8.4 Strain Energy Potentials

8.4.1 The Linear Elastic Strain Energy Potential

The strain energy u was introduced in §8.2[1]. From Eqn 8.2.19, the strain energy can be regarded as a function of the strains:

$$\begin{aligned} u &= u(\varepsilon_{ij}) \\ &= \frac{\mu}{1-2\nu}\left[(1-\nu)\left(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2\right)+2\nu\left(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{zz}\varepsilon_{xx}\right)\right]+2\mu\left(\varepsilon_{xy}^2+\varepsilon_{yz}^2+\varepsilon_{zx}^2\right) \end{aligned} \tag{8.4.1}$$

Differentiating with respect to ε_{xx} (holding the other strains constant),

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \frac{2\mu}{1-2\nu}\left[(1-\nu)(\varepsilon_{xx})+\nu\left(\varepsilon_{yy}+\varepsilon_{zz}\right)\right] \tag{8.4.2}$$

From Hooke's law, Eqn 6.1.9, with Eqn 6.1.5, $\mu = E/[2(1+\nu)]$, the expression on the right is simply σ_{xx}. The strain energy can also be differentiated with respect to the other normal strain components and one has

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz} \tag{8.4.3}$$

The strain energy is a **potential**, meaning that it provides information through a differentiation. Note the similarity between these equations and the equation relating a conservative force and the potential energy seen in §8.1: $dU/dx = F$.

Differentiating Eqn. 8.4.1 with respect to the shear stresses results in

$$\frac{\partial u}{\partial \varepsilon_{xy}} = 2\sigma_{xy}, \quad \frac{\partial u}{\partial \varepsilon_{yz}} = 2\sigma_{yz}, \quad \frac{\partial u}{\partial \varepsilon_{zx}} = 2\sigma_{zx} \tag{8.4.4}$$

The fact that Eqns. 8.4.4 has the factor of 2 on the right hands side but Eqns. 8.4.3 do not is not ideal. There are two common ways of viewing the strain energy potential to overcome this lack of symmetry. First, the strain energy can be taken to be a function of the *six* independent strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$, the latter three being the engineering shear strains, $\gamma_{xy} = 2\varepsilon_{xy}$, etc. Re-writing Eqn. 8.4.1 in terms of the engineering shear strains then leads to the set of equations {▲Problem 1}

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz}, \quad \frac{\partial u}{\partial \gamma_{xy}} = \sigma_{xy}, \quad \frac{\partial u}{\partial \gamma_{yz}} = \sigma_{yz}, \quad \frac{\partial u}{\partial \gamma_{zx}} = \sigma_{zx} \tag{8.4.5}$$

[1] strictly speaking, this is the strain energy *density*, but it should be clear from the context whether it is energy per unit volume or not; the word density will often be omitted henceforth for brevity

The second method is to treat the strain energy as a function of *nine* independent strains, the three normal strains and $\varepsilon_{xy}, \varepsilon_{yx}, \varepsilon_{yz}, \varepsilon_{zy}, \varepsilon_{zx}, \varepsilon_{xz}$. In other words the fact that the strains ε_{xy} and ε_{yx} are the same is ignored and the strain energy is differentiated with respect to these as though they were independent. In order to implement this approach, the strain energy needs to be derived anew treating σ_{xy} and σ_{yx} as independent quantities. This simply means that Fig. 8.2.10 is re-drawn as Fig. 8.4.1 below, and Eqn. 8.2.16 is re-expressed using $\varepsilon_{xy} = (\varepsilon_{xy} + \varepsilon_{yx})/2$ as

$$dW = (\sigma_{yx} dxdz)(d\lambda dy) + (\sigma_{xy} dydz)(d\theta dx) = (dxdydz)\lfloor \sigma_{xy} d\varepsilon_{xy} + \sigma_{yx} d\varepsilon_{yx} \rfloor \qquad (8.4.6)$$

so that

$$u = \frac{1}{2}\sigma_{xy}\varepsilon_{xy} + \frac{1}{2}\sigma_{yx}\varepsilon_{yx} \qquad (8.4.7)$$

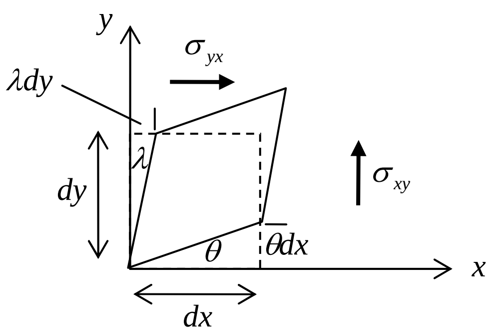

Figure 8.4.1: a volume element under shear stress

Re-writing Eqn. 8.4.1 and differentiation then leads to {▲Problem 2}

$$\begin{gathered} \frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz}, \\ \frac{\partial u}{\partial \varepsilon_{xy}} = \sigma_{xy} \quad , \quad \frac{\partial u}{\partial \varepsilon_{yz}} = \sigma_{yz}, \quad \frac{\partial u}{\partial \varepsilon_{zx}} = \sigma_{zx}, \quad \frac{\partial u}{\partial \varepsilon_{yx}} = \sigma_{yx} \quad , \quad \frac{\partial u}{\partial \varepsilon_{zy}} = \sigma_{zy}, \quad \frac{\partial u}{\partial \varepsilon_{xz}} = \sigma_{xz} \end{gathered} \qquad (8.4.8)$$

These equations can be expressed in the succinct form

$$\boxed{\frac{\partial u}{\partial \varepsilon_{ij}} = \sigma_{ij}} \qquad (8.4.9)$$

8.4.2 The Elastic Strain Energy Potential

Eqns. 8.4.9 was derived for an isotropic linear elastic material. In fact these equations are valid very generally, for non-linear and not-necessarily isotropic materials. Generalising the above discussion, recall that the strain energy is the area beneath the stress-strain curve, Fig. 8.4.2, and

$$dW = du = \sigma d\varepsilon \tag{8.4.10}$$

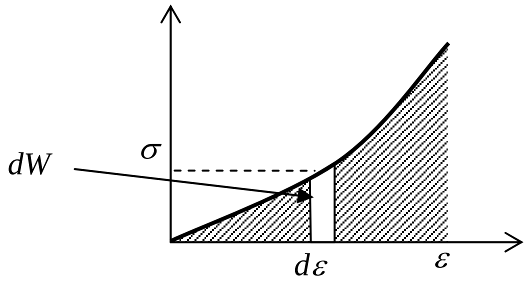

Figure 8.4.2: stress-strain curve for a non-linear material

When the material undergoes increments in strain $d\varepsilon_{xx}$, $d\varepsilon_{xy}$, etc., the increment in strain energy is

$$du = \sigma_{xx} d\varepsilon_{xx} + \sigma_{yy} d\varepsilon_{yy} + \sigma_{xy} d\varepsilon_{xy} + \cdots \tag{8.4.11}$$

If the strain energy is a function of the nine strains ε_{ij}, $u = u(\varepsilon_{ij})$, its increment can also be expressed as

$$du = \frac{\partial u}{\partial \varepsilon_{xx}} d\varepsilon_{xx} + \frac{\partial u}{\partial \varepsilon_{yy}} d\varepsilon_{yy} + \frac{\partial u}{\partial \varepsilon_{xy}} d\varepsilon_{xy} + \cdots \tag{8.4.12}$$

Subtracting Eqns 8.4.12 from 8.4.11 then gives

$$0 = \left(\sigma_{xx} - \frac{\partial u}{\partial \varepsilon_{xx}} \right) d\varepsilon_{xx} + \left(\sigma_{yy} - \frac{\partial u}{\partial \varepsilon_{yy}} \right) d\varepsilon_{yy} + \left(\sigma_{xy} - \frac{\partial u}{\partial \varepsilon_{xy}} \right) d\varepsilon_{xy} + \cdots \tag{8.4.13}$$

Because the strains are independent, that is, any one of them can be adjusted without changing the others, one again arrives at Eqns. 8.4.8-9, only now it has been shown that this result holds very generally.

Note that in the case of an incompressible material, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0$, so that the strains are *not* independent, and Eqns. 8.4.8-9 must be amended.

8.4.3 Symmetry of the Elastic Stiffness Matrix

Consider again the generalised Hooke's Law, Eqn. 6.3.1. Using Eqn. 8.4.9:

$$\frac{\partial}{\partial \varepsilon_{xy}} \left(\frac{\partial u}{\partial \varepsilon_{xx}} \right) = \frac{\partial}{\partial \varepsilon_{xy}} (\sigma_{xx}) = C_{16}, \qquad \frac{\partial}{\partial \varepsilon_{xx}} \left(\frac{\partial u}{\partial \varepsilon_{xy}} \right) = \frac{\partial}{\partial \varepsilon_{xx}} (\sigma_{xy}) = C_{61} \tag{8.4.14}$$

Since the order of partial differentiation for these second partial derivatives should be immaterial, it follows that $C_{16} = C_{61}$. Following the same procedure for the rest of the stresses and strains, it can be seen that the stiffness matrix in Eqn. 6.3.1 is *symmetric* and so there are only 21 independent elastic constants in the most general case of anisotropic elasticity.

8.4.4 The Complementary Energy Potential

Analogous to Eqns. 8.4.10-13, an increment in complementary energy density can be expressed as

$$dc = \varepsilon d\sigma \tag{8.4.15}$$

with

$$dc = \varepsilon_{xx} d\sigma_{xx} + \varepsilon_{yy} d\sigma_{yy} + \varepsilon_{xy} d\sigma_{xy} + \cdots \tag{8.4.16}$$

and

$$dc = \frac{\partial c}{\partial \sigma_{xx}} d\sigma_{xx} + \frac{\partial c}{\partial \sigma_{yy}} d\sigma_{yy} + \frac{\partial c}{\partial \sigma_{xy}} d\sigma_{xy} + \cdots \tag{8.4.17}$$

so that

$$\left(\varepsilon_{xx} - \frac{\partial u}{\partial \sigma_{xx}} \right) d\sigma_{xx} + \left(\varepsilon_{yy} - \frac{\partial u}{\partial \sigma_{yy}} \right) d\sigma_{yy} + \left(\varepsilon_{xy} - \frac{\partial u}{\partial \sigma_{xy}} \right) d\sigma_{xy} + \cdots \tag{8.4.18}$$

With the stresses independent, one has an expression analogous to 8.4.9,

$$\boxed{\frac{\partial c}{\partial \sigma_{ij}} = \varepsilon_{ij}} \tag{8.4.19}$$

8.4.5 Problems

1. Derive equations 8.4.5

2. Derive equations 8.4.8