to the surface of the element under consideration is an "external force". The internal force is the equal and opposite stress on the other side of the surface inside the element. The internal virtual work (per unit volume) is then $\delta W = -\sigma\delta\varepsilon$. Since σ is the *actual* stress, unaffected by the virtual straining,

$$\delta W = -E\varepsilon\delta\varepsilon = -E\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right) = -\frac{1}{2}E\delta\left(\frac{du}{dx}\right)^2 = -\delta\left[\frac{1}{2}E\left(\frac{du}{dx}\right)^2\right] \tag{8.5.12}$$

since the Young's modulus is unaffected by any virtual displacement. The total work done is then

$$\delta W_{\text{int}} = -\delta\int_v \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV \tag{8.5.13}$$

which, comparing with Eqn. 8.5.7, is the desired result, $\delta W_{\text{int}} = -\delta U$.

Example

Two rods with cross sectional areas A_1, A_2, lengths L_1, L_2 and Young's moduli E_1, E_2 and joined together with the other ends fixed, as shown in Fig. 8.5.5. The rods are subjected to a force P where they meet. As the rods elongate/contract, the strain is simply $\varepsilon = u_B / L$, where u_B is the displacement of the point at which the force is applied. The total elastic strain energy is, from Eqn. 8.5.7,

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[2] this holds in general for any function; manipulations with variations form a part of a branch of mathematics known as the **Calculus of Variations**, which is concerned in the main with minima/maxima problems

$$U = \frac{E_1 A_1}{2L_1} u_B^2 + \frac{E_2 A_2}{2L_2} u_B^2 \tag{8.5.14}$$

Introduce now a virtual displacement δu_B at *B*. The external virtual work is $\delta W_{\text{ext}} = P\delta u_B$. The principle of virtual work, Eqn. 8.5.4, states that

$$P\delta u_B = \delta\left\{\left(\frac{E_1 A_1}{2L_1} + \frac{E_2 A_2}{2L_2}\right) u_B^2\right\} \tag{8.5.15}$$

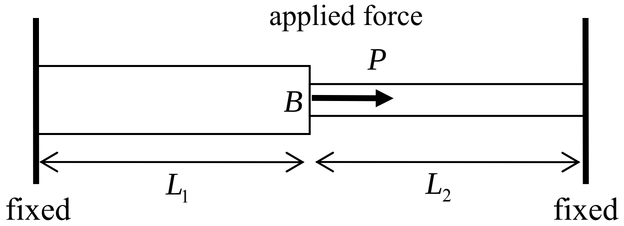

Figure 8.8.5: two rods subjected to a force *P*

From relation 8.5.10,

$$P\delta u_B = \left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2}\right) u_B \delta u_B \tag{8.5.16}$$

The virtual displacement δu_B is arbitrary and so can be cancelled out, giving the result

$$u_B = P\left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2}\right)^{-1} \tag{8.5.17}$$

from which the strains and hence stresses can be evaluated. Note that the reaction forces were not involved in this solution method.

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8.5.4 Virtual Strain Energy for a Beam

The strain energy in a beam is given by Eqn. 8.2.7, *viz*.

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{8.5.18}$$

Using the moment-curvature relation 7.4.37, $M = EI\left(d^2 v / dx^2\right)$, where *v* is the deflection of the beam,

$$U = \int_0^L \frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 dx \tag{8.5.19}$$

and the virtual strain energy is

$$\delta U = \delta \int_0^L \frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 dx \tag{8.5.20}$$

It is not easy to analyse problems using this expression and the principle of virtual work directly, but this expression will be used in the next section in conjunction with the related principle of minimum potential energy.

8.5.5 Problems

1. Consider a uniaxial bar of length *L* with constant cross section *A* and Young's modulus *E*, fixed at one end and subjected to a force *P* at the other. Use the principle of virtual work to show that the displacement at the loaded end is $u = PL / EA$.

2. Consider a uniaxial bar of length *L*, cross sectional area *A* and Young's modulus *E*. What factor of *EAL* is the strain energy when the displacements in the bar are $u = 10^{-3} x$, with *x* measured from one end of the bar? What is the internal virtual work for a virtual displacement $\delta u = 10^{-5} x$? For a constant virtual displacement along the bar?

3. A rigid bar rests upon three columns, a central column with Young's modulus 100GPa and two equidistant outer columns with Young's moduli 200GPa. The columns are of equal length 1m and cross-sectional area 1cm^2. The rigid bar is subjected to a downward force of 10kN. Use the principle of virtual work to evaluate the vertical displacement downward of the rigid bar.

4. Re-solve problem 3 from §8.2.6 using the principle of virtual displacements.

8.6 The Principle of Minimum Potential Energy

The **principle of minimum potential energy** follows directly from the principle of virtual work (for elastic materials).

8.6.1 The Principle of Minimum Potential Energy

Consider again the example given in the last section; in particular re-write Eqn. 8.5.15 as

$$\delta\left\{Pu_B - \left(\frac{E_1A_1}{2L_1} + \frac{E_2A_2}{2L_2}\right)u_B^2\right\} = 0 \tag{8.6.1}$$

The quantity inside the curly brackets is defined to be the **total potential energy** of the system, Π, and the equation states that the variation of Π is zero – that this quantity does not vary when a virtual displacement is imposed:

$$\delta\Pi = 0 \tag{8.6.2}$$

The total potential energy as a function of displacement *u* is sketched in Fig. 8.6.1. With reference to the figure, Eqn. 8.6.2 can be interpreted as follows: the total potential energy attains a stationary value (maximum or minimum) at the *actual* displacement (u_1); for example, $\delta\Pi \neq 0$ for an incorrect displacement u_2. Thus the solution for displacement can be obtained by finding a stationary value of the total potential energy. Indeed, it can be seen that the quantity inside the curly brackets in Fig. 8.6.1 attains a minimum for the solution already derived, Eqn. 8.5.17.

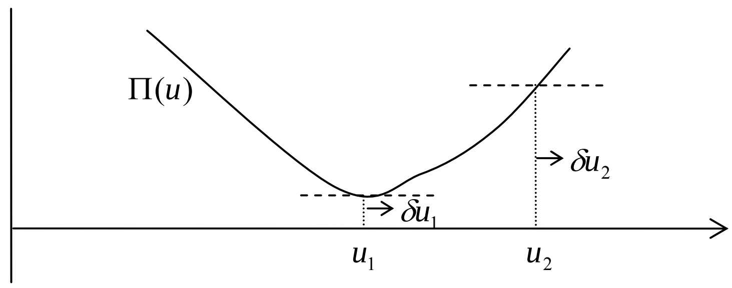

Figure 8.6.1: the total potential energy of a system

To generalise, define the "potential energy" of the applied loads to be $\delta V = -\delta W_{ext}$ so that

$$\delta\Pi = \delta U + \delta V \tag{8.6.3}$$

The external loads must be conservative, precluding for example any sliding frictional loading. Taking the total potential energy to be a function of displacement *u*, one has

$$\delta\Pi = \frac{d\Pi(u)}{du}\delta u = 0 \tag{8.6.4}$$

Thus of all possible displacements *u* satisfying the loading and boundary conditions, the actual displacement is that which gives rise to a stationary point $d\Pi / du = 0$ and the problem reduces to finding a stationary value of the total potential energy $\Pi = U + V$.

Stability

To be precise, Eqn. 8.6.2 only demands that the total potential energy has a stationary point, and in that sense it is called the **principle of stationary potential energy**. One can have a number of stationary points as sketched in Fig. 8.6.2. The true displacement is one of the stationary values u_1, u_2, u_3.

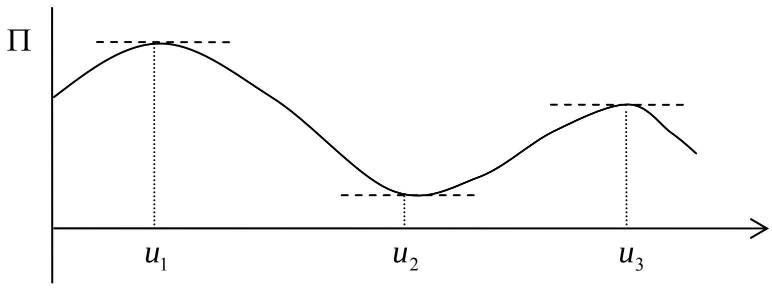

Figure 8.6.2: the total potential energy of a system

Consider the system with displacement u_2. If an external force acts to give the particles of the system some small initial velocity and hence kinetic energy, one has $0 = \Delta\Pi + \Delta K$. The particles will now move and so the displacement u_2 changes. Since Π is a minimum there it must increase and so the kinetic energy must decrease, and so the particles remain close to the equilibrium position. For this reason u_2 is defined as a **stable** equilibrium point of the system. If on the other hand the particles of the body were given small initial velocities from an initial displacement u_1 or u_3, the kinetic energy would increase dramatically; these points are called **unstable** equilibrium points. Only the state of stable equilibrium is of interest here and the principle of stationary potential energy in this case becomes the principle of minimum potential energy.

8.6.2 The Rayleigh-Ritz Method

In applications, the principle of minimum potential energy is used to obtain *approximate* solutions to problems which are otherwise difficult or, more usually, impossible to solve exactly. It forms one basis of the **Finite Element Method** (FEM), a general technique for solving systems of equations which arise in complex mechanics problems.

Example

Consider a uniaxial bar of length *L*, young's modulus *E* and varying cross-section $A = A_0(1 + x/L)$, fixed at one end and subjected to a force *F* at the other. The true solution for displacement to this problem can be shown to be $u = (FL / EA_0)\ln(1 + x / L)$. To see how this might be approximated using the principle, one writes

$$\Pi = U + V = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Fu\big|_{x=L} \tag{8.6.5}$$

First, substituting in the exact solution leads to

$$\Pi = \frac{EA_0}{2}\int_0^L (1 + x / L)\left(\frac{F}{EA_0}\frac{1}{1 + x / L}\right)^2 dx - F\frac{FL}{EA_0}\ln 2 = -\frac{\ln 2}{2}\frac{F^2 L}{EA_0} \tag{8.6.6}$$

According to the principle, any other displacement solution (which satisfies the displacement boundary condition $u(0) = 0$) will lead to a greater potential energy Π.

Suppose now that the solution was unknown. In that case an estimate of the solution can be made in terms of some unknown parameter(s), substituted into Eqn. 8.6.5, and then minimised to find the parameters. This procedure is known as the **Rayleigh Ritz method**. For example, let the guess, or **trial function**, be the linear function $u = \alpha + \beta x$. The boundary condition leads to $\alpha = 0$. Substituting $u = \beta x$ into Eqn. 8.6.5 leads to

$$\Pi = \frac{1}{2}EA_0\beta^2\int_0^L (1 + x / L)dx - F\beta L = \frac{3}{4}EA_0 L\beta^2 - F\beta L \tag{8.6.7}$$

The principle states that $\delta\Pi = (d\Pi / d\beta)\delta\beta = 0$, so that

$$\frac{d\Pi}{d\beta} = \frac{3}{2}EA_0 L\beta - FL = 0 \quad \rightarrow \quad \beta = \frac{2F}{3EA_0} \quad \rightarrow \quad u = \frac{2Fx}{3EA_0} \tag{8.6.8}$$

The exact and approximate Ritz solution are plotted in Fig. 8.6.3.

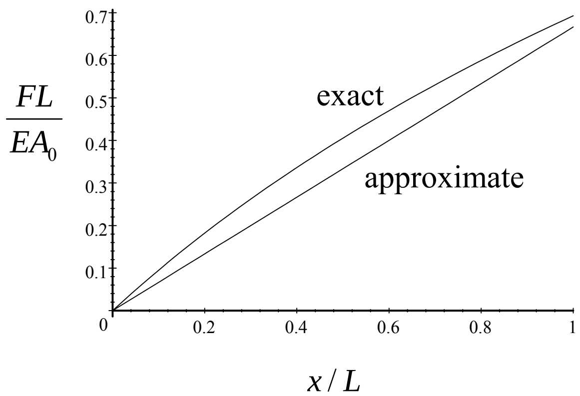

Figure 8.6.3: exact and (Ritz) approximate solution for axial problem

The total potential energy due to this approximate solution $2Fx/3EA_0$ is, from Eqn. 8.6.5,

$$\Pi = -\frac{1}{3}\frac{F^2 L}{EA_0} \tag{8.6.9}$$

which is indeed greater than the minimum value Eqn. 8.6.6 ($\approx -0.347F^2 L/EA_0$).

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The accuracy of the solution 8.6.9 can be improved by using as the trial function a quadratic instead of a linear one, say $u = \alpha + \beta x + \gamma x^2$. Again the boundary condition leads to $\alpha = 0$. Then $u = \beta x + \gamma x^2$ and there are now two unknowns to determine. Since Π is a function of two variables,

$$\delta\Pi(\beta,\gamma) = \frac{\partial\Pi}{\partial\beta}\delta\beta + \frac{\partial\Pi}{\partial\gamma}\delta\gamma = 0 \tag{8.6.10}$$

and the two unknowns can be obtained from the two conditions

$$\frac{\partial\Pi}{\partial\beta} = 0, \qquad \frac{\partial\Pi}{\partial\gamma} = 0 \tag{8.6.11}$$

Example

A beam of length *L* and constant Young's modulus *E* and moment of inertia *I* is supported at its ends and subjected to a uniform distributed force per length *f*. Let the beam undergo deflection $v(x)$. The potential energy of the applied loads is

$$V = -\int_0^L f v(x)dx \tag{8.6.12}$$

and, with Eqn. 8.5.19, the total potential energy is

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx - f\int_0^L v dx \tag{8.6.13}$$

Choose a quadratic trial function $v = \alpha + \beta x + \gamma x^2$. The boundary conditions lead to $v = \gamma x(x - L)$. Substituting into 8.6.13 leads to

$$\Pi = 2\gamma^2 EIL - f\gamma L^3/6 \tag{8.6.14}$$

With $\delta\Pi = (d\Pi/d\gamma)\delta\gamma = 0$, one finds that

$$\gamma = \frac{fL^2}{24EI} \quad \rightarrow \quad v(x) = -\frac{fL^3}{24EI}x + \frac{fL^2}{24EI}x^2 \qquad (8.6.15)$$

which compares with the exact solution

$$v(x) = -\frac{fL^3}{24EI}x + \frac{fL}{12EI}x^3 - \frac{f}{24EI}x^4 \qquad (8.6.16)$$

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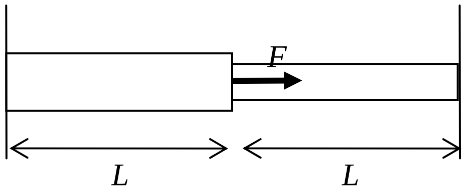
8.6.3 Problems

1. Consider the statically indeterminate uniaxial problem shown below, two bars joined at $x = L$, built in at $x = 0$ and $x = 2L$, and subjected to a force *F* at the join. The cross-sectional area of the bar on the left is 2*A* and that on the right is *A*. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial displacement function of the form $u = \alpha + \beta x + \gamma x^2$ to approximate the exact displacement and in particular the displacement at $x = L$.



2. A beam of length *L* and constant Young's modulus *E* and moment of inertia *I* is supported at its ends and subjected to a uniform distributed force per length *f and* a concentrated force *P* at its centre. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial deflection $v = \alpha \sin(\pi x / L)$, to approximate the exact deflection.

3. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial solution $u = \alpha x$ to approximately solve the problem of axial deformation of an elastic rod of varying cross section, built in at one end and loaded by a uniform distributed force/length *f*, and a force *P* at the free end, as shown below. The cross sectional are is $A(x) = A_0(2 - x/L)$ and the length of the rod is *L*.