7.3 The Thin-walled Pressure Vessel Theory

An important practical problem is that of a cylindrical or spherical object which is subjected to an internal pressure p. Such a component is called a **pressure vessel**, Fig. 7.3.1. Applications arise in many areas, for example, the study of cellular organisms, arteries, aerosol cans, scuba-diving tanks and right up to large-scale industrial containers of liquids and gases.

In many applications it is valid to assume that

(i) the material is isotropic
(ii) the strains resulting from the pressures are small
(iii) the wall thickness t of the pressure vessel is much smaller than some characteristic radius: $t = r_o - r_i << r_o, r_i$

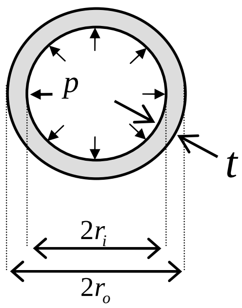


Figure 7.3.1: A pressure vessel (cross-sectional view)

Because of (i,ii), the isotropic linear elastic model is used. Because of (iii), it will be assumed that there is negligible variation in the stress field across the thickness of the vessel, Fig. 7.3.2.

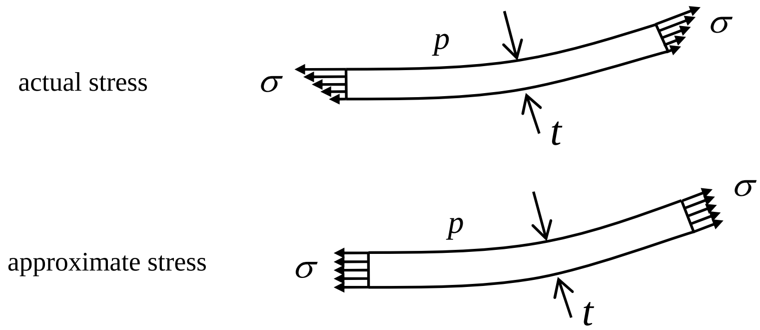


Figure 7.3.2: Approximation to the stress arising in a pressure vessel

As a rule of thumb, if the thickness is less than a tenth of the vessel radius, then the actual stress will vary by less than about 5% through the thickness, and in these cases the constant stress assumption is valid.

Note that a pressure $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p_i$ means that the stress on *any* plane drawn inside the vessel is subjected to a normal stress $-p_i$ and zero shear stress (see problem 6 in section 3.5.7).

7.3.1 Thin Walled Spheres

A thin-walled spherical shell is shown in Fig. 7.3.3. Because of the symmetry of the sphere and of the pressure loading, the **circumferential** (or **tangential** or **hoop) stress** σ_t at any location and in any tangential orientation must be the same (and there will be zero shear stresses).

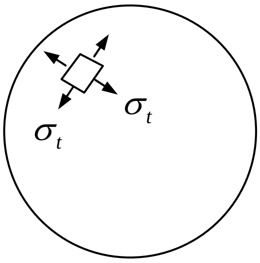


Figure 7.3.3: a thin-walled spherical pressure vessel

Considering a free-body diagram of one half of the sphere, Fig. 7.3.4, force equilibrium requires that

$$\pi\left(r_o^2 - r_i^2\right)\sigma_t - \pi r_i^2 p = 0 \tag{7.3.1}$$

and so, with $r_0 = r_i + t$,

$$\sigma_t = \frac{r_i^2 p}{2r_i t + t^2} \tag{7.3.2}$$

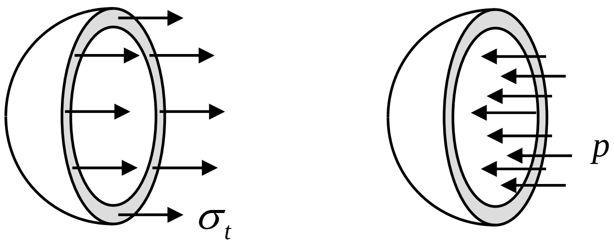


Figure 7.3.4: a free body diagram of one half of the spherical pressure vessel

One can now take as a characteristic radius the dimension *r*. This could be the inner radius, the outer radius, or the average of the two – results for all three should be close. Setting $r = r_i$ and neglecting the small terms $t^2 \ll 2r_i t$,

$$\boxed{\sigma_t = \frac{pr}{2t}} \quad \textbf{Tangential stress in a thin-walled spherical pressure vessel} \qquad (7.3.3)$$

This tangential stress accounts for the stress in the plane of the surface of the sphere. The stress normal to the walls of the sphere is called the **radial stress**, σ_r. The radial stress is zero on the outer wall since that is a free surface. On the inner wall, the normal stress is $\sigma_r = -p$, Fig. 7.3.5. From Eqn. 7.3.3, since $t/r << 1$, $p << \sigma_t$, and it is reasonable to take $\sigma_r = 0$ not only on the outer wall, but on the inner wall also. The stress state in the spherical wall is then one of plane stress.

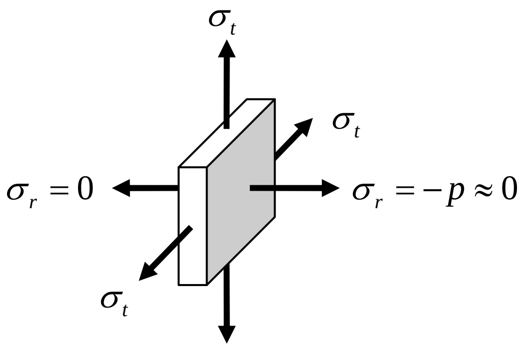


Figure 7.3.5: An element at the surface of a spherical pressure vessel

There are no in-plane shear stresses in the spherical pressure vessel and so the tangential and radial stresses are the principal stresses: $\sigma_1 = \sigma_2 = \sigma_t$, and the minimum principal stress is $\sigma_3 = \sigma_r = 0$. Thus the radial direction is one principal direction, and any two perpendicular directions in the plane of the sphere's wall can be taken as the other two principal directions.

Strain in the Thin-walled Sphere

The thin-walled pressure vessel expands when it is internally pressurised. This results in three principal strains, the **circumferential strain** ε_c (or **tangential strain** ε_t) in two perpendicular in-plane directions, and the **radial strain** ε_r. Referring to Fig. 7.3.6, these strains are

$$\varepsilon_c = \frac{A'C' - AC}{AC} = \frac{C'D' - CD}{CD}, \quad \varepsilon_r = \frac{A'B' - AB}{AB} \qquad (7.3.4)$$

From Hooke's law (Eqns. 6.1.8 with *z* the radial direction, with $\sigma_r = 0$),

$$\begin{bmatrix} \varepsilon_c \\ \varepsilon_c \\ \varepsilon_r \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E \\ -\nu/E & 1/E & -\nu/E \\ -\nu/E & -\nu/E & 1/E \end{bmatrix} \begin{bmatrix} \sigma_t \\ \sigma_t \\ \sigma_r \end{bmatrix} = \frac{1}{E}\frac{pr}{2t} \begin{bmatrix} 1-\nu \\ 1-\nu \\ -2\nu \end{bmatrix} \qquad (7.3.5)$$

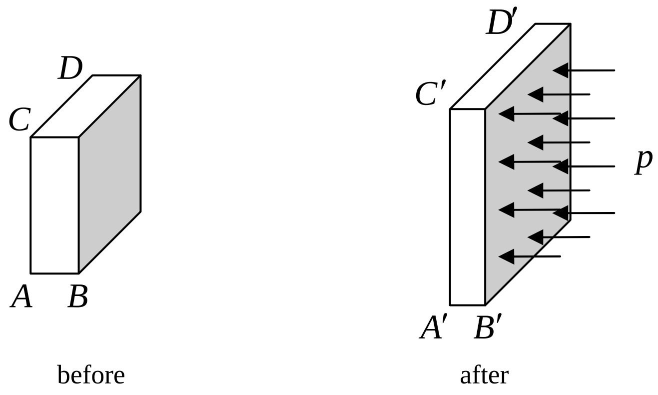


Figure 7.3.6: Strain of an element at the surface of a spherical pressure vessel

To determine the amount by which the vessel expands, consider a circumference at average radius *r* which moves out with a displacement δ_r, Fig. 7.3.7. From the definition of normal strain

$$\varepsilon_c = \frac{(r+\delta_r)\Delta\theta - r\Delta\theta}{r\Delta\theta} = \frac{\delta_r}{r} \tag{7.3.6}$$

This is the circumferential strain for points on the mid-radius. The strain at other points in the vessel can be approximated by this value.

The expansion of the sphere is thus

$$\delta_r = r\varepsilon_c = \frac{1-\nu}{E}\frac{pr^2}{2t} \tag{7.3.7}$$

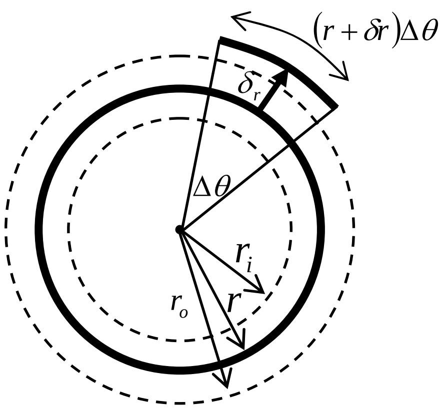


Figure 7.3.7: Deformation in the thin-walled sphere as it expands

To determine the amount by which the circumference increases in size, consider Fig. 7.3.8, which shows the original circumference at radius *r* of length *c* increase in size by an amount δ_c. One has

$$\delta_c = c\varepsilon_c = 2\pi r\varepsilon_c = 2\pi \frac{1-\nu}{E}\frac{pr^2}{2t} \tag{7.3.8}$$

It follows from Eqn. 7.3.7-8 that the circumference and radius increases are related through

$$\delta_c = 2\pi\delta_r \tag{7.3.9}$$

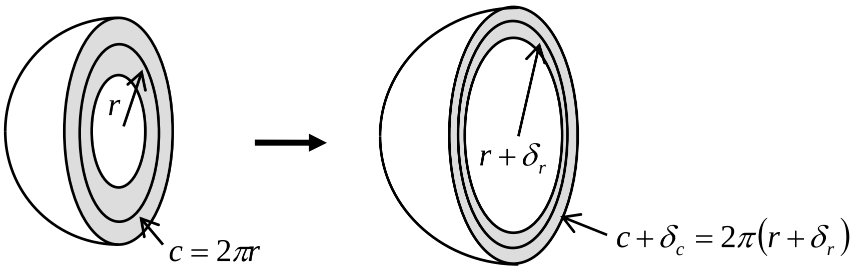


Figure 7.3.8: Increase in circumference length as the vessel expands

Note that the circumferential strain is *positive*, since the circumference is increasing in size, but the radial strain is *negative* since, as the vessel expands, the thickness decreases.

7.3.2 Thin Walled Cylinders

The analysis of a thin-walled internally-pressurised cylindrical vessel is similar to that of the spherical vessel. The main difference is that the cylinder has three different principal stress values, the circumferential stress, the radial stress, and the **longitudinal stress** σ_l, which acts in the direction of the cylinder axis, Fig. 7.3.9.

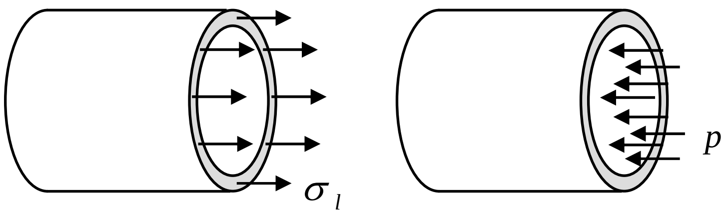


Figure 7.3.9: free body diagram of a cylindrical pressure vessel

Again taking a free-body diagram of the cylinder and carrying out an equilibrium analysis, one finds that, as for the spherical vessel,

$$\boxed{\sigma_l = \frac{pr}{2t}} \quad \textbf{Longitudinal stress in a thin-walled cylindrical pressure vessel} \tag{7.3.10}$$

Note that this analysis is only valid at positions sufficiently far away from the cylinder ends, where it might be closed in by caps – a more complex stress field would arise there.

The circumferential stress can be evaluated from an equilibrium analysis of the free body diagram in Fig. 7.3.10:

$$-\sigma_c 2tL + 2r_i Lp = 0 \tag{7.3.11}$$

and so

$$\boxed{\sigma_c = \frac{pr}{t}}$$ **Circumferential stress in a thin-walled cylindrical pressure vessel**

(7.3.12)

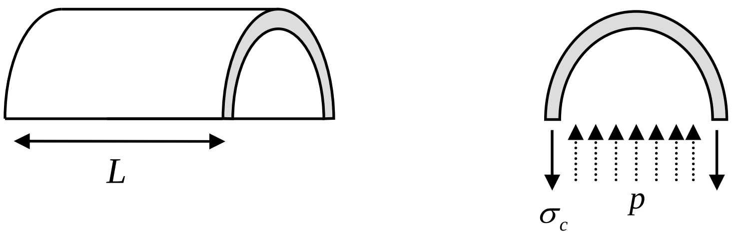


Figure 7.3.10: free body diagram of a cylindrical pressure vessel

As with the sphere, the radial stress varies from $-p$ at the inner surface to zero at the outer surface, but again is small compared with the other two stresses, and so is taken to be $\sigma_r = 0$.

Strain in the Thin-walled cylinder

The analysis of strain in the cylindrical pressure vessel is very similar to that of the spherical vessel. Eqns. 7.3.6 and 7.3.9 hold also here. Eqn. 7.3.5 would need to be amended to account for the three different principal stresses in the cylinder.

7.3.3 External Pressure

The analysis given above can be extended to the case where there is also an external pressure acting on the vessel. The internal pressure is now denoted by p_i and the external pressure is denoted by p_o, Fig. 7.3.11.

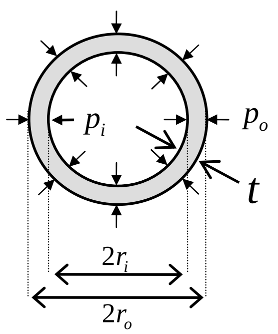


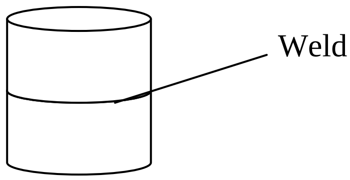
Figure 7.3.11: A pressure vessel subjected to internal and external pressure

In this case, the pressure p in formulae derived above can simply be replaced by $(p_i - p_o)$, which is known as the **gage pressure** (see the Appendix to this section, §7.3.5, for justification).

7.3.4 Problems

1. A 20m diameter spherical tank is to be used to store gas. The shell plating is 10 mm thick and the working stress of the material, that is, the maximum stress to which the material should be subjected, is 125 MPa. What is the maximum permissible gas pressure?

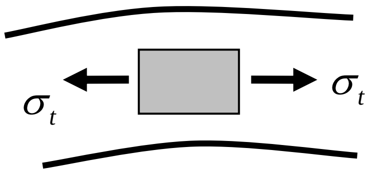
2. A steel propane tank for a BBQ grill has a 25cm diameter[1] and a wall thickness of 5mm (see figure). The tank is pressurised to 1.2 MPa.
   (a) determine the longitudinal and circumferential stresses in the cylindrical body of the tank
   (b) determine the absolute maximum shear stress in the cylindrical portion of the tank
   (c) determine the tensile force per cm length being supported by a weld joining the upper and lower sections of the tank.



3. What are the strains in the BBQ tank of question 2? What is the radial displacement? [take the steel to be isotropic with $E = 200\text{GPa}, \nu = 0.3$]

4. What are the strains in the cylindrical pressure vessel, in terms of E, ν, p, t and r?

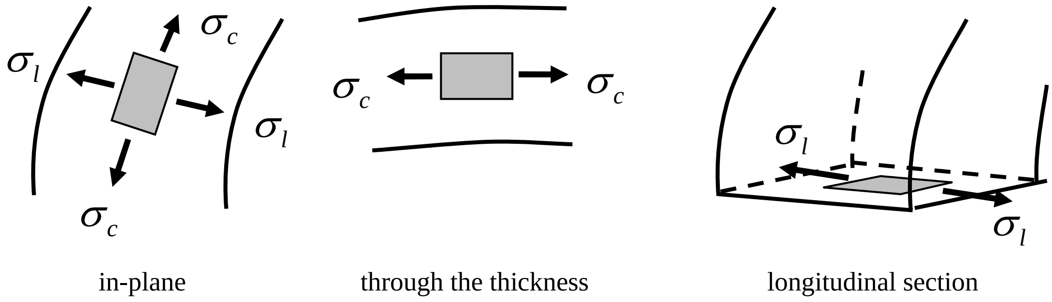
5. There are no shear stresses in the tangential plane of the spherical pressure vessel. However, there are shear stresses acting on planes through the thickness of the wall. A cross-section through the thickness is shown below. Take it that the radial stresses are zero. What are the maximum shear stresses occurring on this cross section?



6. The three perpendicular planes in the cylindrical pressure vessel are the in-plane, through the thickness and longitudinal sections, as shown below. The non-zero (principal) stresses acting on these planes are also shown. Evaluate the maximum

[1] this is an average diameter – the inside is 250-5mm and the outside is 250+5mm

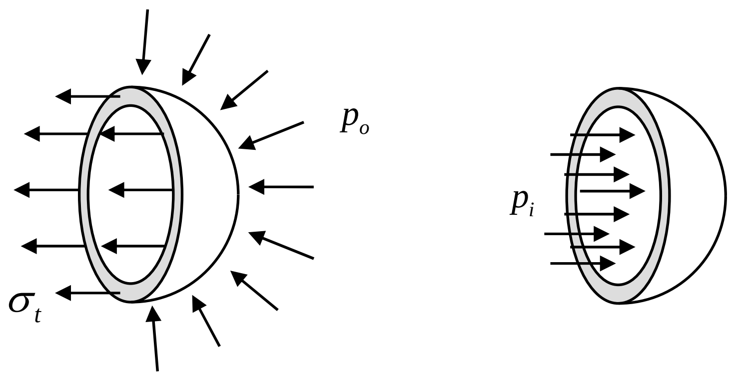
shear stresses on each of these three planes. Which of these three maxima is the overall maximum shear stress acting in the vessel?



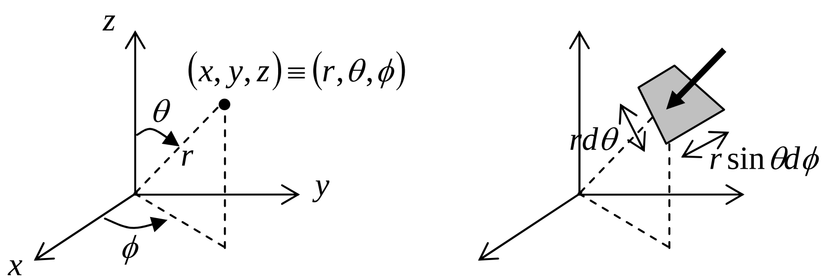
7.3.5 Appendix to §7.3

Equilibrium of a Pressure Vessel with both internal and external pressure

Consider the spherical pressure vessel. An external pressure p_o is distributed around its outer surface. Consider a free-body diagram of one half of the vessel, as shown below.



The force due to the external pressure acting in the horizontal direction can be evaluated using the spherical coordinates shown below.



An element of surface area upon which the pressure acts, swept out when the angles change by $d\theta$ and $d\phi$, has sides $r d\theta$ and $r \sin\theta d\phi$. The force acting on this area is then $p_o r^2 \sin\theta d\theta d\phi$. Force equilibrium in the horizontal (y) direction then leads to

$$-r_o^2 p_o \int_0^{\pi} \sin^2 \theta d\theta \int_0^{\pi} \sin \phi d\phi - \pi\left(r_o^2 - r_i^2\right)\sigma_t + \pi r_i^2 p_i = 0$$

and so,

$$\sigma_t = \frac{r_i^2 p_i - r_o^2 p_o}{\left(r_0 + r_i\right)t}$$

or $\sigma_t \approx (p_i - p_o)r / 2t$ – see Eqn. 7.3.3.