7 Applications of Elasticity

The linear elastic model was introduced in the previous chapter and some elementary problems involving elastic materials were solved there (in particular in section 6.2). In this Chapter, five important, practical, theories are presented concerning elastic materials; they all have specific geometries and are subjected to particular types of load. In §7.1, the geometry is that of a long slender bar and the load is one which acts along the length of the bar; in §7.2, the geometry is that of a long slender circular bar and the load is one which twists the bar; in §7.3 the geometry is that of a thin-walled cylindrical or spherical component, and the load is normal to these walls; in §7.4 the geometry is that of a long and slender beam, and the load is transverse to the beam length. Finally, in §7.5, the geometry is a column, fixed at one end and loaded at the other so that it deflects. These five particular situations allow for simplifications (or approximations) to be made to the full three-dimensional linear elastic stress-strain relations; this allows one to write down simple expressions for the stress and strain and so solve some important practical problems analytically.

7.1 One Dimensional Axial Deformations

In this section, a specific simple geometry is considered, that of a long and thin straight component loaded in such a way that it deforms in the axial direction only. The *x*-axis is taken as the longitudinal axis, with the cross-section lying in the $x-y$ plane, Fig. 7.1.1.

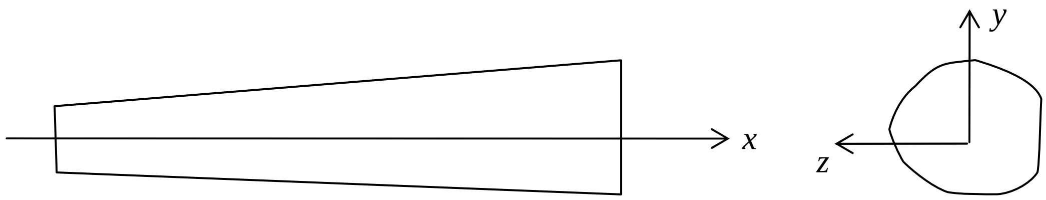


Figure 7.1.1: A slender straight component; (a) longitudinal axis, (b) cross-section

7.1.1 Basic relations for Axial Deformations

Any static analysis of a structural component involves the following three considerations:
(1) constitutive response
(2) kinematics
(3) equilibrium

In this Chapter, it is taken for (1) that the material responds as an isotropic linear elastic solid. It is assumed that the only significant stresses and strains occur in the axial direction, and so the stress-strain relations 6.1.8-9 reduce to the one-dimensional equation $\sigma_{xx} = E\varepsilon_{xx}$ or, dropping the subscripts,

$$\sigma = E\varepsilon \tag{7.1.1}$$

Kinematics (2), the study of deformation, was the subject of Chapter 4. In the theory developed here, known as **axial deformation**, it is assumed that the axis of the component remains straight and that cross-sections that are initially perpendicular to the axis remain perpendicular after deformation. This implies that, although the strain might vary along the axis, it remains *constant over any cross section*. The axial strain occurring over any section is defined by Eqn. 4.1.2,

$$\varepsilon = \frac{L - L_0}{L_0} \tag{7.1.2}$$

This is illustrated in Fig. 7.1.2, which shows a (shaded) region undergoing a compressive (negative) strain.

Recall that individual particles/points undergo displacements whereas regions/line-elements undergo strain. In Fig. 7.1.2, the particle originally at *A* has undergone a displacement $u(A)$ whereas the particle originally at *B* has undergone a displacement $u(B)$. From Fig. 7.1.2, another way of expressing the strain in the shaded region is (see Eqn. 4.1.3)

$$\varepsilon = \frac{u(B) - u(A)}{L_0} \tag{7.1.3}$$

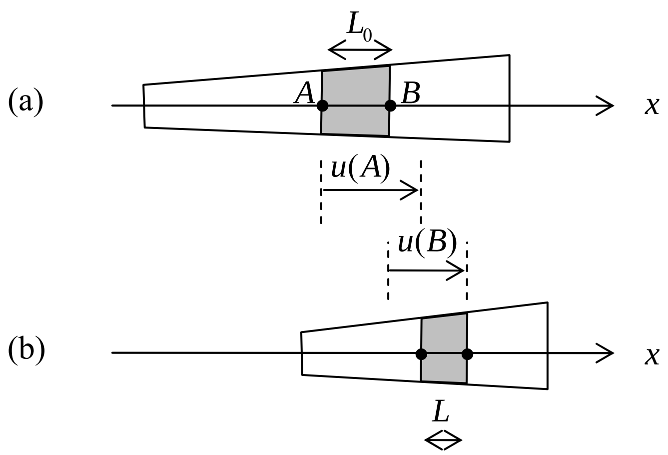


Figure 7.1.2: axial strain; (a) before deformation, (b) after deformation

Both displacements $u(A)$ and $u(B)$ of Fig. 7.1.2 are *positive*, since the particles displace in the positive x direction – if they moved to the left, for consistency, one would say they underwent *negative* displacements. Further, positive stresses are as shown in Fig. 7.1.3a and negative stresses are as shown in Fig. 7.1.3b. From Eqn. 7.1.1, a positive stress implies a positive strain (lengthening) and a compressive stress implies a negative strain (contracting)

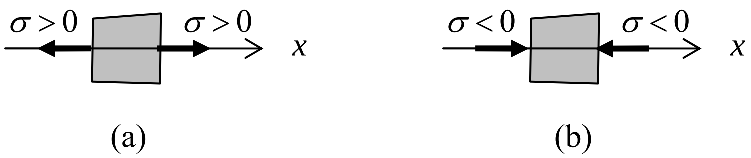


Figure 7.1.3: Stresses arising in the slender component; (a) positive (tensile) stress, (b) negative (compressive) stress

Equilibrium, (3), will be considered in the individual examples below.

Note that, in the previous Chapter, problems were solved using only the stress-strain law (1). Kinematics (2) and equilibrium (3) were not considered, the reason being the problems were so simple, with uniform (homogeneous) stress and strain (as indeed also in the first example which follows). Whenever more complex problems are encountered, with non-uniform stress and strains, (3) and perhaps (2) need to be considered to solve for the stress and strain.

7.1.2 Structures with Uniform Members

A uniform axial member is one with cross-section A and modulus E constant along its length, and loaded with axial forces at its ends only.

Example

Consider the bar of initial length L shown in Fig. 7.1.4, subjected to equal and opposite end-forces F. The free-body (equilibrium) diagram of a section of the bar shown in Fig. 7.1.4b shows that the internal force is also F everywhere along the bar. The stress is thus everywhere $\sigma = F / A$ and the strain is everywhere

$$\varepsilon = \frac{F}{EA} \tag{7.1.4}$$

and, from Eq. 7.1.2, the bar extends in length by an amount

$$\Delta = \frac{FL}{EA} \tag{7.1.5}$$

Note that, although the force acting on the left-hand end is negative (acting in the $-x$ direction), the stress there is positive (see Fig. 7.1.3).

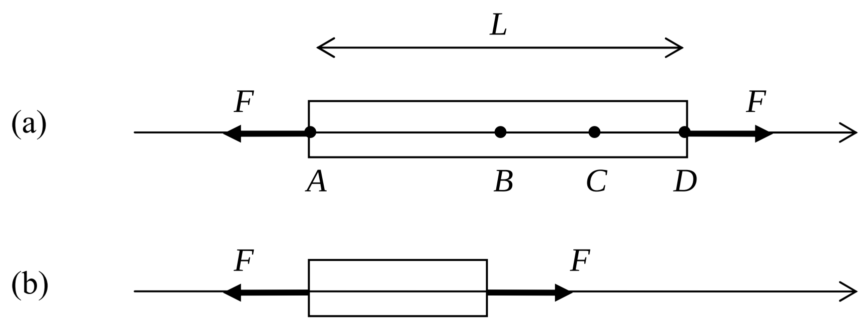


Figure 7.1.4: A uniform axial member; (a) subjected to axial forces *F*, (b) free-body diagram

Displacements need to be calculated relative to some datum displacement[1]. For example, suppose that the displacement at the centre of the bar is zero, $u(B) = 0$, Fig. 7.1.4. Then, from Eqn. 7.1.3,

$$\begin{aligned} u(C) &= u(B) + \varepsilon(C - B) = \frac{F}{EA}\frac{L}{4} \\ u(D) &= u(B) + \varepsilon(D - B) = \frac{F}{EA}\frac{L}{2} \\ u(A) &= u(B) + \varepsilon(A - B) = -\frac{F}{EA}\frac{L}{2} \end{aligned} \tag{7.1.6}$$

■

[1] which is another way of saying that one can translate the bar left or right as a rigid body without affecting the stress or strain – but it does affect the displacements

Example

Consider the two-element structure shown in Fig. 7.1.5. The first element is built-in to a wall at end *A*, is of length L_1, cross-sectional area A_1 and Young's modulus E_1. The second element is attached at *B* and has properties L_2, A_2, E_2. External loads F and *P* are applied at *B* and *C* as shown. An unknown reaction force *R* acts at the wall, at *A*. This can be determined from the force equilibrium equation for the complete structure:

$$R - F + P = 0 \tag{7.1.7}$$

Note that, as is usual, the reaction is assumed to act in the positive (*x*) direction. With *R* known, the stress $\sigma^{(1)}$ in the first element can be evaluated using the free-body diagram 7.1.5b, and $\sigma^{(2)}$ using Fig. 7.1.5c:

$$\sigma^{(1)} = \frac{P-F}{A_1}, \qquad \sigma^{(2)} = \frac{P}{A_2} \tag{7.1.8}$$

and so the strain is

$$\varepsilon^{(1)} = \frac{P-F}{E_1 A_1}, \qquad \varepsilon^{(2)} = \frac{P}{E_2 A_2} \tag{7.1.9}$$

Note that the stress and strain are *discontinuous* at *B* [2].

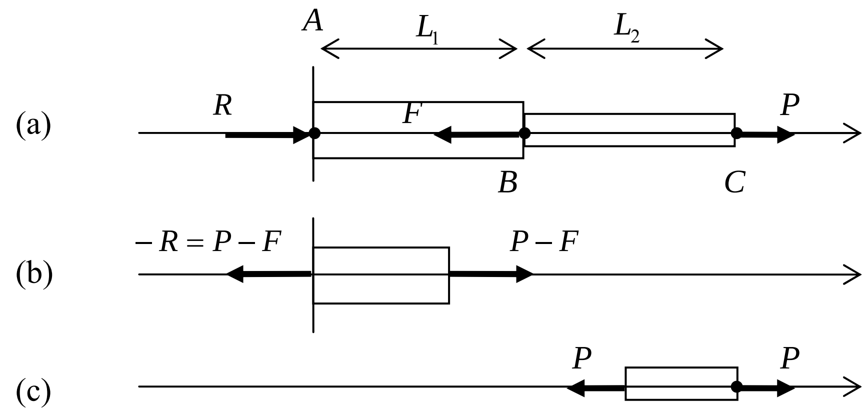


Figure 7.1.5: A two-element structure (a) subjected to axial forces *F* and *P*, (b,c) free-body diagrams

For each element, the total elongations Δ_i are

[2] this result, which can be viewed as a violation of equilibrium at *B*, is a result of the one-dimensional approximation of what is really a two-dimensional problem

$$\Delta_1 = u(B) - u(A) = \frac{(P-F)L_1}{E_1 A_1}$$
$$\Delta_2 = u(C) - u(B) = \frac{PL_2}{E_2 A_2} \qquad (7.1.10)$$

If $P > F$, then $\Delta_1 > 0$ as expected, with $R < 0$ and $\sigma > 0$.

Thus far, the stress and strain (and elongations) have been obtained. If one wants to evaluate the displacements, then one needs to ensure that the strains in each of the two elements are **compatible**, that is, that the elements fit together after deformation just like they did before deformation. In this example, the displacements at *B* and *C* are

$$u(B) = u(A) + \Delta_1, \quad u(C) = u(B) + \Delta_2 \qquad (7.1.11)$$

A **compatibility condition**, bringing together the separate relations in 7.1.11, is then

$$u(C) = u(A) + \frac{(P-F)L_1}{E_1 A_1} + \frac{PL_2}{E_2 A_2} \qquad (7.1.12)$$

ensuring that $u(B)$ is unique. As in the previous example, the displacements can now be calculated if the displacement at any one (datum) point is known. Indeed, it is known that $u(A) = 0$.

■

Example

Consider next the similar situation shown in Fig. 7.1.6. Here, both ends of the two-element structure are built-in and there is only one applied force, *F*, at *B*. There are now two reaction forces, at ends *A* and *C*, but there is only one equilibrium equation to determine them:

$$R_A + F + R_C = 0 \qquad (7.1.13)$$

Any structure for which there are more unknowns than equations of equilibrium, so that the stresses cannot be determined without considering the deformation of the structure, is called a **statically indeterminate** structure[3].

[3] See the end of §2.3.3

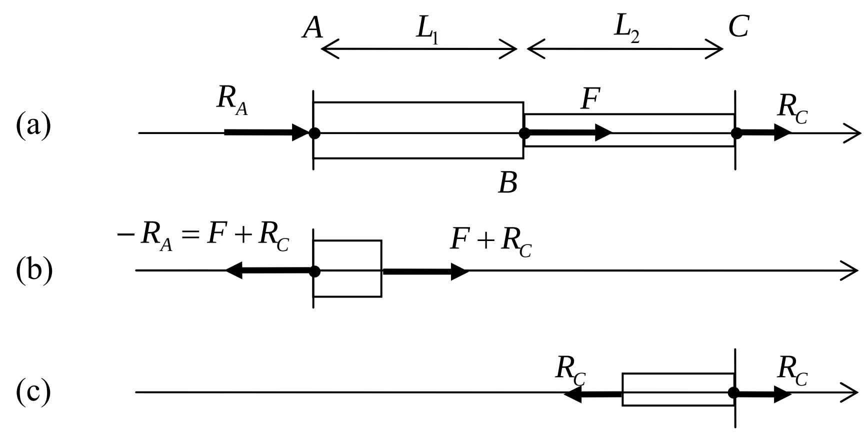


Figure 7.1.6: A two-element structure built-in and both ends; (a) subjected to an axial force *F*, (b,c) free-body diagrams

In terms of the unknown reactions, the strains are

$$\varepsilon^{(1)} = \frac{\sigma^{(1)}}{E_1} = -\frac{R_A}{E_1 A_1} = \frac{F + R_C}{E_1 A_1}, \qquad \varepsilon^{(2)} = \frac{\sigma^{(2)}}{E_2} = \frac{R_C}{E_2 A_2} \tag{7.1.14}$$

and, for each element, the total elongations are

$$\Delta_1 = \frac{R_A L_1}{E_1 A_1}, \qquad \Delta_2 = \frac{R_C L_2}{E_2 A_2} \tag{7.1.15}$$

Finally, compatibility of both elements implies that the total elongation $\Delta_1 + \Delta_2 = 0$. Using this relation with Eqn. 7.1.13-14 then gives

$$R_A = +F \frac{L_2 E_1 A_1}{L_1 E_2 A_2 + L_2 E_1 A_1}, \qquad R_C = -F \frac{L_1 E_2 A_2}{L_1 E_2 A_2 + L_2 E_1 A_1} \tag{7.1.16}$$

The displacements can now be evaluated, for example,

$$u(B) = +F \frac{1}{E_1 A_1 / L_1 + E_2 A_2 / L_2} \tag{7.1.17}$$

so that a positive *F* displaces *B* to the right and a negative *F* displaces *B* to the left.

■

Note the general solution procedure in this last example, known as the **basic force method**:

Equilibrium + Compatibility of Strain in terms of unknown Forces
→ Solve equations for unknown Forces

The Stiffness Method

The **stiffness method** (also known as the **displacement method**) is a slight modification of the above solution procedure, where the final equations to be solved involve known forces and unknown displacements only:

Equilibrium in terms of Displacement
→ Solve equations for unknown Displacements

If one deals in displacements, one does not need to ensure compatibility (it will automatically be satisfied); compatibility only needs to be considered when dealing in strains (as in the previous example)[4].

Example (The Stiffness Method)

Consider a series of three bars of cross-sectional areas A_1, A_2, A_3, Young's moduli E_1, E_2, E_3 and lengths L_1, L_2, L_3, Fig. 7.1.7. The first and third bars are built-in at points *A* and *D*, bars one and two meet at *B* and bars two and three meet at *C*. Forces P_B and P_C act at *B* and *C* respectively.

The force is constant in each bar, and for each bar there is a relation between the force F_i, and elongation, Δ_i, Eqn. 7.1.5:

$$F_i = k_i \Delta_i \quad \text{where} \quad k_i = \frac{A_i E_i}{L_i} \tag{7.1.18}$$

Here, k_i is the effective **stiffness** of each bar. The elongations are related to the displacements, $\Delta_1 = u_B - u_A$ etc., so that, with $u_A = u_D = 0$,

$$F_1 = k_1 u_B, \qquad F_2 = k_2 (u_C - u_B), \qquad F_3 = -k_3 u_C \tag{7.1.19}$$

There are two **degrees of freedom** in this problem, that is, two nodes are free to move. One therefore needs two equilibrium equations. One could use any two of

$$-F_1 + P_B + P_C + F_3 = 0, \qquad -F_1 + P_B + F_2 = 0, \qquad -F_2 + P_C + F_3 = 0 \tag{7.1.20}$$

In the stiffness method, one uses the second and third of these; the second is the "node *B*" equation and the third is the "node *C*" equation. Substituting Eqns. 7.1.19 into 7.1.20 leads to the system of two equations

$$\begin{aligned} -(k_1 + k_2)u_B + k_2 u_C &= -P_B \\ +k_2 u_B - (k_2 + k_3)u_C &= -P_C \end{aligned} \tag{7.1.21}$$

[4] the reason is: if you know the displacements, you know where every particle is and you know the strains and everything else; if you *only know the strains*, you know the *change in displacement*, but you do not know the actual displacements. You need some extra information to know the displacements – this is the compatibility equation

which can be solved for the two unknown nodal displacements.

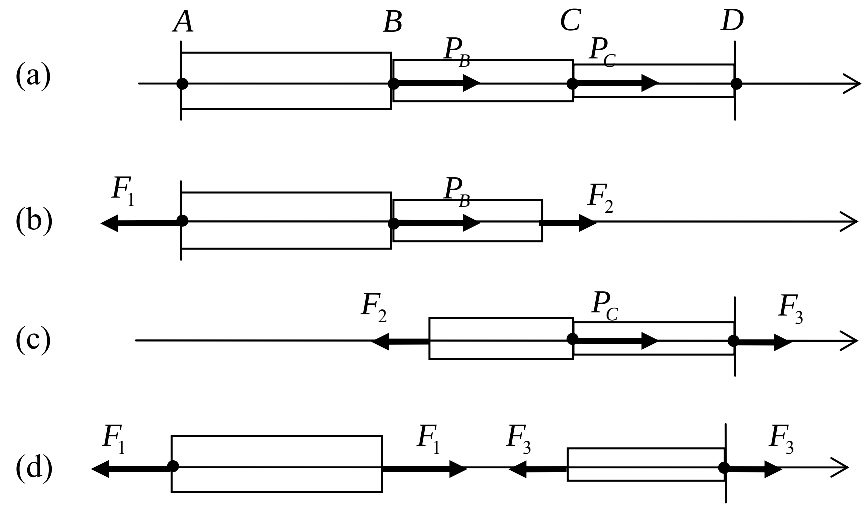


Figure 7.1.7: three bars in series; (a) subjected to external loads, (b,c,d) free-body diagrams

Equations 7.1.21 can also be written in the matrix form

$$\begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{bmatrix} u_B \\ u_C \end{bmatrix} = \begin{bmatrix} -P_B \\ -P_C \end{bmatrix} \tag{7.1.22}$$

Note that it was not necessary to evaluate the reactions to obtain a solution. Once the forces have been found, the reactions can be found using the free-body diagram of Fig. 7.1.7d.

The stiffness method is a very systematic procedure. It can be used to solve for structures with many elements, with the two equations 7.1.21, 7.1.22, replaced by a large system of equations which can be solved numerically using a computer.

7.1.3 Structures with Non-uniform Members

Consider the structure shown in Fig. 7.1.8, an axial bar consisting of two separate components bonded together. The components have Young's moduli E_1, E_2 and cross-sectional areas A_1, A_2. The bar is subjected to equal and opposite forces F as shown, in such a way that axial deformations occur, that is, the cross-sections remain perpendicular to the x axis throughout the deformation.

Since there are only axial deformations, the strain is constant over a cross-section. However, the stress is not uniform, with $\sigma_1 = E_1\varepsilon$ and $\sigma_2 = E_2\varepsilon$; on any cross-section, the stress is higher in the stiffer component. The resultant force acting on each component is $F_1 = E_1 A_1 \varepsilon$ and $F_2 = E_2 A_2 \varepsilon$. Since $F_1 + F_2 = F$, the total elongation is

$$\Delta = \frac{FL}{E_1 A_1 + E_2 A_2} \tag{7.1.23}$$

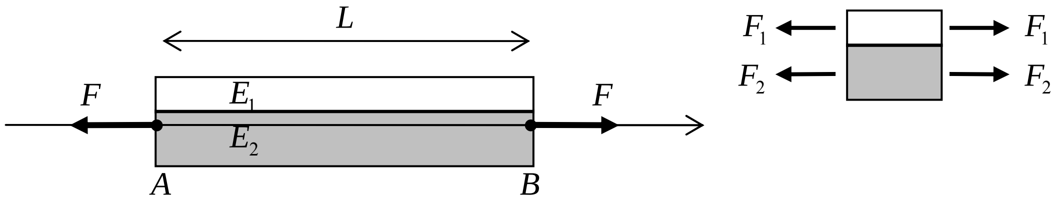


Figure 7.1.8: A bar consisting of two separate materials bonded together

7.1.4 Resultant Force and Moment

Consider the force and moments acting over any cross-section, Fig. 7.1.9. The resultant force is the integral of the stress times elemental area over the cross section, Eqn. 3.1.2,

$$F = \int_A \sigma dA \tag{7.1.24}$$

There are two moments; the moment M_y about the *y* axis and M_z about the *z* axis,

$$M_y = \int_A z\sigma dA, \quad M_z = -\int_A y\sigma dA \tag{7.1.25}$$

Positive moments are defined through the **right hand rule**, i.e. with the thumb of the right hand pointing in the positive *y* direction, the closing of the fingers indicates the positive M_y; the negative sign in Eqn. 7.1.25b is due to the fact that a positive stress with $y > 0$ would lead to a negative moment M_z.

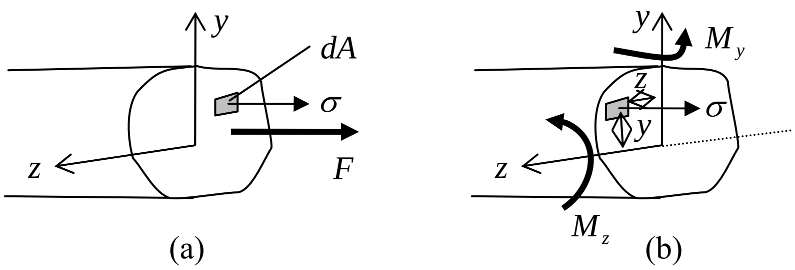


Figure 7.1.9: Resultants on a cross-section; (a) resultant force, (b) resultant moments

Consider now the case where *the stress is constant over a cross-section*. (Since it is assumed that the strain is constant over the cross-section, from Eqn. 7.1.1 this will occur when the Young's modulus is constant.) In that case, Eqns. 7.1.24-25 can be re-written as

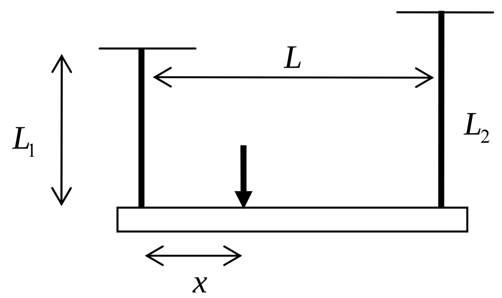
$$F = \sigma A, \quad M_y = \sigma \int_A z dA, \quad M_z = -\sigma \int_A y dA \tag{7.1.25}$$

The quantities $\int_A z dA$ and $\int_A y dA$ are the **first moments of area** about, respectively, the *y* and *z* axes. These are equal to $\bar{z}A$ and $\bar{y}A$, where $(\bar{y}, \bar{z})$ are the coordinates of the centroid of the section (see Eqn. 3.2.2). Taking the *x* axis to run through the centroid, $\bar{y} = \bar{z} = 0$, which results in $M_y = M_z = 0$. Thus, a resultant axial force which acts through the centroid of the cross-section ensures that there is no moment/rotation of that cross-section, the main assumption of this section.

For the non-uniform member of Fig. 7.1.8, since the resultant of a constant stress over an area is a force acting through the centroid of that area, the forces F_1, F_2 act through the centroids of the respective areas A_1, A_2. The precise location of the total resultant force *F* can be determined by taking the moments of the forces F_1, F_2 about the *y* and *z* axes, and equating this to the moment of the force *F* about these axes.

7.1.5 Problems

1. Consider the *rigid* beam supported by two deformable bars shown below. The bars have properties L_1, A_1 and L_2, A_2 and have the same Young's modulus *E*. They are separated by a distance *L*. The beam supports an arbitrary load at position *x*, as shown. What is *x* if the beam is to remain horizontal after deformation.



7.2 Torsion

In this section, the geometry to be considered is that of a long slender circular bar and the load is one which twists the bar. Such problems are important in the analysis of twisting components, for example lug wrenches and transmission shafts.

7.2.1 Basic relations for Torsion of Circular Members

The theory of torsion presented here concerns **torques**[1] which twist the members but which *do not induce any warping*, that is, cross sections which are perpendicular to the axis of the member remain so after twisting. Further, radial lines remain straight and radial as the cross-section rotates – they merely rotate with the section.

For example, consider the member shown in Fig. 7.2.1, built-in at one end and subject to a torque T at the other. The x axis is drawn along its axis. The torque shown is positive, following the right-hand rule (see §7.1.4). The member twists under the action of the torque and the radial plane $ABCD$ moves to $ABC'D$.

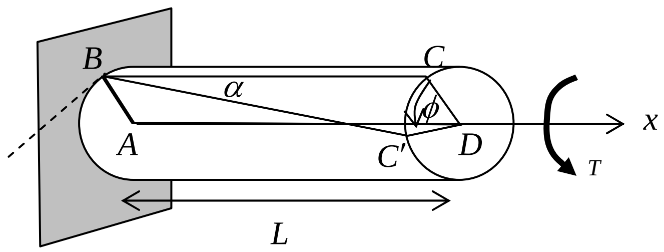


Figure 7.2.1: A cylindrical member under the action of a torque

Whereas in the last section the measure of deformation was elongation of the axial members, here an appropriate measure is the amount by which the member twists, the rotation angle ϕ. The rotation angle will vary along the member – the sign convention is that ϕ is positive in the same direction as positive T as indicated by the arrow in Fig. 7.2.1. Further, whereas the measure of strain used in the previous section was the normal strain ε_{xx}, here it will be the engineering shear strain γ_{xy} (twice the tensorial shear strain ε_{xy}). A relationship between γ (dropping the subscripts) and ϕ will next be established.

As the line BC deforms into BC', Fig. 7.2.1, it undergoes an angle change α. As defined in §4.1.2, the shear strain γ is the change in the original right angle formed by BC and a tangent at B (indicated by the dotted line – this is the y axis to be used in γ_{xy}). If α is small, then

$$\gamma \approx \alpha \approx \tan\alpha \approx \frac{CC'}{BC} \approx \frac{R\varphi(L)}{L} \tag{7.2.1}$$

[1] the term torque is usually used instead of moment in the context of twisting shafts such as those considered in this section

where L is the length, R the radius of the member and $\phi(L)$ means the magnitude of ϕ at L. Note that the strain is constant along the length of the member although ϕ is not. Considering a general cross-section within the member, as in Fig. 7.2.2, one has

$$\gamma \approx \alpha \approx \frac{R\varphi(x)}{x} \tag{7.2.2}$$

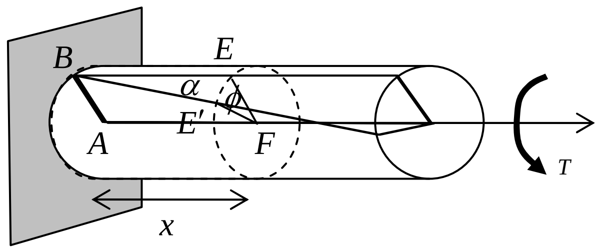


Figure 7.2.2: A section of a twisting cylindrical member

The shear strain at an arbitrary radial location r, $0 < r < R$, is

$$\gamma(r) = \frac{r\phi(x)}{x} \tag{7.2.3}$$

showing that the shear strain varies from zero at the centre of the shaft to a maximum $R\phi(L)/L \;\left(= R\phi(x)/x\right)$ on the outer surface of the shaft.

The only strain is this shear strain and so the only stress which will arise is a shear stress τ. From Hooke's Law

$$\tau = G\gamma \tag{7.2.4}$$

where G is the shear modulus (the μ of Eqn. 6.1.5). Following the shear strain, the shear stress is zero at the centre of the shaft and a maximum on the outer surface.

Considering a free-body diagram of any portion of the shaft of Fig. 7.2.1, a torque T acts on all cross-sections. This torque must equal the resultant of the shear stresses acting over the section, as schematically illustrated in Fig. 7.2.3a.

The elemental force acting over an element of area dA is τdA and so the resultant moment about $r = 0$ is

$$T = \int_{dA} r\tau(r)dr \tag{7.2.5}$$

But γ / r is a constant and so therefore also is τ / r (provided G is) and Eqn. 7.2.5 can be re-written as

$$T = \frac{\tau(r)}{r}\left[\int_A r^2 dA\right] = \frac{\tau(r)J}{r} \tag{7.2.6}$$

The quantity in square brackets is called the **polar moment of inertia** of the cross-section (also called the **polar second moment of area**) and is denoted by *J*:

$$\boxed{J = \int_A r^2 dA} \qquad \textbf{Polar Moment of Area} \tag{7.2.7}$$

where dA is an element of area and the integration is over the complete cross-section.

For the circular cross-section under consideration, the area element has sides dr and $rd\theta$, Fig. 7.2.3c, so

$$J = \int_0^{2\pi}\int_0^R r^3 dr d\theta = 2\pi \int_0^R r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \tag{7.2.8}$$

where *D* is the diameter.

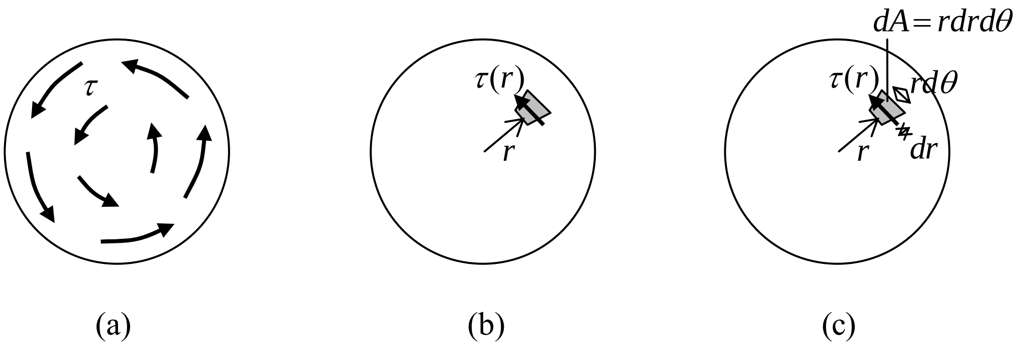


Figure 7.2.3: Shear stresses acting over a cross-section; (a) shear stress, (b,c) moment for an elemental area

From Eqn. 7.2.6, the shear stress at any radial location is given by

$$\boxed{\tau(r) = \frac{rT}{J}} \tag{7.2.9}$$

From Eqn. 7.2.1, 7.2.4, 7.26 and 7.2.9, the angle of twist at the end of the member – or the twist at one end relative to that at the other end – is

$$\boxed{\phi = \frac{TL}{GJ}} \tag{7.2.10}$$

Example

Consider the problem shown in Fig.7.2.4, two torsion members of lengths L_1, L_2, diameters d_1, d_2 and shear moduli G_1, G_2, built-in at *A* and subjected to torques T_B and T_C. Equilibrium of moments can be used to determine the unknown torques acting in each member:

$$-T_1 + T_B + T_C = 0, \qquad -T_2 + T_C = 0 \tag{7.2.11}$$

so that $T_1 = T_B + T_C$ and $T_2 = T_C$.

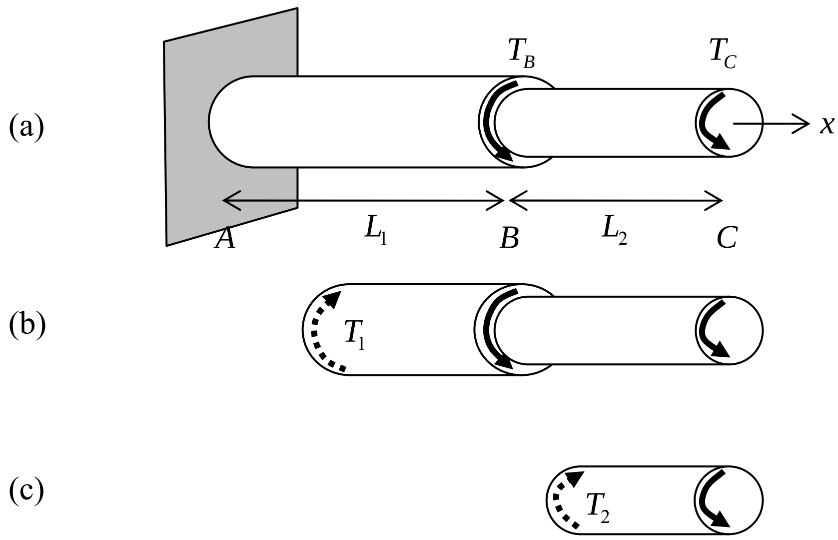


Figure 7.2.4: A structure consisting of two torsion members; (a) subjected to torques T_B and T_C, (b,c) free-body diagrams

The shear stresses in each member are therefore

$$\tau_1 = \frac{r(T_B + T_C)}{J_1}, \qquad \tau_2 = \frac{rT_C}{J_2} \tag{7.2.12}$$

where $J_1 = \pi d_1^4 / 32$ and $J_2 = \pi d_2^4 / 32$.

From Eqn. 7.2.10, the angle of twist at *B* is given by $\phi_B = T_1 L_1 / G_1 J_1$. The angle of twist at *C* is then

$$\phi_C = \frac{T_2 L_2}{G_2 J_2} - \phi_B \tag{7.2.13}$$

■

Statically indeterminate problems can be solved using methods analogous to those used in the section 7.1 for uniaxial members.

Example

Consider the structure in Fig. 7.2.5, similar to that in Fig. 7.2.4 only now both ends are built-in and there is only a single applied torque, T_B.

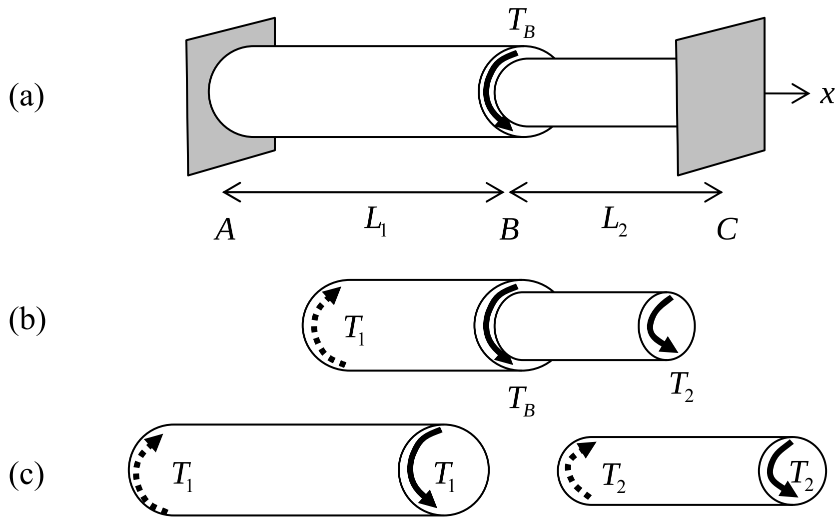


Figure 7.2.5: A structure consisting of two torsion members; (a) subjected to a Torque T_B, (b) free-body diagram, (c) separate elements

Referring to the free-body diagram of Fig. 7.2.5b, there is only one equation of equilibrium with which to determine the two unknown member torques:

$$-T_1 + T_B + T_2 = 0 \tag{7.2.14}$$

and so the deformation of the structure needs to be considered. A systematic way of dealing with this situation is to consider each element separately, as in Fig. 7.2.5c. The twist in each element is

$$\phi_1 = \frac{T_1 L_1}{G_1 J_1}, \quad \phi_2 = \frac{T_2 L_2}{G_2 J_2} \tag{7.2.15}$$

The total twist is zero and so $\phi_1 + \phi_2 = 0$ which, with Eqn. 7.2.14, can be solved to obtain

$$T_1 = +\frac{L_2 G_1 J_1}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B, \quad T_2 = -\frac{L_1 G_2 J_2}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B \tag{7.2.16}$$

The rotation at *B* can now be determined, $\phi_B = \phi_1 = -\phi_2$.

■

7.2.2 Stress Distribution in Torsion Members

The shear stress in Eqn. 7.2.9 is acting over a cross-section of a torsion member. From the symmetry of the stress, it follows that shear stresses act also along the length of the member, as illustrated to the left of Fig. 7.2.6. Shear stresses do not act *on* the surface of the element shown, as it is a free surface.

Any element of material not aligned with the axis of the cylinder will undergo a complex stress state, as shown to the right of Fig. 7.2.6. The stresses acting on an element are given by the stress transformation equations, Eqns. 3.4.9:

$$\sigma'_{xx} = +\sin 2\theta\tau, \qquad \sigma'_{yy} = -\sin 2\theta\tau, \qquad \sigma'_{xy} = +\cos 2\theta\tau \qquad (7.2.17)$$

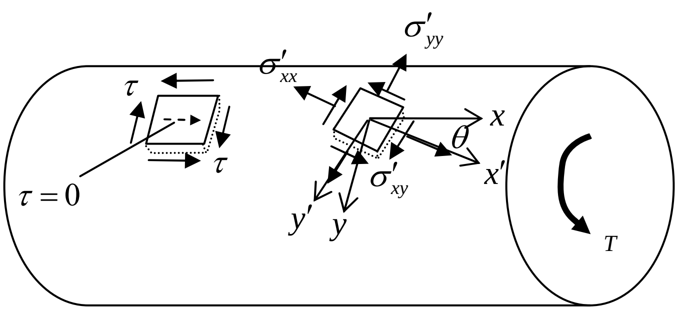
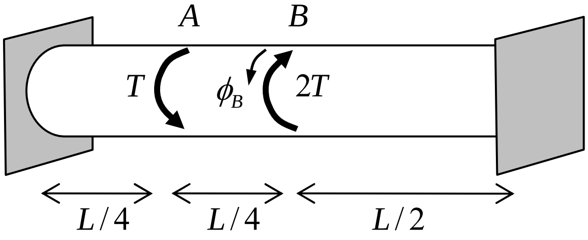


Figure 7.2.6: Stress distribution in a torsion member

From Eqns. 3.5.4-5, the maximum normal (principal) stresses arise on planes at $\theta = \pm 45^o$ and are $\sigma_1 = +\tau$ and $\sigma_2 = -\tau$. Thus the maximum tensile stress in the member occurs at 45^o to the axis and arises at the surface. The maximum shear stress is simply τ, with $\theta = 0$.

7.2.3 Problems

1. A shaft of length *L* and built-in at both ends is subjected to two external torques, T at *A* and $2T$ at *B*, as shown below. The shaft is of diameter *d* and shear modulus *G*. Determine the maximum (absolute value of) shear stress in the shaft and determine the angle of twist at *B*.



7.3 The Thin-walled Pressure Vessel Theory

An important practical problem is that of a cylindrical or spherical object which is subjected to an internal pressure p. Such a component is called a **pressure vessel**, Fig. 7.3.1. Applications arise in many areas, for example, the study of cellular organisms, arteries, aerosol cans, scuba-diving tanks and right up to large-scale industrial containers of liquids and gases.

In many applications it is valid to assume that

(i) the material is isotropic
(ii) the strains resulting from the pressures are small
(iii) the wall thickness t of the pressure vessel is much smaller than some characteristic radius: $t = r_o - r_i << r_o, r_i$

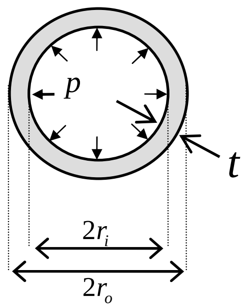


Figure 7.3.1: A pressure vessel (cross-sectional view)

Because of (i,ii), the isotropic linear elastic model is used. Because of (iii), it will be assumed that there is negligible variation in the stress field across the thickness of the vessel, Fig. 7.3.2.

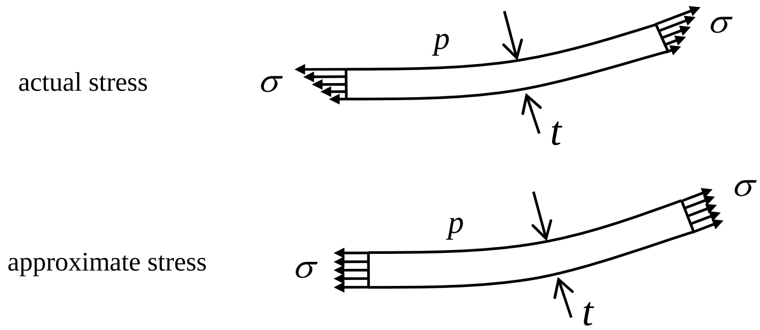


Figure 7.3.2: Approximation to the stress arising in a pressure vessel

As a rule of thumb, if the thickness is less than a tenth of the vessel radius, then the actual stress will vary by less than about 5% through the thickness, and in these cases the constant stress assumption is valid.

Note that a pressure $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p_i$ means that the stress on *any* plane drawn inside the vessel is subjected to a normal stress $-p_i$ and zero shear stress (see problem 6 in section 3.5.7).

7.3.1 Thin Walled Spheres

A thin-walled spherical shell is shown in Fig. 7.3.3. Because of the symmetry of the sphere and of the pressure loading, the **circumferential** (or **tangential** or **hoop) stress** σ_t at any location and in any tangential orientation must be the same (and there will be zero shear stresses).

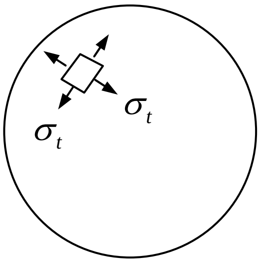


Figure 7.3.3: a thin-walled spherical pressure vessel

Considering a free-body diagram of one half of the sphere, Fig. 7.3.4, force equilibrium requires that

$$\pi\left(r_o^2 - r_i^2\right)\sigma_t - \pi r_i^2 p = 0 \tag{7.3.1}$$

and so, with $r_0 = r_i + t$,

$$\sigma_t = \frac{r_i^2 p}{2r_i t + t^2} \tag{7.3.2}$$

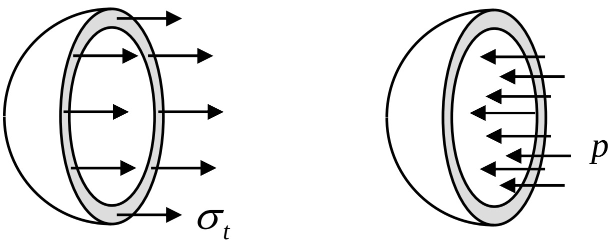


Figure 7.3.4: a free body diagram of one half of the spherical pressure vessel

One can now take as a characteristic radius the dimension *r*. This could be the inner radius, the outer radius, or the average of the two – results for all three should be close. Setting $r = r_i$ and neglecting the small terms $t^2 \ll 2r_i t$,

$$\boxed{\sigma_t = \frac{pr}{2t}} \quad \textbf{Tangential stress in a thin-walled spherical pressure vessel} \qquad (7.3.3)$$

This tangential stress accounts for the stress in the plane of the surface of the sphere. The stress normal to the walls of the sphere is called the **radial stress**, σ_r. The radial stress is zero on the outer wall since that is a free surface. On the inner wall, the normal stress is $\sigma_r = -p$, Fig. 7.3.5. From Eqn. 7.3.3, since $t/r << 1$, $p << \sigma_t$, and it is reasonable to take $\sigma_r = 0$ not only on the outer wall, but on the inner wall also. The stress state in the spherical wall is then one of plane stress.

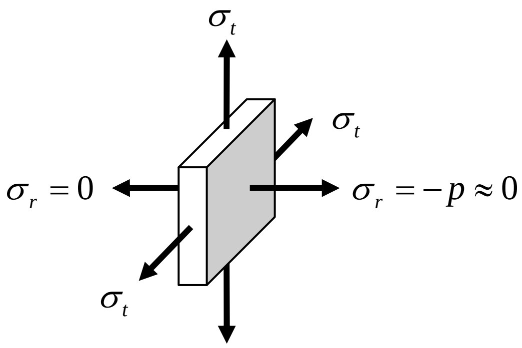


Figure 7.3.5: An element at the surface of a spherical pressure vessel

There are no in-plane shear stresses in the spherical pressure vessel and so the tangential and radial stresses are the principal stresses: $\sigma_1 = \sigma_2 = \sigma_t$, and the minimum principal stress is $\sigma_3 = \sigma_r = 0$. Thus the radial direction is one principal direction, and any two perpendicular directions in the plane of the sphere's wall can be taken as the other two principal directions.

Strain in the Thin-walled Sphere

The thin-walled pressure vessel expands when it is internally pressurised. This results in three principal strains, the **circumferential strain** ε_c (or **tangential strain** ε_t) in two perpendicular in-plane directions, and the **radial strain** ε_r. Referring to Fig. 7.3.6, these strains are

$$\varepsilon_c = \frac{A'C' - AC}{AC} = \frac{C'D' - CD}{CD}, \quad \varepsilon_r = \frac{A'B' - AB}{AB} \qquad (7.3.4)$$

From Hooke's law (Eqns. 6.1.8 with *z* the radial direction, with $\sigma_r = 0$),

$$\begin{bmatrix} \varepsilon_c \\ \varepsilon_c \\ \varepsilon_r \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E \\ -\nu/E & 1/E & -\nu/E \\ -\nu/E & -\nu/E & 1/E \end{bmatrix} \begin{bmatrix} \sigma_t \\ \sigma_t \\ \sigma_r \end{bmatrix} = \frac{1}{E}\frac{pr}{2t} \begin{bmatrix} 1-\nu \\ 1-\nu \\ -2\nu \end{bmatrix} \qquad (7.3.5)$$

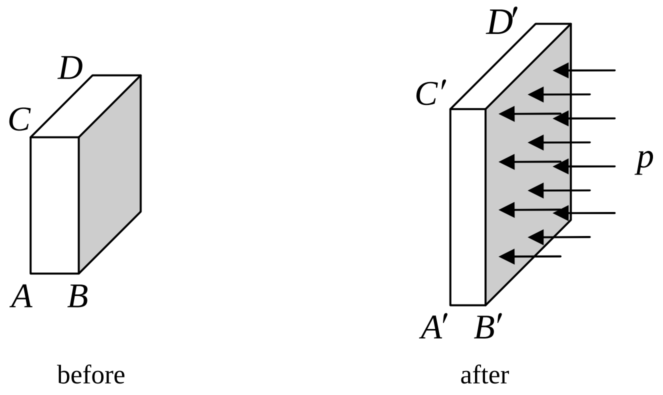


Figure 7.3.6: Strain of an element at the surface of a spherical pressure vessel

To determine the amount by which the vessel expands, consider a circumference at average radius *r* which moves out with a displacement δ_r, Fig. 7.3.7. From the definition of normal strain

$$\varepsilon_c = \frac{(r+\delta_r)\Delta\theta - r\Delta\theta}{r\Delta\theta} = \frac{\delta_r}{r} \tag{7.3.6}$$

This is the circumferential strain for points on the mid-radius. The strain at other points in the vessel can be approximated by this value.

The expansion of the sphere is thus

$$\delta_r = r\varepsilon_c = \frac{1-\nu}{E}\frac{pr^2}{2t} \tag{7.3.7}$$

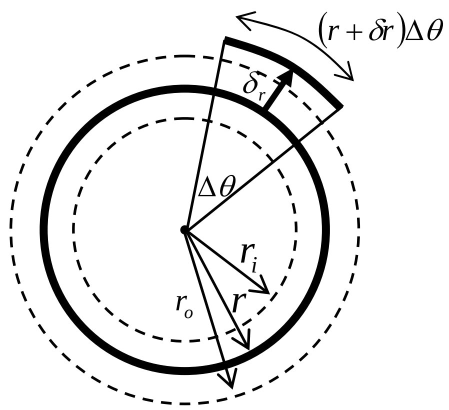


Figure 7.3.7: Deformation in the thin-walled sphere as it expands

To determine the amount by which the circumference increases in size, consider Fig. 7.3.8, which shows the original circumference at radius *r* of length *c* increase in size by an amount δ_c. One has

$$\delta_c = c\varepsilon_c = 2\pi r\varepsilon_c = 2\pi \frac{1-\nu}{E} \frac{pr^2}{2t} \tag{7.3.8}$$

It follows from Eqn. 7.3.7-8 that the circumference and radius increases are related through

$$\delta_c = 2\pi\delta_r \tag{7.3.9}$$

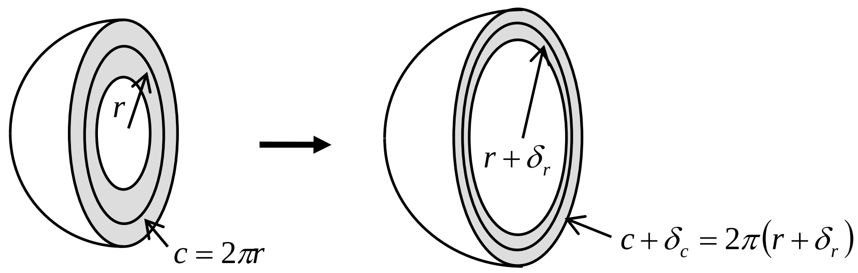


Figure 7.3.8: Increase in circumference length as the vessel expands

Note that the circumferential strain is *positive*, since the circumference is increasing in size, but the radial strain is *negative* since, as the vessel expands, the thickness decreases.

7.3.2 Thin Walled Cylinders

The analysis of a thin-walled internally-pressurised cylindrical vessel is similar to that of the spherical vessel. The main difference is that the cylinder has three different principal stress values, the circumferential stress, the radial stress, and the **longitudinal stress** σ_l, which acts in the direction of the cylinder axis, Fig. 7.3.9.

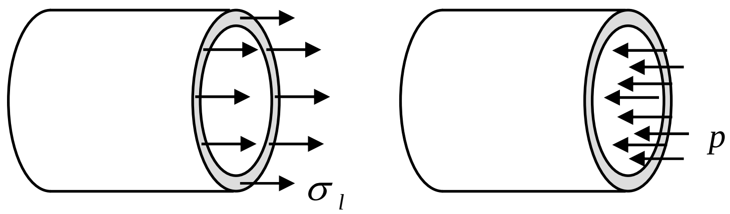


Figure 7.3.9: free body diagram of a cylindrical pressure vessel

Again taking a free-body diagram of the cylinder and carrying out an equilibrium analysis, one finds that, as for the spherical vessel,

$$\boxed{\sigma_l = \frac{pr}{2t}} \qquad \textbf{Longitudinal stress in a thin-walled cylindrical pressure vessel} \tag{7.3.10}$$

Note that this analysis is only valid at positions sufficiently far away from the cylinder ends, where it might be closed in by caps – a more complex stress field would arise there.

The circumferential stress can be evaluated from an equilibrium analysis of the free body diagram in Fig. 7.3.10:

$$-\sigma_c 2tL + 2r_i Lp = 0 \tag{7.3.11}$$

and so

$$\boxed{\sigma_c = \frac{pr}{t}}$$ **Circumferential stress in a thin-walled cylindrical pressure vessel** (7.3.12)

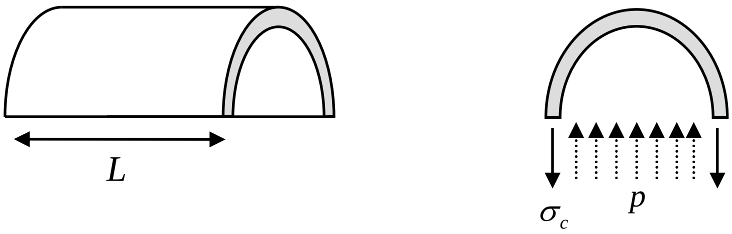


Figure 7.3.10: free body diagram of a cylindrical pressure vessel

As with the sphere, the radial stress varies from $-p$ at the inner surface to zero at the outer surface, but again is small compared with the other two stresses, and so is taken to be $\sigma_r = 0$.

Strain in the Thin-walled cylinder

The analysis of strain in the cylindrical pressure vessel is very similar to that of the spherical vessel. Eqns. 7.3.6 and 7.3.9 hold also here. Eqn. 7.3.5 would need to be amended to account for the three different principal stresses in the cylinder.

7.3.3 External Pressure

The analysis given above can be extended to the case where there is also an external pressure acting on the vessel. The internal pressure is now denoted by p_i and the external pressure is denoted by p_o, Fig. 7.3.11.

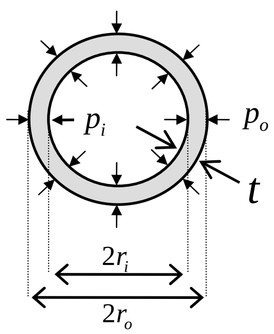


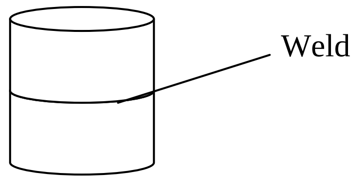
Figure 7.3.11: A pressure vessel subjected to internal and external pressure

In this case, the pressure p in formulae derived above can simply be replaced by $(p_i - p_o)$, which is known as the **gage pressure** (see the Appendix to this section, §7.3.5, for justification).

7.3.4 Problems

1. A 20m diameter spherical tank is to be used to store gas. The shell plating is 10 mm thick and the working stress of the material, that is, the maximum stress to which the material should be subjected, is 125 MPa. What is the maximum permissible gas pressure?

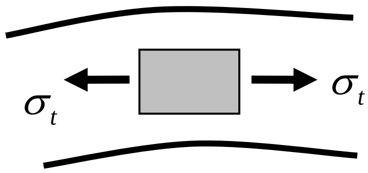
2. A steel propane tank for a BBQ grill has a 25cm diameter[1] and a wall thickness of 5mm (see figure). The tank is pressurised to 1.2 MPa.
   (a) determine the longitudinal and circumferential stresses in the cylindrical body of the tank
   (b) determine the absolute maximum shear stress in the cylindrical portion of the tank
   (c) determine the tensile force per cm length being supported by a weld joining the upper and lower sections of the tank.



3. What are the strains in the BBQ tank of question 2? What is the radial displacement? [take the steel to be isotropic with $E = 200\text{GPa}, \nu = 0.3$]

4. What are the strains in the cylindrical pressure vessel, in terms of E, ν, p, t and r?

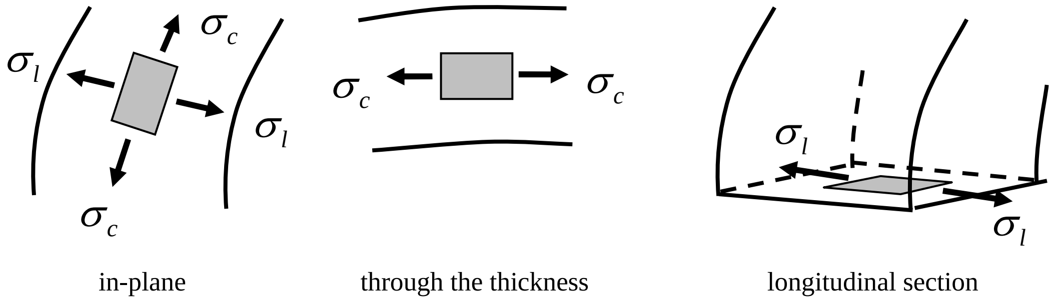
5. There are no shear stresses in the tangential plane of the spherical pressure vessel. However, there are shear stresses acting on planes through the thickness of the wall. A cross-section through the thickness is shown below. Take it that the radial stresses are zero. What are the maximum shear stresses occurring on this cross section?



6. The three perpendicular planes in the cylindrical pressure vessel are the in-plane, through the thickness and longitudinal sections, as shown below. The non-zero (principal) stresses acting on these planes are also shown. Evaluate the maximum

[1] this is an average diameter – the inside is 250-5mm and the outside is 250+5mm

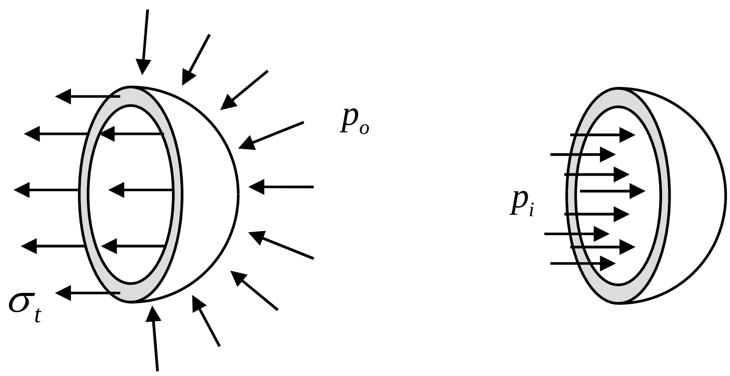
shear stresses on each of these three planes. Which of these three maxima is the overall maximum shear stress acting in the vessel?



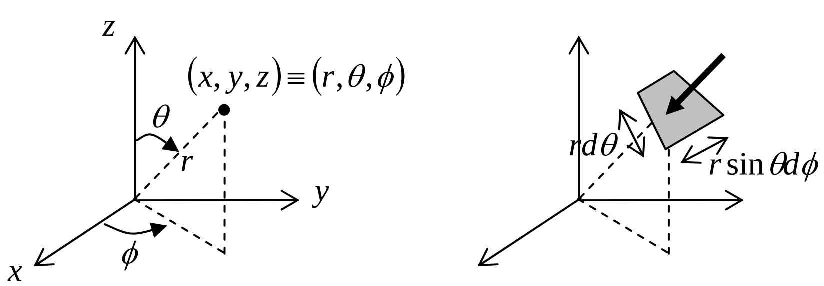
7.3.5 Appendix to §7.3

Equilibrium of a Pressure Vessel with both internal and external pressure

Consider the spherical pressure vessel. An external pressure p_o is distributed around its outer surface. Consider a free-body diagram of one half of the vessel, as shown below.



The force due to the external pressure acting in the horizontal direction can be evaluated using the spherical coordinates shown below.



An element of surface area upon which the pressure acts, swept out when the angles change by $d\theta$ and $d\phi$, has sides $rd\theta$ and $r\sin\theta d\phi$. The force acting on this area is then $p_o r^2 \sin\theta d\theta d\phi$. Force equilibrium in the horizontal (y) direction then leads to

$$-r_o^2 p_o \int_0^{\pi} \sin^2 \theta d\theta \int_0^{\pi} \sin \phi d\phi - \pi\left(r_o^2 - r_i^2\right)\sigma_t + \pi r_i^2 p_i = 0$$

and so,

$$\sigma_t = \frac{r_i^2 p_i - r_o^2 p_o}{\left(r_0 + r_i\right)t}$$

or $\sigma_t \approx (p_i - p_o)r/2t$ – see Eqn. 7.3.3.

7.4 The Elementary Beam Theory

In this section, problems involving long and slender beams are addressed. As with pressure vessels, the geometry of the beam, and the specific type of loading which will be considered, allows for approximations to be made to the full three-dimensional linear elastic stress-strain relations.

The beam theory is used in the design and analysis of a wide range of structures, from buildings to bridges to the load-bearing bones of the human body.

7.4.1 The Beam

The term **beam** has a very specific meaning in engineering mechanics: it is a component that is designed to support **transverse loads**, that is, loads that act perpendicular to the longitudinal axis of the beam, Fig. 7.4.1. The beam supports the load by *bending only*. Other mechanisms, for example twisting of the beam, are not allowed for in this theory.

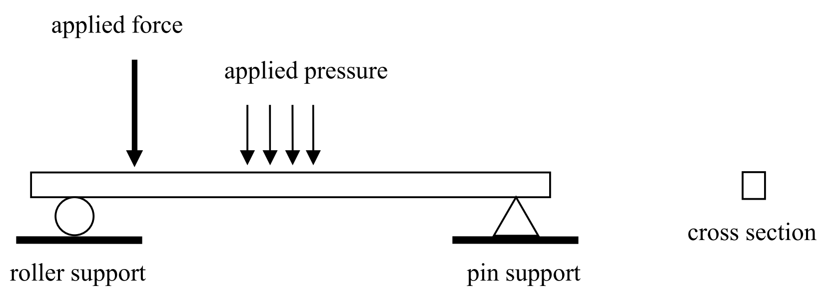


Figure 7.4.1: A supported beam loaded by a force and a distribution of pressure

It is convenient to show a two-dimensional cross-section of the three-dimensional beam together with the beam cross section, as in Fig. 7.4.1. The beam can be supported in various ways, for example by roller supports or pin supports (see section 2.3.3). The cross section of this beam happens to be rectangular but it can be any of many possible shapes.

It will assumed that the beam has a **longitudinal plane of symmetry**, with the cross section symmetric about this plane, as shown in Fig. 7.4.2. Further, it will be assumed that the loading and supports are also symmetric about this plane. With these conditions, the beam has no tendency to twist and will undergo bending only[1].

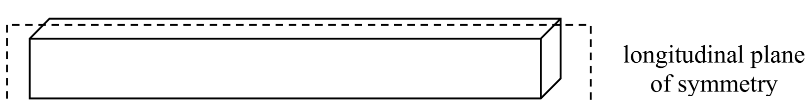


Figure 7.4.2: The longitudinal plane of symmetry of a beam

[1] certain very special cases, where there is *not* a plane of symmetry for geometry and/or loading, can lead also to bending with no twist, but these are not considered here

Imagine now that the beam consists of many fibres aligned longitudinally, as in Fig. 7.4.3. When the beam is bent by the action of downward transverse loads, the fibres near the top of the beam contract in length whereas the fibres near the bottom of the beam extend. Somewhere in between, there will be a plane where the fibres do not change length. This is called the **neutral surface**. The intersection of the longitudinal plane of symmetry and the neutral surface is called the **axis of the beam**, and the deformed axis is called the **deflection curve**.

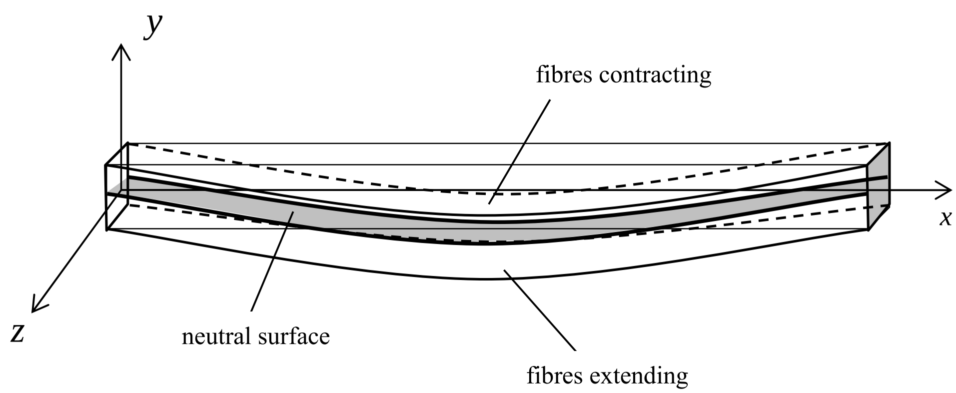


Figure 7.4.3: the neutral surface of a beam

A conventional coordinate system is attached to the beam in Fig. 7.4.3. The *x* axis coincides with the (longitudinal) axis of the beam, the *y* axis is in the transverse direction and the longitudinal plane of symmetry is in the $x - y$ plane, also called the **plane of bending**.

7.4.2 Moments and Forces in a Beam

Normal and shear stresses act over any cross section of a beam, as shown in Fig. 7.4.4. The normal and shear stresses acting on each side of the cross section are equal and opposite for equilibrium, Fig. 7.4.4b. The normal stresses σ will vary over a section during bending. Referring again to Fig. 7.4.3, over one part of the section the stress will be tensile, leading to extension of material fibres, whereas over the other part the stresses will be compressive, leading to contraction of material fibres. This distribution of normal stress results in a moment *M* acting on the section, as illustrated in Fig. 7.4.4c. Similarly, shear stresses τ act over a section and these result in a shear force *V*.

The beams of Fig. 7.4.3 and Fig. 7.4.4 show the normal stress and deflection one would expect when a beam bends downward. There are situations when parts of a beam bend upwards, and in these cases the signs of the normal stresses will be opposite to those shown in Fig. 7.4.4. However, the moments (and shear forces) shown in Fig. 7.4.4 will be regarded as *positive*. This sign convention to be used is shown in Fig. 7.4.5.

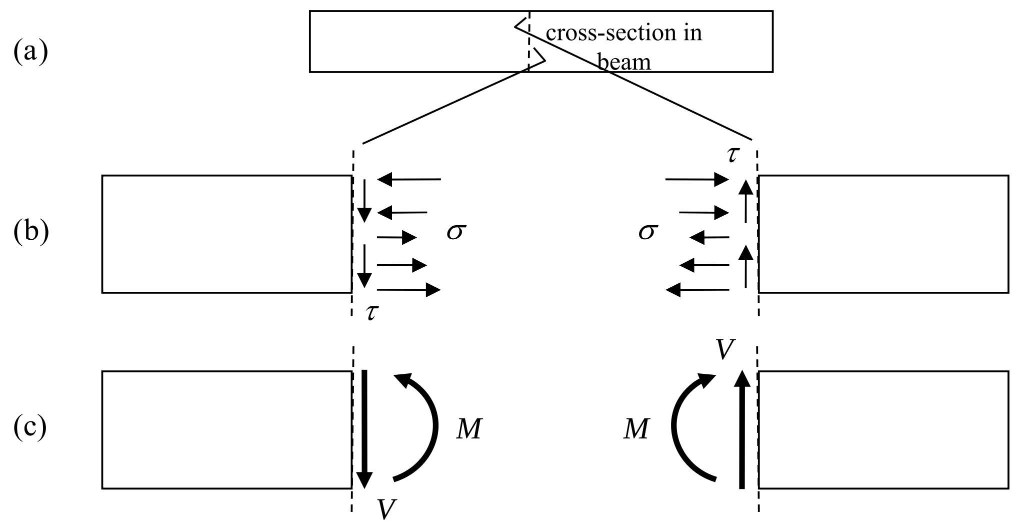


Figure 7.4.4: stresses and moments acting over a cross-section of a beam; (a) a cross-section, (b) normal and shear stresses acting over the cross-section, (c) the moment and shear force resultant of the normal and shear stresses

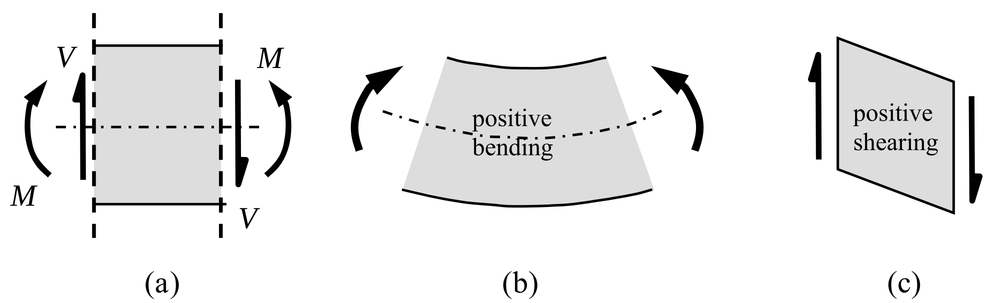


Figure 7.4.5: sign convention for moments and shear forces

Note that the sign convention for the shear stress conventionally used the beam theory conflicts with the sign convention for shear stress used in the rest of mechanics, introduced in Chapter 3. This is shown in Fig. 7.4.6.

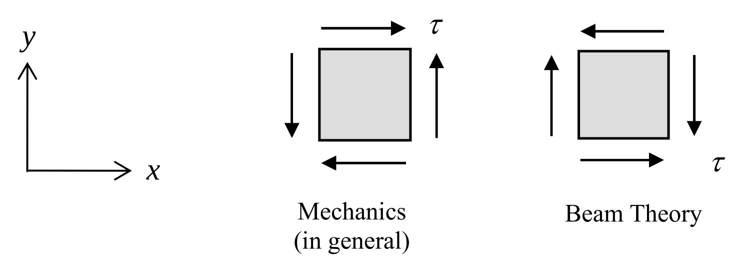


Figure 7.4.6: sign convention for shear stress in beam theory

The moments and forces acting within a beam can in many simple problems be evaluated from equilibrium considerations alone. Some examples are given next.

Example 1

Consider the **simply supported** beam in Fig. 7.4.7. From the loading, one would expect the beam to deflect something like as indicated by the deflection curve drawn. The reaction at the roller support, end A, and the vertical reaction at the pin support[2], end B, can be evaluated from the equations of equilibrium, Eqns. 2.3.3:

$$R_{Ay} = P/3, \quad R_{By} = 2P/3 \tag{7.4.1}$$

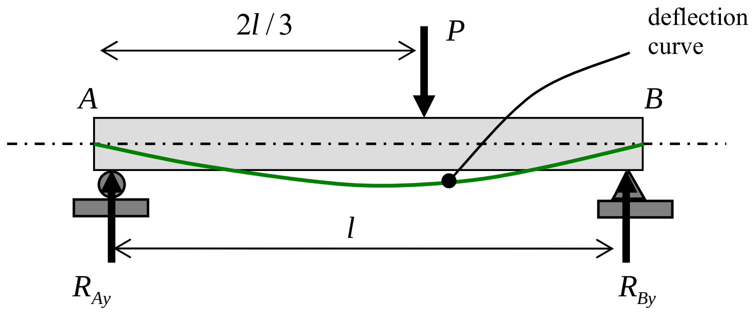
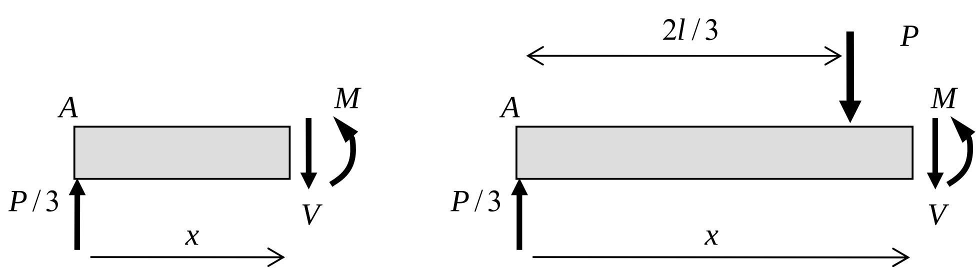


Figure 7.4.7: a simply supported beam

The moments and forces acting *within* the beam can be evaluated by taking free-body diagrams of sections of the beam. There are clearly two distinct regions in this beam, to the left and right of the load. Fig. 7.4.8a shows an arbitrary portion of beam representing the left-hand side. A coordinate system has been introduced, with x measured from A.[3] An unknown moment M and shear force V act at the end. A *positive* moment and force have been drawn in Fig. 7.4.8a. From the equilibrium equations, one finds that the shear force is constant but that the moment varies linearly along the beam:

$$V = \frac{P}{3}, \qquad M = \frac{P}{3}x \qquad \left(0 < x < \frac{2l}{3}\right) \tag{7.4.2}$$



(a) (b)

Figure 7.4.8: free body diagrams of sections of a beam

[2] the horizontal reaction at the pin is zero since there are no applied forces in this direction; the beam theory does not consider such types of (axial) load; further, one does not have a pin at each support, since this would prevent movement in the horizontal direction which in turn would give rise to forces in the horizontal direction – hence the pin at one end and the roller support at the other end
[3] the coordinate x can be measured from any point in the beam; in this example it is convenient to measure it from point A

Cutting the beam to the right of the load, Fig. 7.4.8b, leads to

$$V = -\frac{2P}{3}, \qquad M = \frac{2P}{3}(l - x) \qquad (\frac{2l}{3} < x < l) \tag{7.4.3}$$

The shear force is negative, so acts in the direction opposite to that initially assumed in Fig. 7.4.8b.

The results of the analysis can be displayed in what are known as a **shear force diagram** and a **bending moment diagram**, Fig. 7.4.9. Note that there is a "jump" in the shear force at $x = 2l/3$ equal to the applied force, and in this example the bending moment is everywhere positive.

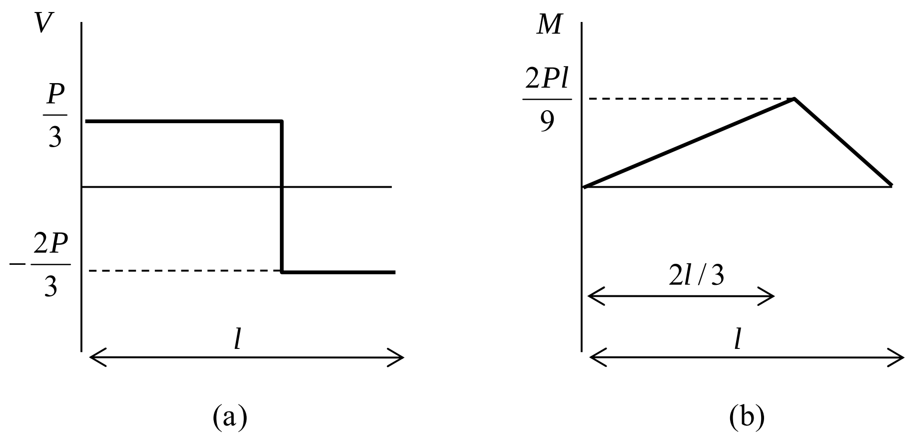


Figure 7.4.9: results of analysis; (a) shear force diagram, (b) bending moment diagram

■

Example 2

Fig. 7.4.10 shows a **cantilever**, that is, a beam supported by clamping one end (refer to Fig. 2.3.8). The cantilever is loaded by a force at its mid-point and a (negative) moment at its end.

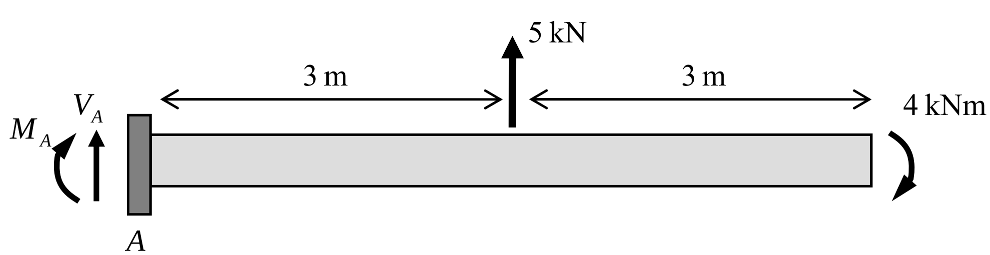


Figure 7.4.10: a cantilevered beam loaded by a force and moment

Again, positive unknown reactions M_A and V_A are considered at the support *A*. From the equilibrium equations, one finds that

$$M_A = 11\,\text{kNm}, \quad V_A = -5\,\text{kN} \tag{7.4.4}$$

As in the previous example, there are two distinct regions along the beam, to the left and to the right of the applied concentrated force. Again, a coordinate x is introduced and the beam is sectioned as in Fig. 7.4.11. The unknown moment M and shear force V can then be evaluated from the equilibrium equations:

$$\begin{array}{lll} V = -5\,\text{kN}, & M = 11 - 5x\ \text{kNm} & (0 < x < 3) \\ V = 0, & M = -4\ \text{kNm} & (3 < x < 6) \end{array} \tag{7.4.5}$$

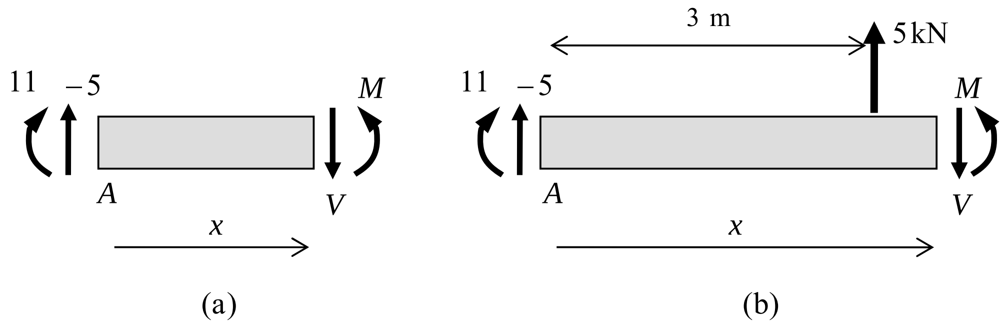


Figure 7.4.11: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.12.

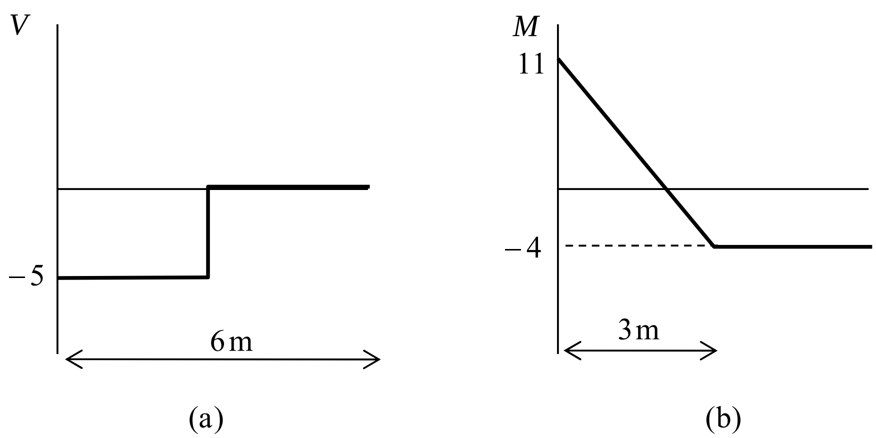


Figure 7.4.12: results of analysis; (a) shear force diagram, (b) bending moment diagram

In this example the beam experiences negative bending moment over most of its length.

■

Example 3

Fig. 7.4.13 shows a simply supported beam subjected to a distributed load (force per unit length). The load is uniformly distributed over half the length of the beam, with a triangular distribution over the remainder.

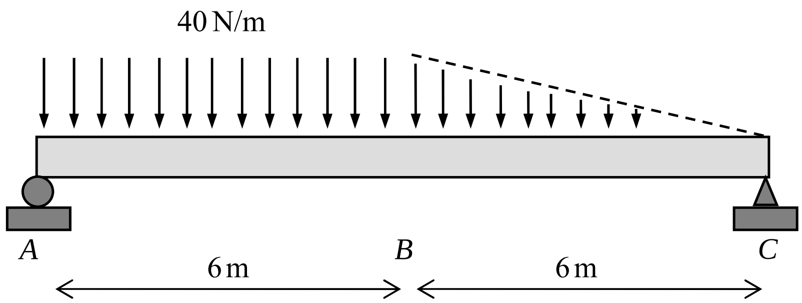


Figure 7.4.13: a beam subjected to a distributed load

The unknown reactions can be determined by replacing the distributed load with statically equivalent forces as in Fig. 7.4.14 (see §3.1.2). The equilibrium equations then give

$$R_A = 220 \text{ N}, \quad R_C = 140 \text{ N} \tag{7.4.6}$$

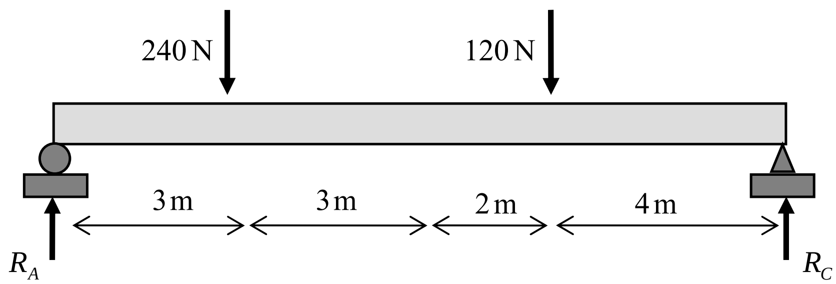


Figure 7.4.14: equivalent forces acting on the beam of Fig. 7.4.13

Referring again to Fig. 7.4.13, there are two distinct regions in the beam, that under the uniform load and that under the triangular distribution of load. The first case is considered in Fig. 7.4.15.

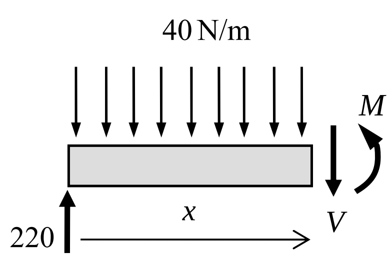


Figure 7.4.15: free body diagram of a section of a beam

The equilibrium equations give

$$V = 220 - 40x, \qquad M = 220x - 20x^2 \qquad (0 < x < 6) \tag{7.4.7}$$

The region beneath the triangular distribution is shown in Fig. 7.4.16. Two possible approaches are illustrated: in Fig. 7.4.16a, the free body diagram consists of the complete length of beam to the left of the cross-section under consideration; in Fig. 7.4.16b, only the portion to the right is considered, with distance measured from the right hand end, as $12-x$. The problem is easier to solve using the second option; from Fig. 7.4.16b then, with the equilibrium equations, one finds that

$$V=-140+10(12-x)^2/3, \quad M=140(12-x)-10(12-x)^3/9 \quad \left(6<\mathrm{x}<12\right) \tag{7.4.8}$$

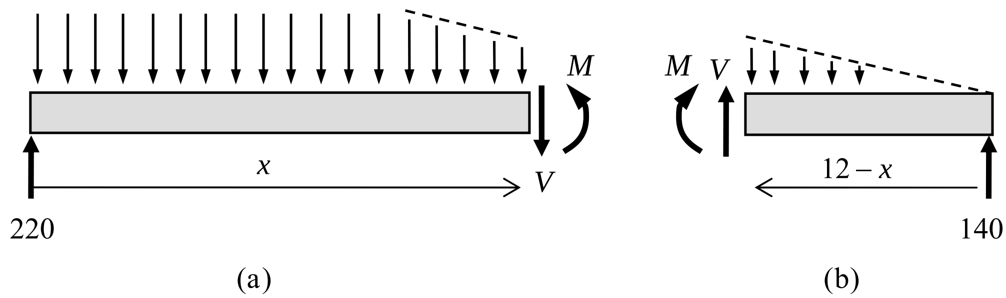


Figure 7.4.16: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.17.

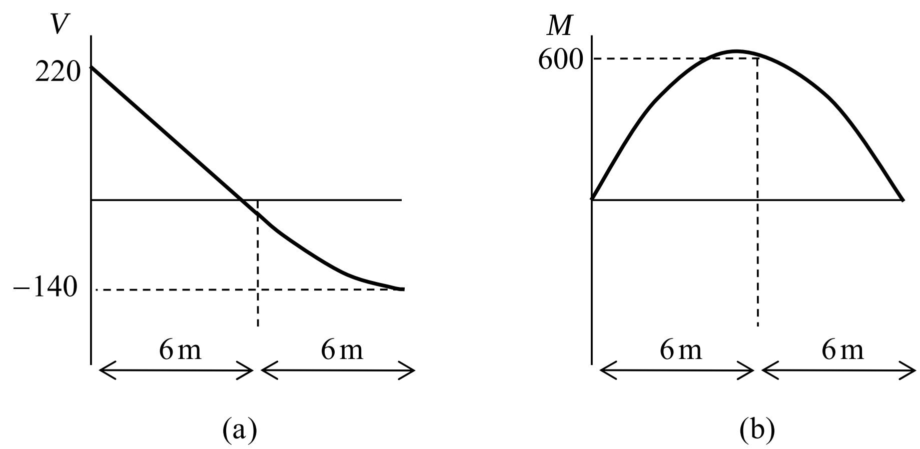


Figure 7.4.17: results of analysis; (a) shear force diagram, (b) bending moment diagram

■

7.4.3 The Relationship between Loads, Shear Forces and Bending Moments

Relationships between the applied loads and the internal shear force and bending moment in a beam can be established by considering a small beam element, of width Δx, and subjected to a distributed load $p(x)$ which varies along the section of beam, and which is *positive upward*, Fig. 7.4.18.

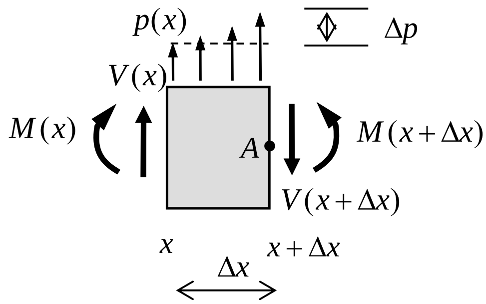


Figure 7.4.18: forces and moments acting on a small element of beam

At the left-hand end of the free body, at position *x*, the shear force, moment and distributed load have values $V(x)$, $M(x)$ and $p(x)$ respectively. On the right-hand end, at position $x+\Delta x$, their values are slightly different: $V(x+\Delta x)$, $M(x+\Delta x)$ and $p(x+\Delta x)$. Since the element is very small, the distributed load, even if it is varying, can be approximated by a *linear* variation over the element. The distributed load can therefore be considered to be a uniform distribution of intensity $p(x)$ over the length Δx together with a triangular distribution, 0 at x and Δp say, a *small* value, at $x+\Delta x$. Equilibrium of vertical forces then gives

$$\begin{aligned} &V(x)+p(x)\Delta x+\frac{1}{2}\Delta p\Delta x-V(x+\Delta x)=0 \\ &\rightarrow \frac{V(x+\Delta x)-V(x)}{\Delta x}=p(x)+\frac{1}{2}\Delta p \end{aligned} \tag{7.4.9}$$

Now let the size of the element decrease towards zero. The left-hand side of Eqn. 7.4.9 is then the definition of the derivative, and the second term on the right-hand side tends to zero, so

$$\boxed{\frac{dV}{dx}=p(x)} \tag{7.4.10}$$

This relation can be seen to hold in Eqn. 7.4.7 and Fig. 7.4.17a, where the shear force over $0<x<6$ has a slope of -40 and the pressure distribution is uniform, of intensity $-40\,\text{N/m}$. Similarly, over $6<x<12$, the pressure decreases linearly and so does the slope in the shear force diagram, reaching zero slope at the end of the beam.

It also follows from 7.4.10 that the change in shear along a beam is equal to the area under the distributed load curve:

$$V(x_2)-V(x_1)=\int_{x_1}^{x_2} p(x)dx \tag{7.4.11}$$

Consider now moment equilibrium, by taking moments about the point A in Fig. 7.4.18:

$$\begin{aligned}&-M(x)-V(x)\Delta x+M(x+\Delta x)-p(x)\Delta x\frac{\Delta x}{2}-\frac{1}{2}\Delta p\Delta x\frac{\Delta x}{3}=0\\&\rightarrow\frac{M(x+\Delta x)-M(x)}{\Delta x}=V(x)+p(x)\frac{\Delta x}{2}+\Delta p\frac{\Delta x}{6}\end{aligned} \tag{7.4.12}$$

Again, as the size of the element decreases towards zero, the left-hand side becomes a derivative and the second and third terms on the right-hand side tend to zero, so that

$$\boxed{\frac{dM}{dx}=V(x)} \tag{7.4.13}$$

This relation can be seen to hold in Eqns. 7.4.2-3, 7.4.5 and 7.4.7-8. It also follows from Eqn. 7.4.13 that the change in moment along a beam is equal to the area under the shear force curve:

$$M(x_2)-M(x_1)=\int_{x_1}^{x_2}V(x)dx \tag{7.4.14}$$

7.4.4 Deformation and Flexural Stresses in Beams

The moment at any given cross-section of a beam is due to a distribution of normal stress, or **flexural stress** (or **bending stress**) across the section (see Fig. 7.4.4). As mentioned, the stresses to one side of the neutral axis are tensile whereas on the other side of the neutral axis they are compressive. To determine the distribution of normal stress over the section, one must determine the precise location of the neutral axis, and to do this one must consider the *deformation* of the beam.

Apart from the assumption of there being a longitudinal plane of symmetry and a neutral axis along which material fibres do not extend, the following two assumptions will be made concerning the deformation of a beam:

1. Cross-sections which are plane and are perpendicular to the axis of the undeformed beam remain plane and remain perpendicular to the deflection curve of the deformed beam. In short: “plane sections remain plane”. This is illustrated in Fig. 7.4.19. It will be seen later that this assumption is a valid one provided the beam is sufficiently long and slender.

2. Deformation in the vertical direction, i.e. the transverse strain ε_{yy}, may be neglected in deriving an expression for the longitudinal strain ε_{xx}. This assumption is summarised in the deformation shown in Fig. 7.4.20, which shows an element of length l and height h undergoing transverse and longitudinal strain.

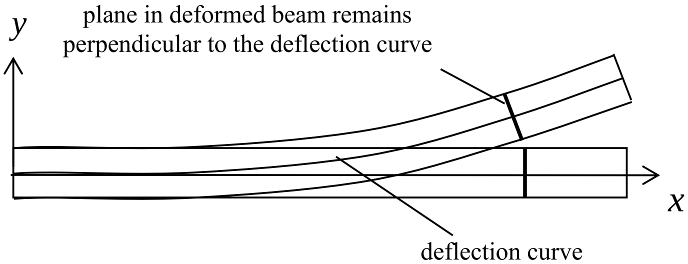
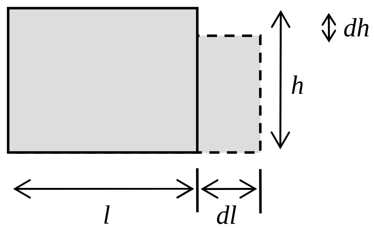


Figure 7.4.19: plane sections remain plane in the elementary beam theory



$$\varepsilon_{xx} = \frac{dl}{l}, \quad \varepsilon_{yy} = -\frac{dh}{h} \approx 0$$

Figure 7.4.20: transverse strain is neglected in the elementary beam theory

With these assumptions, consider now the element of beam shown in Fig. 7.4.21. Here, two material fibres ab and pq, of length Δx in the undeformed beam, deform to $a'b'$ and $p'q'$. The deflection curve has a radius of curvature *R*. The above two assumptions imply that, referring to the figure:

$$\begin{aligned} &\angle p'a'b' = \angle a'b'q' = \pi / 2 && \text{(assumption 1)} \\ &|ap| = |a'p'|, \quad |bq| = |b'q'| && \text{(assumption 2)} \end{aligned} \tag{7.4.15}$$

Since the fibre ab is on the neutral axis, by definition $|a'b'| = |ab|$. However the fibre pq, a distance *y* from the neutral axis, extends in length from Δx to length $\Delta x'$. The longitudinal strain for this fibre is

$$\varepsilon_{xx} = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(R-y)\Delta\theta - R\Delta\theta}{R\Delta\theta} = -\frac{y}{R} \tag{7.4.16}$$

As one would expect, this relation implies that a small *R* (large curvature) is related to a large strain and a large *R* (small curvature) is related to a small strain. Further, for $y > 0$ (above the neutral axis), the strain is negative, whereas if $y < 0$ (below the neutral axis), the strain is positive[4], and the variation across the cross-section is linear.

---

[4] this is under the assumption that *R* is positive, which means that the beam is concave up; a negative *R* implies that the centre of curvature is below the beam

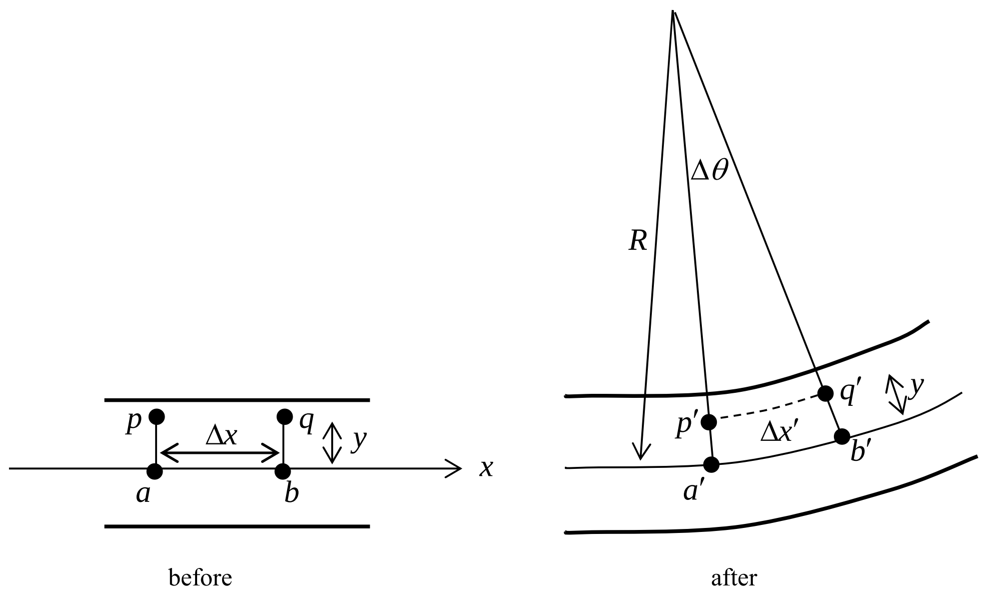


Figure 7.4.21: deformation of material fibres in an element of beam

To relate this deformation to the stresses arising in the beam, it is necessary to postulate the stress-strain law for the material out of which the beam is made. Here, it is assumed that the beam is isotropic linear elastic[5].

The beam is a three-dimensional object, and so will in general experience a fairly complex three-dimensional stress state. We will show in what follows that a simple one-dimensional approximation, $\sigma_{xx} = E\varepsilon_{xx}$, whilst disregarding all other stresses and strains, will be sufficiently accurate for our purposes.

Since there are no forces acting in the *z* direction, the beam is in a state of plane stress, and the stress-strain equations are (see Eqns. 6.1.10)

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{zz} &= -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}, \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0
\end{aligned}
\tag{7.4.17}
$$

Yet another assumption is now made, that the transverse normal stresses, σ_{yy}, may be neglected in comparison with the flexural stresses σ_{xx}. This is similar to the above assumption #2 concerning the deformation, where the transverse normal strain was neglected in comparison with the longitudinal strain. It might seem strange at first that the transverse stress is neglected, since all loads are in the transverse direction. However,

[5] the beam theory can be extended to incorporate more complex material models (constitutive equations)

just as the tangential stresses are much larger than the radial stresses in the pressure vessel, it is found that the longitudinal stresses in a beam are very much greater than the transverse stresses. With this assumption, the first of Eqn. 7.4.17 reduces to a one-dimensional equation:

$$\varepsilon_{xx} = \sigma_{xx} / E \qquad (7.4.18)$$

and, from Eqn. 7.4.16, dropping the subscripts on σ,

$$\sigma = -\frac{E}{R} y \qquad (7.4.19)$$

Finally, the resultant force of the normal stress distribution over the cross-section must be zero, and the resultant moment of the distribution is M, leading to the conditions

$$\begin{aligned} 0 &= \int_A \sigma dA = -\frac{E}{R}\int_A y dA \\ M &= -\int_A \sigma y dA = \frac{E}{R}\int_A y^2 dA = -\frac{\sigma}{y}\int_A y^2 dA \end{aligned} \qquad (7.4.20)$$

and the integration is over the complete cross-sectional area A. The minus sign in the second of these equations arises because a positive moment and a positive y imply a compressive (negative) stress (see Fig. 7.4.4).

The quantity $\int_A y dA$ is the first moment of area about the neutral axis, and is equal to $\bar{y}A$, where $\bar{y}$ is the centroid of the section (see, for example, §3.2.1). Note that the horizontal component ("in-out of the page") of the centroid will always be at the centre of the beam due to the symmetry of the beam about the plane of bending. Since the first moment of area is zero, it follows that $\bar{y} = 0$: *the neutral axis passes through the centroid of the cross-section*.

The quantity $\int_A y^2 dA$ is called the **second moment of area** or the **moment of inertia** about the neutral axis, and is denoted by the symbol I. It follows that the flexural stress is related to the moment through

$$\boxed{\sigma = -\frac{My}{I}} \qquad \textbf{Flexural stress in a beam} \qquad (7.4.21)$$

This is one of the most famous and useful formulas in mechanics.

The Moment of Inertia

The moment of inertia depends on the shape of a beam's cross-section. Consider the important case of a rectangular cross section. Before determining the moment of inertia one must locate the centroid (neutral axis). Due to symmetry, the neutral axis runs through the centre of the cross-section. To evaluate I for a rectangle of height h and width b, consider a small strip of height dy at location y, Fig. 7.4.22. Then

$$I = \int_A y^2 dA = b \int_{-h/2}^{+h/2} y^2 dy = \frac{bh^3}{12} \tag{7.4.22}$$

This relation shows that the "taller" the cross-section, the larger the moment of inertia, something which holds generally for I. Further, the larger is I, the smaller is the flexural stress, which is always desirable.

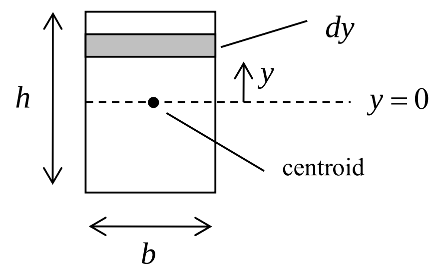


Figure 7.4.22: Evaluation of the moment of inertia for a rectangular cross-section

For a circular cross-section with radius R, consider Fig. 7.4.23. The moment of inertia is then

$$I = \int_A y^2 dA = \int_0^{2\pi}\int_0^R r^3 \sin^2\theta dr d\theta = \frac{\pi R^4}{4} \tag{7.4.23}$$

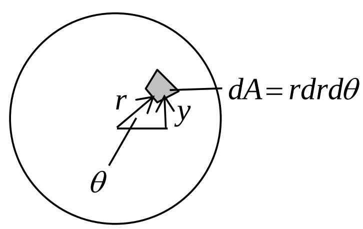


Figure 7.2.23: Moment of inertia for a circular cross-section

Example

Consider the beam shown in Fig. 7.4.24. It is loaded symmetrically by two concentrated forces each of magnitude 100N and has a circular cross-section of radius 100mm. The reactions at the two supports are found to be 100N. Sectioning the beam to the left of the forces, and then to the right of the first force, one finds that

$$\begin{aligned} V &= 100, \qquad M = 100x \qquad (0 < x < 250) \\ V &= 0, \qquad M = 25000 \quad (250 < x < l/2) \end{aligned} \tag{7.4.24}$$

where l is the length of the beam.

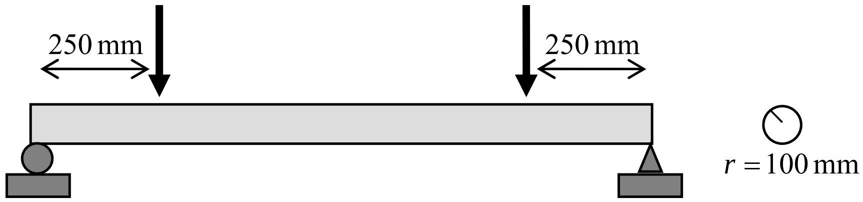


Figure 7.4.24: a loaded beam with circular cross-section

The maximum tensile stress is then

$$\sigma_{\max} = -\frac{M_{\max}(-y_{\max})}{I} = \frac{25000r}{\pi r^4/4} = 31.8\,\text{MPa} \tag{7.4.25}$$

and occurs at all sections between the two loads (at the base of the beam).

■

7.4.5 Shear Stresses in Beams

In the derivation of the flexural stress formula, Eqn. 7.4.21, it was assumed that plane sections remain plane. This implies that there is no shear strain and, for an isotropic elastic material, no shear stress, as indicated in Fig. 7.4.25.

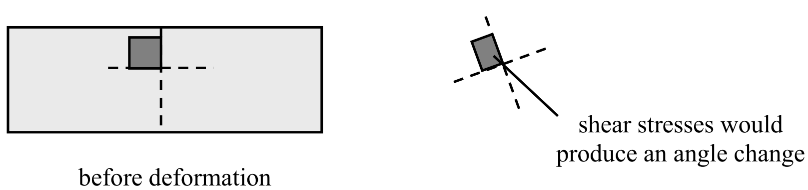


Figure 7.4.25: a section of beam before and after deformation

This fact will now be ignored, and an expression for the shear stress τ within a beam will be developed. It is implicitly assumed that this shear stress has little effect on the (calculation of the) flexural stress.

As in Fig. 7.4.18, consider the equilibrium of a thin section of beam, as shown in Fig. 7.4.26. The beam has *rectangular* cross-section (although the theory developed here is strictly for rectangular cross sections only, it can be used to give approximate shear stress values in any beam with a plane of symmetry). Consider the equilibrium of a section *of* this section, at the upper surface of the beam, shown hatched in Fig. 7.4.26. The stresses acting on this section are as shown. Again, the normal stress is compressive at the surface, consistent with the sign convention for a positive moment. Note that there are no shear stresses acting at the surface – there may be distributed normal loads or forces acting at the surface but, for clarity, these are not shown, and they are not necessary for the following calculation.

From equilibrium of forces in the horizontal direction of the surface section:

$$\left[-\int_A \sigma dA\right]_x + \left[\int_A \sigma dA\right]_{x+\Delta x} + \tau b \Delta x = 0 \tag{7.4.26}$$

The third term on the left here assumes that the shear stress is uniform over the section – this is similar to the calculations of §7.4.3 – for a very small section, the variation in stress is a small term and may be neglected. Using the bending stress formula, Eqn. 7.4.21,

$$-\int_A \frac{M(x+\Delta x)-M(x)}{\Delta x}\frac{y}{I}dA + \tau b = 0 \tag{7.4.27}$$

and, with Eqn. 7.4.13, as $\Delta x \to 0$,

$$\boxed{\tau = \frac{VQ}{Ib}} \quad \textbf{Shear stress in a beam} \tag{7.4.28}$$

where Q is the first moment of area $\int_A ydA$ of the *surface section* of the cross-section.

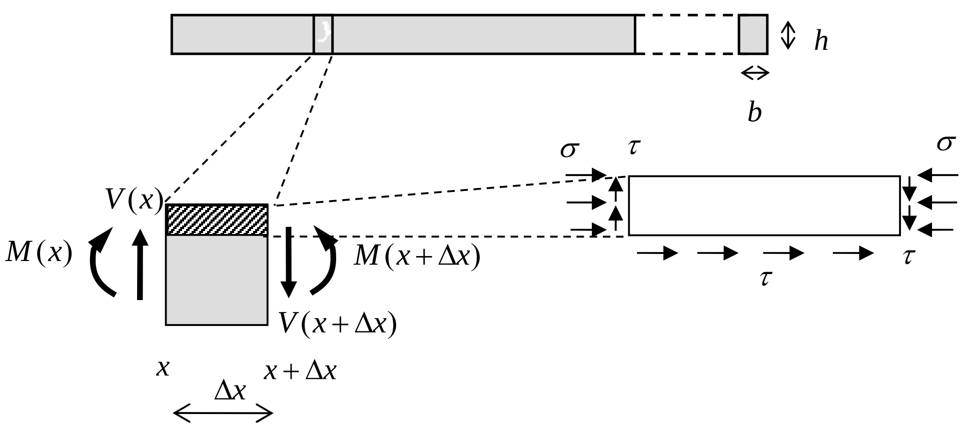


Figure 7.4.26: stresses and forces acting on a small section of material at the surface of a beam

As mentioned, this formula 7.4.28 can be used as an approximation of the shear stress in a beam of arbitrary cross-section, in which case b can be regarded as the depth of the beam at that section. For the rectangular beam, one has

$$Q = b\int_y^{h/2} ydy = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \tag{7.4.29}$$

so that

$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) \tag{7.4.30}$$

The maximum shear stress in the cross-section arises at the neutral surface:

$$\tau_{\max} = \frac{3V}{2bh} = \frac{3V}{2A} \tag{7.4.31}$$

and the shear stress dies away towards the upper and lower surfaces. Note that the average shear stress over the cross-section is V / A and the maximum shear stress is 150% of this value.

Finally, since the shear stress on a vertical cross-section has been evaluated, the shear stress on a longitudinal section has been evaluated, since the shear stresses on all four sides of an element are the same, as in Fig.7.4.6.

Example

Consider the simply supported beam loaded by a concentrated force shown in Fig. 7.4.27. The cross-section is rectangular with height $100\,\text{mm}$ and width $50\,\text{mm}$. The reactions at the supports are $5\,\text{kN}$ and $15\,\text{kN}$. To the left of the load, one has $V = 5\,\text{kN}$ and $M = 5x\,\text{kNm}$. To the right of the load, one has $V = -15\,\text{kN}$ and $M = 30 - 15x\,\text{kNm}$.

The maximum shear stress will occur along the neutral axis and will clearly occur where V is largest, so anywhere to the right of the load:

$$\tau_{\max} = \frac{3V_{\max}}{2A} = 4.5\,\text{MPa} \tag{7.4.32}$$

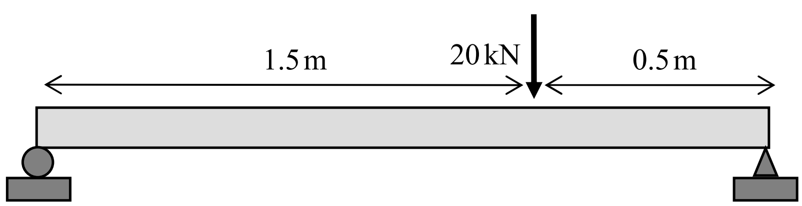


Figure 7.4.27: a simply supported beam

As an example of general shear stress evaluation, the shear stress at a point 25 mm below the top surface and 1 m in from the left-hand end is, from Eqn 7.4.30, $\tau = +1.125\,\text{MPa}$. The shear stresses acting on an element at this location are shown in Fig. 7.4.28.

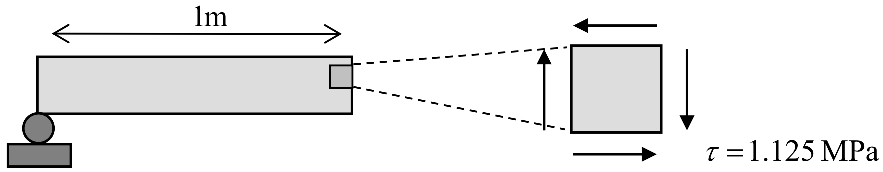


Figure 7.4.28: shear stresses acting at a point in the beam

■

7.4.6 Approximate nature of the beam theory

The beam theory is only an approximate theory, with a number of simplifications made to the full equations of elasticity. A more advanced (and exact) mechanics treatment of the beam problem would not make any assumptions regarding plane sections remaining plane, etc. The accuracy of the beam theory can be explored by comparing the beam theory results with the results of the more exact theory.

When a beam is in **pure bending**, that is when the shear force is everywhere zero, the full elasticity solution shows that plane sections *do* actually remain plane and the beam theory is exact. For more complex loadings, plane sections *do* actually deform. For example, it can be shown that the initially plane sections of a cantilever subjected to an end force, Fig. 7.4.29, do not remain plane. Nevertheless, the beam theory prediction for normal and shear stress is exact in this simple case.

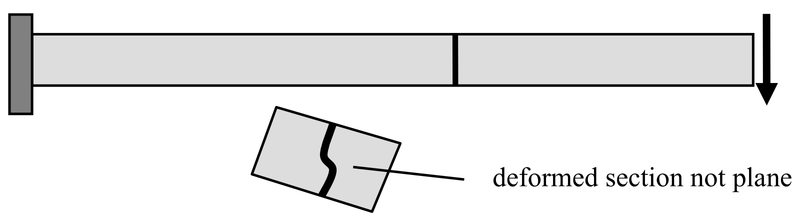


Figure 7.4.29: a cantilevered beam loaded by a force and moment

Consider next a cantilevered beam of length l and rectangular cross section, height h and width b, subjected to a uniformly distributed load p. With x measured from the cantilevered end, the shear force and moment are given by $V = p(l-x)$ and $M = (pl^2/2)\left(-1+2x/l-(x/l)^2\right)$. The shear stress is

$$\tau = \frac{6p}{bh^3}\left(\frac{h^2}{4} - y^2\right)(l-x) \tag{7.4.33}$$

and the flexural stresses at the cantilevered end, at the upper surface, are

$$\frac{\sigma}{p} = \frac{3}{b}\left(\frac{l}{h}\right)^2 \tag{7.4.34}$$

The solution for shear stress, Eqn 7.4.33, turns out to be exact; however, the exact solution corresponding to Eqn 7.4.34 is[6]

$$\frac{\sigma}{p} = \frac{1}{b}\left[3\left(\frac{l}{h}\right)^2 - \frac{1}{5}\right] \tag{7.4.35}$$

It can be seen that the beam theory is a good approximation for the case when l/h is large, in which case the term 1/5 is negligible.

Following this type of analysis, a general rule of thumb is this: for most configurations, the elementary beam theory formulae for flexural stress and transverse shear stress are accurate to within about 3% for beams whose length-to-height ratio is greater than about 4.

7.4.7 Beam Deflection

Consider the deflection curve of a beam. The displacement of the neutral axis is denoted by v, positive upwards, as in Fig. 7.4.30. The slope at any point is then given by the first derivative, dv/dx.

For any type of material, provided the slope of the deflection curve is small, it can be shown that the radius of curvature R is related to the second derivative d^2v/dx^2 through (see the Appendix to this section, §7.4.10)

$$\frac{1}{R} = \frac{d^2v}{dx^2} \tag{7.4.36}$$

and for this reason d^2v/dx^2 is called the **curvature** of the beam. Using Eqn. 7.4.19, $\sigma = -Ey/R$, and the flexural stress expression, Eqn. 7.4.21, $\sigma = -My/I$, one has the **moment-curvature equation**

$$\boxed{M(x) = EI\frac{d^2v}{dx^2}} \qquad \textbf{moment-curvature equation} \tag{7.4.37}$$

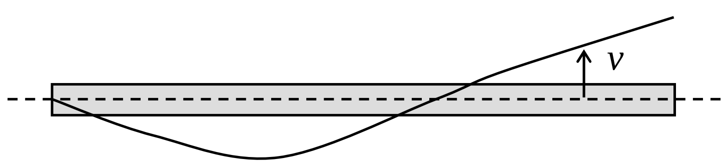


Figure 7.4.30: the deflection of a beam

With the moment known, this differential equation can be integrated twice to obtain the deflection. Boundary conditions must be supplied to obtain constants of integration.

[6] this can be derived using the Stress Function method discussed in Book 2, section 3.2

Example

Consider the cantilevered beam of length L shown in Fig. 7.4.31, subjected to an end-force F and end-moment M_0. The moment is found to be $M(x) = F(L - x) + M_0$, with x measured from the clamped end. The moment-curvature equation is then

$$\begin{aligned} & EI\frac{d^2 v}{dx^2} = (FL + M_0) - Fx \\ \rightarrow \quad & EI\frac{dv}{dx} = (FL + M_0)x - \frac{1}{2}Fx^2 + C_1 \\ \rightarrow \quad & EIv = \frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3 + C_1 x + C_2 \end{aligned} \tag{7.4.38}$$

The boundary conditions are that the displacement and slope are both zero at the clamped end, from which the two constant of integration can be obtained:

$$\begin{aligned} v(0) = 0 \quad & \rightarrow \quad C_2 = 0 \\ v'(0) = 0 \quad & \rightarrow \quad C_1 = 0 \end{aligned} \tag{7.4.39}$$

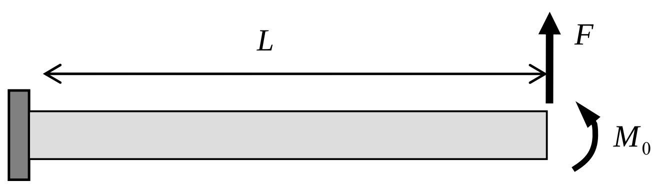


Figure 7.4.31: a cantilevered beam loaded by an end-force and moment

The slope and deflection are therefore

$$v = \frac{1}{EI}\left[\frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3\right], \qquad \frac{dv}{dx} = \frac{1}{EI}\left[(FL + M_0)x - \frac{1}{2}Fx^2\right] \tag{7.4.40}$$

The maximum deflection occurs at the end, where

$$v(L) = \frac{1}{EI}\left[\frac{1}{2}M_0 L^2 + \frac{1}{3}FL^3\right] \tag{7.4.41}$$

■

The term EI in Eqns. 7.4.40-41 is called the **flexural rigidity**, since it is a measure of the resistance of the beam to deflection.

Example

Consider the simply supported beam of length L shown in Fig. 7.4.32, subjected to a uniformly distributed load p over half its length. In this case, the moment is given by

$$M(x) = \begin{cases} \dfrac{3}{8}pLx - \dfrac{1}{2}px^2 & 0 < x < \dfrac{L}{2} \\ \dfrac{1}{8}pL(L-x) & \dfrac{L}{2} < x < L \end{cases} \tag{7.4.42}$$

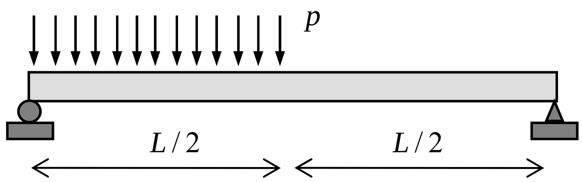


Figure 7.4.32: a simply supported beam subjected to a uniformly distributed load over half its length

It is necessary to apply the moment-curvature equation to each of the two regions $0 < x < L/2$ and $L/2 < x < L$ separately, since the expressions for the moment in these regions differ. Thus there will be four constants of integration:

$$\begin{aligned} &EI\frac{d^2v}{dx^2} = \frac{3}{8}pLx - \frac{1}{2}px^2 && EI\frac{d^2v}{dx^2} = \frac{1}{8}pL^2 - \frac{1}{8}pLx \\ &\to EI\frac{dv}{dx} = \frac{3}{16}pLx^2 - \frac{1}{6}px^3 + C_1 && \to EI\frac{dv}{dx} = \frac{1}{8}pL^2x - \frac{1}{16}pLx^2 + D_1 \\ &\to EIv = \frac{3}{48}pLx^3 - \frac{1}{24}px^4 + C_1x + C_2 && \to EIv = \frac{1}{16}pL^2x^2 - \frac{1}{48}pLx^3 + D_1x + D_2 \end{aligned} \tag{7.4.43}$$

The boundary conditions are: (i) no deflection at roller support, $v(0) = 0$, from which one finds that $C_2 = 0$, and (ii) no deflection at pin support, $v(L) = 0$, from which one finds that $D_2 = -pL^4/24 - D_1L$. The other two necessary conditions are the **continuity conditions** where the two solutions meet. These are that (i) the deflection of both solutions agree at $x = L/2$ and (ii) the slope of both solutions agree at $x = L/2$. Using these conditions, one finds that

$$C_1 = -\frac{9pL^3}{384}, \quad C_2 = -\frac{17pL^3}{384} \tag{7.4.44}$$

so that

$$v = \begin{cases} \dfrac{wL^4}{384EI}\left[-9\left(\dfrac{x}{L}\right) + 24\left(\dfrac{x}{L}\right)^3 - 16\left(\dfrac{x}{L}\right)^4\right] & 0 < x < \dfrac{L}{2} \\ \dfrac{wL^4}{384EI}\left[1 - 17\left(\dfrac{x}{L}\right) + 24\left(\dfrac{x}{L}\right)^2 - 8\left(\dfrac{x}{L}\right)^3\right] & \dfrac{L}{2} < x < L \end{cases} \tag{7.4.45}$$

The deflection is shown in Fig. 7.4.33. Note that the maximum deflection occurs in $0 < x < L/2$; it can be located by setting $dv/dx = 0$ there and solving.

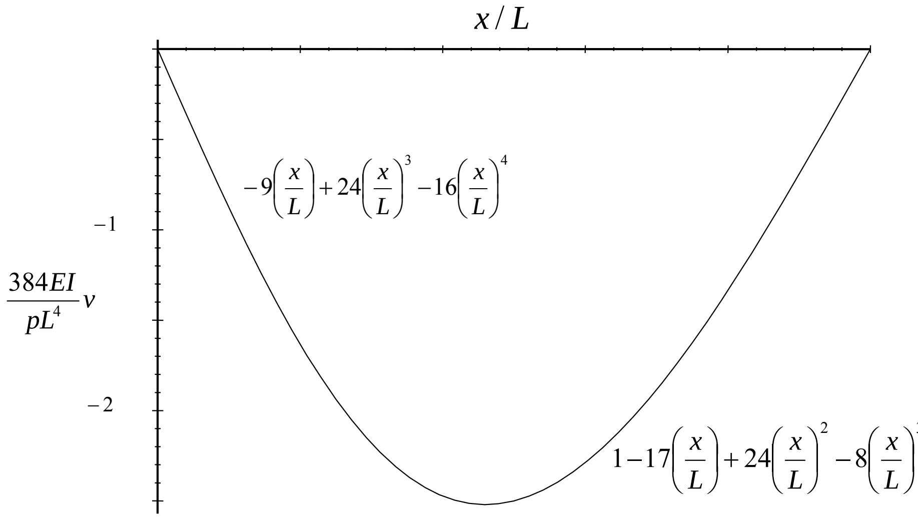


Figure 7.4.33: deflection of a beam

■

7.4.8 Statically Indeterminate Beams

Consider the beam shown in Fig. 7.4.34. It is cantilevered at one end and supported by a roller at its other end. A moment is applied at its centre. There are three unknown reactions in this problem, the reaction force at the roller and the reaction force and moment at the built-in end. There are only two equilibrium equations with which to determine these three unknowns and so it is not possible to solve the problem from equilibrium considerations alone. The beam is therefore statically indeterminate (see the end of section 2.3.3).

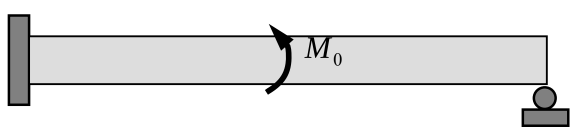


Figure 7.4.34: a cantilevered beam supported also by a roller

More examples of statically indeterminate beam problems are shown in Fig. 7.4.35. To solve such problems, one must consider the deformation of the beam. The following example illustrates how this can be achieved.

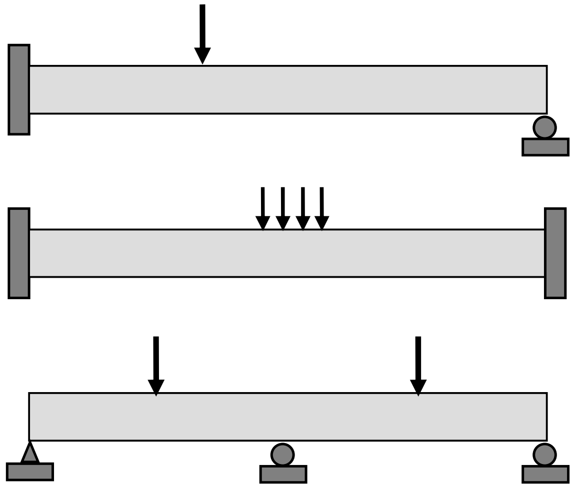


Figure 7.4.35: examples of statically indeterminate beams

Example

Consider the beam of length L shown in Fig. 7.4.36, cantilevered at end A and supported by a roller at end B. A moment M_0 is applied at B.

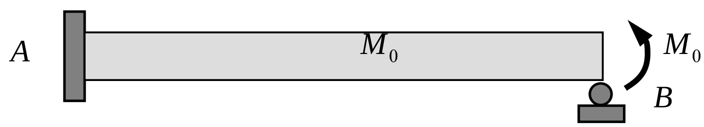


Figure 7.4.36: a statically indeterminate beam

The moment along the beam can be expressed in terms of the *unknown* reaction force at end B: $M(x) = R_B(L-x) + M_0$. As before, one can integrate the moment-curvature equation:

$$\begin{aligned} &EI\frac{d^2v}{dx^2} = R_B(L-x) + M_0 \\ &\rightarrow EI\frac{dv}{dx} = (R_B L + M_0)x - \frac{1}{2}R_B x^2 + C_1 \\ &\rightarrow EIv = \frac{1}{2}(R_B L + M_0)x^2 - \frac{1}{6}R_B x^3 + C_1 x + C_2 \end{aligned} \tag{7.4.46}$$

There are three boundary conditions, two to determine the constants of integration and one can be used to determine the unknown reaction R_B. The boundary conditions are (i) $v(0) = 0 \rightarrow C_2 = 0$, (ii) $dv/dx(0) = 0 \rightarrow C_1 = 0$ and (iii) $v(L) = 0$ from which one finds that $R_B = -3M_0/2L$. The slope and deflection are therefore

$$v = \frac{M_B L^2}{4EI}\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right]$$
$$\frac{dv}{dx} = \frac{M_B L}{4EI}\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)\right] \tag{7.4.47}$$

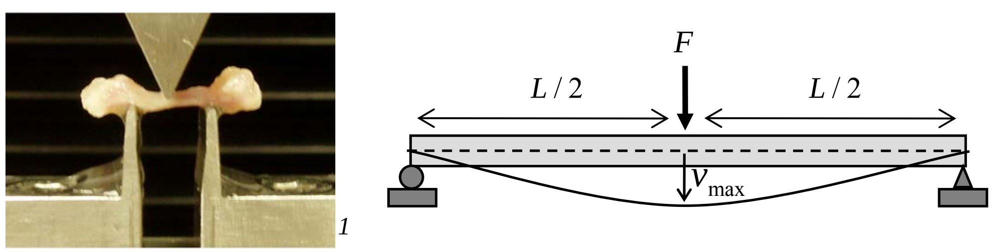
One can now return to the equilibrium equations to find the remaining reactions acting on the beam, which are $R_A = -R_B$ and $M_A = M_0 + LR_B$

■

7.4.9 The Three-point Bending Test

The 3-point bending test is a very useful experimental procedure. It is used to gather data on materials which are subjected to bending in service. It can also be used to get the Young's Modulus of a material for which it might be more difficult to get *via* a tension or other test.

A mouse bone is shown in the standard 3-point bend test apparatus in Fig. 7.4.37a. The idealised beam theory model of this test is shown in Fig. 7.4.37b. The central load is *F*, so the reactions at the supports are $F/2$. The moment is zero at the supports, varying linearly to a maximum $FL/4$ at the centre.



(a) (b)

Figure 7.4.37: the three-point bend test; (a) a mouse bone specimen, (b) idealised model

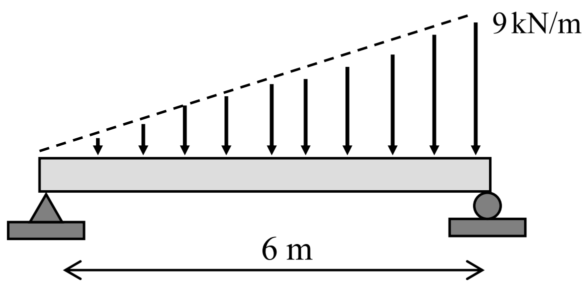
The maximum flexural stress then occurs at the outer fibres at the centre of the beam: for a circular cross-section, $\sigma_{\max} = FL/\pi R^3$. Integrating the moment-curvature equation, and using the fact that the deflection is zero at the supports and, from symmetry, the slope is zero at the centre, the maximum deflection is seen to be $v_{\max} = FL^3/12\pi R^4 E$. If one plots the load *F* against the deflection $v_{\max}$, one will see a straight line (initially, before the elastic limit is reached); let the slope of this line be $\hat{E}$. The Young's modulus can then be evaluated through

$$E = \frac{L^3}{12\pi R^4}\hat{E} \tag{7.4.48}$$

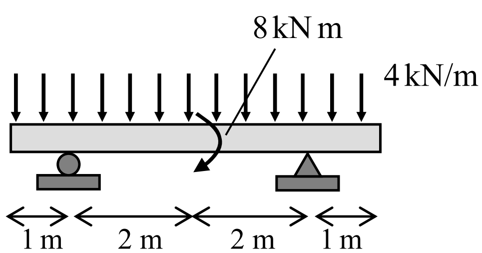
With $\sigma = E\varepsilon$, the maximum strain is $\varepsilon_{\max} = FL/\pi ER^3 = 12Rv_{\max}/L^2$. By carrying the test on beyond the elastic limit, the strength of the material at failure can be determined.

7.4.10 Problems

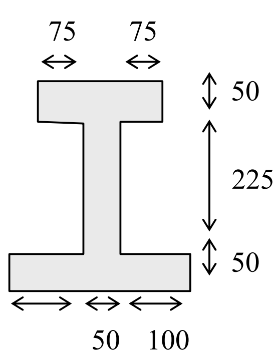
1. The simply supported beam shown below carries a vertical load that increases uniformly from zero at the left end to a maximum value of 9 kN/m at the right end. Draw the shearing force and bending moment diagrams



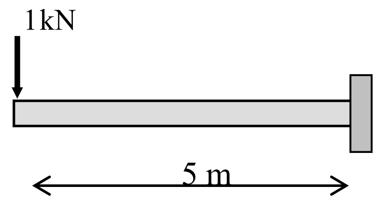
2. The beam shown below is simply supported at two points and overhangs the supports at each end. It is subjected to a uniformly distributed load of 4 kN/m as well as a couple of magnitude 8 kN m applied to the centre. Draw the shearing force and bending moment diagrams



3. Evaluate the centroid of the beam cross-section shown below (all measurements in mm)

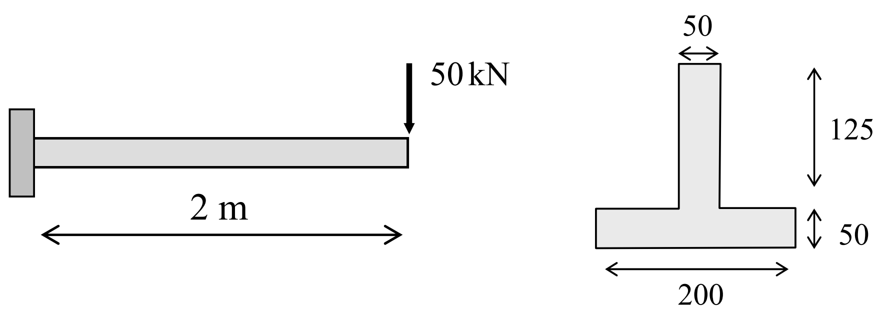


4. Determine the maximum tensile and compressive stresses in the following beam (it has a rectangular cross-section with height 75 mm and depth 50 mm)

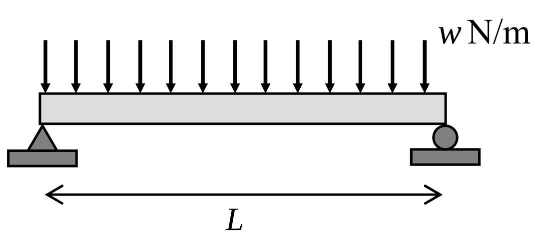


5. Consider the cantilever beam shown below. Determine the maximum shearing stress in the beam and determine the shearing stress 25 mm from the top surface of the beam at a section adjacent to the supporting wall. The cross-section is the "T" shape shown, for which $I = 40 \times 10^6 \text{ mm}^4$.

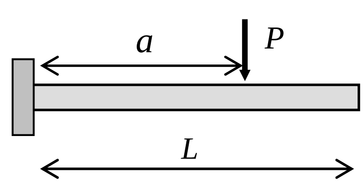
   [note: use the shear stress formula derived for rectangular cross-sections – as mentioned above, in this formula, *b* is the thickness of the beam *at the point where the shear stress is being evaluated*]



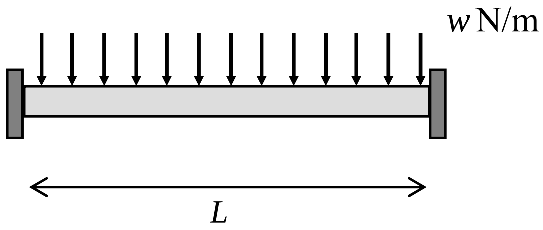
6. Obtain an expression for the maximum deflection of the simply supported beam shown here, subject to a uniformly distributed load of $w\,\text{N/m}$.



7. Determine the equation of the deflection curve for the cantilever beam loaded by a concentrated force *P* as shown below.



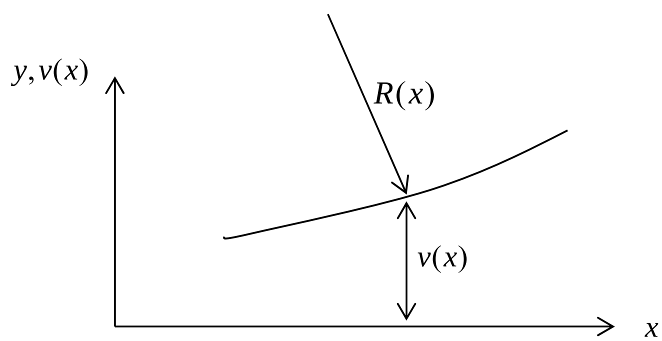
8. Determine the reactions for the following uniformly loaded beam clamped at both ends.



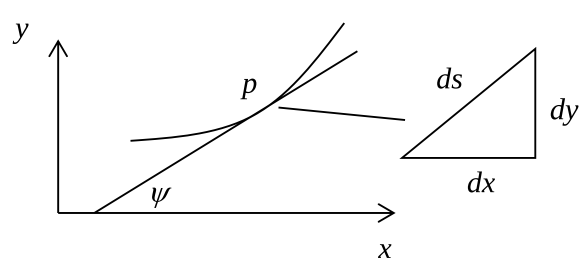
7.4.11 Appendix to §7.4

Curvature of the deflection curve

Consider a deflection curve with deflection $v(x)$ and radius of curvature $R(x)$, as shown in the figure below. Here, *deflection* is the transverse displacement (in the *y* direction) of the points that lie along the axis of the beam. A relationship between $v(x)$ and $R(x)$ is derived in what follows.



First, consider a curve (arc) *s*. The tangent to some point *p* makes an angle ψ with the *x* – axis, as shown below. As one move along the arc, ψ changes.



Define the **curvature** κ of the curve to be the rate at which ψ increases relative to *s*,

$$\kappa = \frac{d\psi}{ds}$$

Thus if the curve is very "curved", ψ is changing rapidly as one moves along the curve (as one increase *s*) and the curvature will be large.

From the above figure,

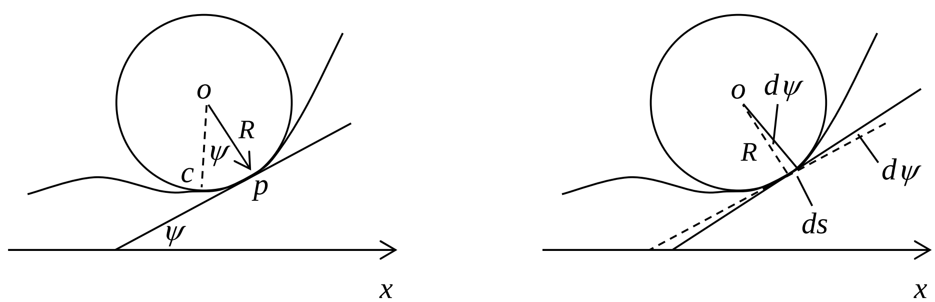
$$\tan\psi = \frac{dy}{dx}, \quad \frac{ds}{dx} = \frac{\sqrt{(dx)^2 + (dy)^2}}{dx} = \sqrt{1 + (dy/dx)^2},$$

so that

$$\kappa = \frac{d\psi}{ds} = \frac{d\psi}{dx}\frac{dx}{ds} = \frac{d\left(\arctan(dy/dx)\right)}{dx}\frac{dx}{ds} = \frac{1}{1+(dy/dx)^2}\frac{d^2y}{dx^2}\frac{dx}{ds}$$
$$= \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}}$$

Finally, it will be shown that the curvature is simply the reciprocal of the radius of curvature. Draw a circle to the point *p* with radius R. Arbitrarily measure the arc length *s* from the point *c*, which is a point on the circle such that $\angle cop = \psi$. Then arc length $s = R\psi$, so that

$$\kappa = \frac{d\psi}{ds} = \frac{1}{R}$$



Thus

$$\frac{1}{R} = \frac{\frac{d^2v}{dx^2}}{\left[1+\left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$$

If one assumes now that the slopes of the deflection curve are small, then $dv/dx << 1$ and

$$\frac{1}{R} \approx \frac{d^2v}{dx^2}$$

Images used:

1. http://www.mc.vanderbilt.edu/root/vumc.php?site=CenterForBoneBiology&doc=20412

7.5 Elastic Buckling

The initial theory of the buckling of columns was worked out by Euler in 1757, a nice example of a theory preceding the application, the application mainly being for the later "invented" metal and concrete columns in modern structures.

7.5.1 Columns and Buckling

A **column** is a long slender bar under axial compression, Fig. 7.5.1. A column can be horizontal, vertical or inclined; in the latter cases it is termed a **strut**.

The column under axial compression responds elastically in exactly the same way as the axial bar of §7.1. For example, it decreases in length under a compressive force P by an amount given by Eqn. 7.1.5, $\Delta = PL/EA$. However, when the compressive force is large enough, the column will **buckle** with lateral deflection. This possibility is the subject of this section.

Euler's Theory of Buckling

Consider an elastic column of length L, pin-ended so free to rotate at its ends, subjected to an axial load P, Fig. 7.5.1. Assume that it undergoes a lateral deflection denoted by v. Moment equilibrium of a section of the deflected column cut at a typical point x, and using the moment-curvature Eqn. 7.4.37, results in

$$-Pv(x) = M(x) = EI\frac{d^2 v}{dx^2} \tag{7.5.1}$$

Hence the deflection v satisfies the differential equation

$$\frac{d^2 v}{dx^2} + k^2 v(x) = 0 \tag{7.5.2}$$

where

$$k^2 = \frac{P}{EI} \tag{7.5.3}$$

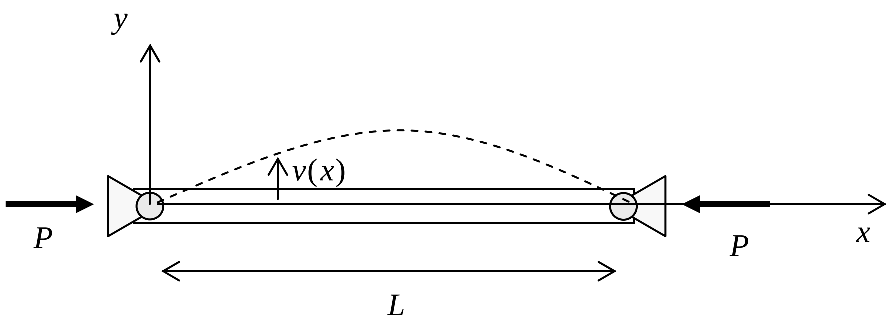


Fig. 7.5.1: a column with deflection *v*

The ordinary differential equation 7.5.2 is linear, homogeneous and with constant coefficients. Its solution can be found in any standard text on differential equations and is given by (for $k^2 > 0$)

$$v(x) = A\cos(kx) + B\sin(kx) \tag{7.5.4}$$

where A and B are as yet unknown constants. The boundary conditions for pinned-ends are

$$v(0) = 0, \qquad v(L) = 0 \tag{7.5.5}$$

The first condition requires A to be zero and the second leads to

$$B\sin(kL) = 0 \tag{7.5.6}$$

It follows that either:
(a) $B = 0$, in which case $v(x) = 0$ for all x and the column is not deflected

or

(b) $\sin(kL) = 0$, which holds when kL is an integer number of π's, i.e.

$$k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots, \tag{7.5.7}$$

As mentioned, the solution (a) is governed by the axial deformation theory discussed in §7.1. Concentrating on (b), the corresponding solution for the deflection is

$$v_n(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{7.5.8}$$

The parameter k is defined by Eqn. 7.5.3, so that, using 7.5.7,

$$P_n = EI\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{7.5.9}$$

It has hence been shown that buckling, i.e. $v \neq 0$, can only occur at a discreet set of applied loads - the **buckling loads** - given by 7.5.9. In practice the most important buckling load is the first, corresponding to $n = 1$, since this will be the first of the loads reached as the applied load P is increased from zero; this is called the **critical buckling load**:

$$\boxed{P_c = EI\left(\frac{\pi}{L}\right)^2} \tag{7.5.10}$$

with associated deflection

$$v_1(x) = B\sin\left(\frac{\pi x}{L}\right) \tag{7.5.11}$$

The column hence deforms into a single sine wave, which is termed the **mode** or **mode shape** of the deflected column. Note that B, the amplitude of the deflection, can not be determined by this model. This is a consequence of assuming the deflection is small; of **linearising** the problem (which is inherent in the derivation of the moment-deflection curve, Eqn. 7.5.1). A more exact finite deformation theory has been worked out and is called the **theory of the elastica**, but this is not pursued here.

This mathematical structure, where one finds one can only get non-zero solutions of an equation for certain values of a parameter is very common in engineering and theoretical physics. The critical values of the parameter, in this case *k*, are termed the **eigenvalues** of the problem, and the corresponding non-zero solutions, $v(x)$, are the **eigenfunctions**.

The second moment of area *I* has dimensions of $(\text{length})^4$, and for columns is often written in the form $I = Ar^2$ where *A* is the cross-sectional area of the column and the length *r* is called the **radius of gyration**. For example in the case of a circular shaft of radius *a*, $I = \pi a^4/4$ (see Eqn. 7.4.23) so $r = a/2$.

Failure of the Column

The expression 7.5.10 for the critical buckling load can be written in terms of the radius of gyration:

$$P_{cr} = EAr^2\left(\frac{\pi}{L}\right)^2 \qquad \text{or} \qquad \frac{\sigma_{cr}}{E} = \frac{\pi^2}{(L/r)^2} \tag{7.5.12}$$

where σ_{cr} is the mean compressive stress on the loaded end of the column.

The second equation in 7.5.12 is the most convenient non-dimensional form of presenting theoretical and experimental results for buckling problems. The ratio L/r is called the **slenderness ratio**.

Failure of the column will occur in purely axial compression if the stress in the column reaches the yield stress of the material (see §5.2). On the other hand, if the critical buckling stress σ_{cr} is less than the yield stress, then the column will fail by buckling before the yield stress is reached.

Eqn. 7.5.12 is plotted in Fig. 7.5.2. The yield stress of the material is denoted by *Y*. A critical slenderness ratio is denoted by $(L/r)_{cr}$. For slenderness ratios less than the critical value, that is, for relatively squat columns, the stress in the column will reach the yield stress before buckling occurs.

For example, consider a steel column for which $E = 210\text{ GPa}$ and $Y = 210\text{ MPa}$. The critical value of the slenderness ratio is then $L/r = 99.35$, which is a length to diameter ratio of about 25 for a circular column. Buckling will then occur in such columns which have $L/r > 99.35$, for sufficiently high applied axial compressive force.

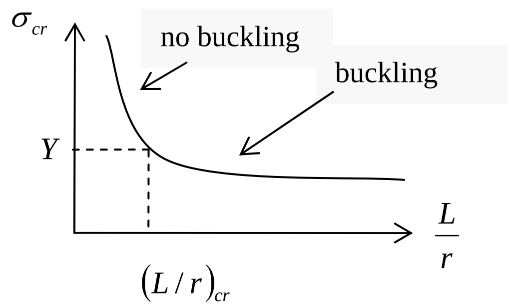


Fig. 7.5.2: critical values of the slenderness ratio

7.5.2 A General Approach to Buckling

The model developed above only applies to columns simply supported at each end. To discuss the more general case one can return to the formulation of the bending of a beam discussed in §7.4.3, but include also axial forces. Fig. 7.4.18 is reproduced as Fig. 7.5.3 but now with compressive axial forces, the forces offset by a small increment in deflection Δv.

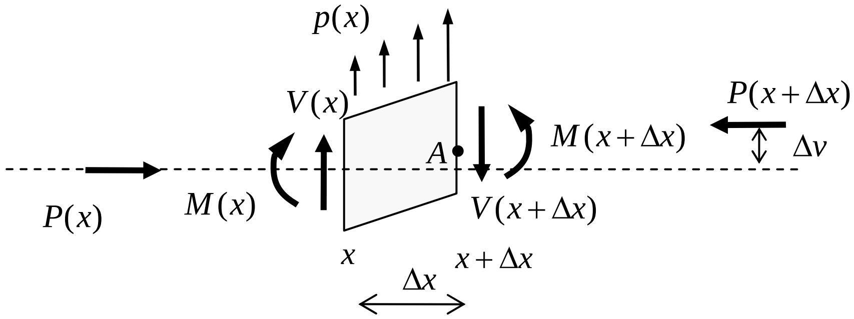


Figure 7.5.3: forces and moments acting on a column

Resolving vertically, one again arrives at Eqn. 7.4.10:

$$\frac{dV}{dx} = p(x) \tag{7.5.13}$$

Resolving horizontally, one simply gets $P(x) = P(x + \Delta x)$, so that P is constant. Taking moments, one has, instead of 7.4.13,

$$\frac{dM}{dx} + P\frac{dv}{dx} = V \tag{7.5.14}$$

Note the extra term involving *P*, which is not present in pure bending theory. Eliminating *M* between 7.5.14 and the moment-curvature equation 7.4.37 then leads to an expression for the shear force:

$$\frac{V}{EI} = \frac{d^3 v}{dx^3} + \frac{P}{EI}\frac{dv}{dx} \tag{7.5.15}$$

Note that, in the beam theory, where $P = 0$, the third derivative of the deflection is zero whenever the shear force is zero, in particular at a free, i.e. unsupported, end. Here, however, it is no longer true that the third derivative is zero.

The final differential equation is now obtained by differentiating 7.5.15 and using 7.5.13:

$$\frac{d^4 v}{dx^4} + \frac{P}{EI}\frac{d^2 v}{dx^2} = \frac{p}{EI} \tag{7.5.16}$$

Concentrating on the buckling behaviour and so neglecting the transverse load $p(x)$ [1], one arrives at the differential equation

$$\frac{d^4 v}{dx^4} + k^2 \frac{d^2 v}{dx^2} = 0 \tag{7.5.17}$$

where again $k^2 = P / EI$ (Eqn. 7.5.3). Eqn. 7.5.17 is a homogeneous fourth-order differential equation and its solution is

$$v(x) = A\cos(kx) + B\sin(kx) + Cx + D \tag{7.5.18}$$

The four constants are determined by the end conditions on $v(x)$, two conditions at each end. There are three cases:

(1) Pinned end:
boundary conditions are $v = 0$ and $M = 0$; from the moment-curvature equation, $M = 0$ can be replaced with $d^2 v / dx^2 = 0$

(2) Fixed end:
boundary conditions are $v = 0$, $dv / dx = 0$

(3) Free end:
Boundary conditions are $M = 0$ and $V = 0$; again, this implies that $d^2 v / dx^2 = 0$ and, from Eqn. 7.5.15, $V = 0$ can be replaced with $d^3 v / dx^3 + k^2 (dv / dx) = 0$

The case of pinned-pinned results again in the Euler solution given above. Consider now the case where one end is clamped and the other, loaded, end, is unrestrained ("fixed-free"), Fig. 7.5.4.

[1] bars subjected to both axial compressive loads and transverse loads are called **beam-columns**

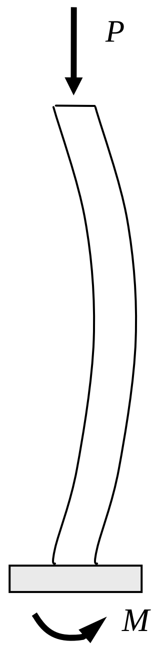


Fig. 7.5.4: a fixed-free column

At the clamped end, $v(0) = v'(0) = 0$, giving

$$A + D = 0, \qquad C + kB = 0 \tag{7.5.19}$$

At the free end, $v''(L) = 0$ and $v'''(L) + k^2 v'(L) = 0$, leading to

$$A\cos(kx) + B\sin(kx) = 0, \qquad C = 0 \tag{7.5.20}$$

Thus, from 7.5.19, *B* too is zero and *A* satisfies

$$A\cos(kL) = 0 \tag{7.5.21}$$

Buckling hence can only occur when $\cos(kL) = 0$, i.e. when

$$kL = \pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{7.5.22}$$

Using the definition of the parameter *k* the buckling loads are given by

$$P = EI\left[\frac{\pi\left(n + \frac{1}{2}\right)}{L}\right]^2, \quad n = 0, 1, 2, \ldots \tag{7.5.23}$$

with the critical buckling load now

$$P_{cr} = EI\left(\frac{\pi}{2L}\right)^2 \tag{7.5.24}$$

which is one quarter of the value for a pinned strut, Eqn. 7.5.10. The buckling modes are given by 7.5.18:

$$v(x) = D\left\{1 - \cos\left[\left(n + \tfrac{1}{2}\right)\pi \frac{x}{L}\right]\right\}, \quad n = 0, 1, 2, \ldots \tag{7.5.25}$$

The first three modes are sketched in Fig. 7.5.5; again, the amplitude is unknown, only the shape.

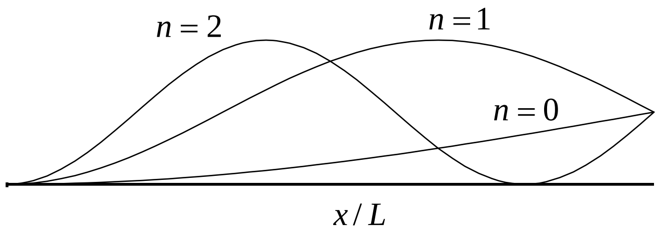


Figure 7.5.5: mode shapes for the fixed-free column

Other cases of end-support can be treated in the same way. Results for the critical buckling stress for various cases are sketched in Fig. 7.5.6.

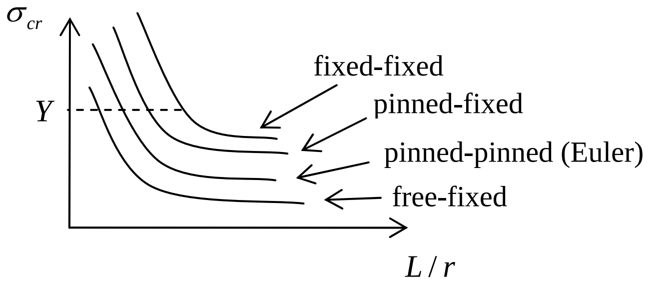


Fig. 7.5.6: critical values of the slenderness ratio for different end-cases