6 Linear Elasticity

The simplest constitutive law for solid materials is the linear elastic law, which assumes a linear relationship between stress and engineering strain. This assumption turns out to be an excellent predictor of the response of components which undergo small deformations, for example steel and concrete structures under large loads, and also works well for practically any material at a sufficiently small load.

The linear elastic model is discussed in this chapter and some elementary problems involving elastic materials are solved. Anisotropic elasticity is discussed in Section 6.3.

6.1 The Linear Elastic Model

6.1.1 The Linear Elastic Model

Repeating some of what was said in Section 5.3: the Linear Elastic model is used to describe materials which respond as follows:

(i) the strains in the material are small[1] (**linear**)
(ii) the stress is proportional to the strain, $\sigma \propto \varepsilon$ (**linear**)
(iii) the material returns to its original shape when the loads are removed, and the unloading path is the same as the loading path (**elastic**)
(iv) there is no dependence on the rate of loading or straining (**elastic**)

From the discussion in the previous chapter, this model well represents the engineering materials up to their elastic limit. It also models well almost any material provided the stresses are sufficiently small.

The stress-strain (loading and unloading) curve for the Linear Elastic solid is shown in Fig. 6.1.1a. Other possible responses are shown in Figs. 6.1.1b,c. Fig. 6.1.1b shows the typical response of a rubbery-type material and many biological tissues; these are **non-linear elastic** materials. Fig. 6.1.1c shows the typical response of **viscoelastic** materials (see Chapter 10) and that of many plastically and viscoplastically deforming materials (see Chapters 11 and 12).

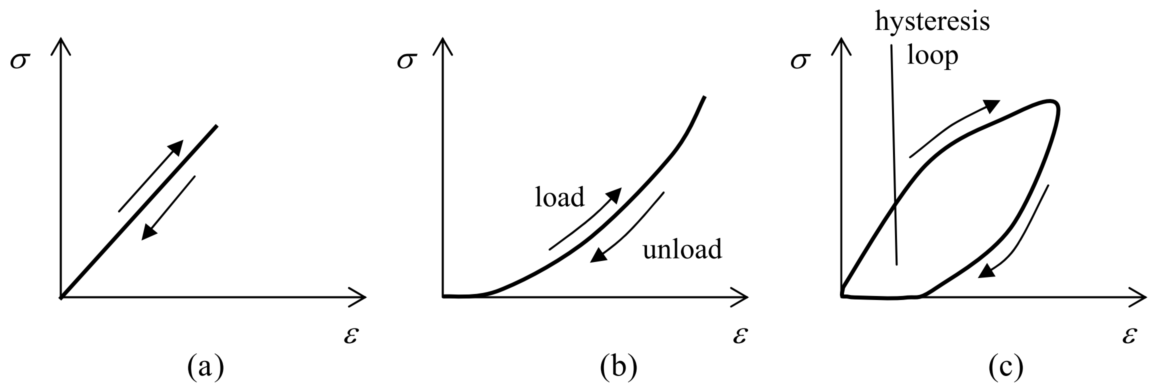


Figure 6.1.1: Different stress-strain relationships; (a) linear elastic, (b) non-linear elastic, (c) viscoelastic/plastic/viscoplastic

It will be assumed at first that the material is isotropic and homogeneous. The case of an anisotropic elastic material is discussed in Section 6.3.

[1] if the small-strain approximation is not made, the stress-strain relationship will be inherently non-linear; the actual strain, Eqn. 4.1.7, involves (non-linear) squares and square-roots of lengths

6.1.2 Stress-Strain Law

Consider a cube of material subjected to a uniaxial tensile stress σ_{xx}, Fig. 6.1.2a. One would expect it to respond by extending in the *x* direction, $\varepsilon_{xx} > 0$, and to contract laterally, so $\varepsilon_{yy} = \varepsilon_{zz} < 0$, these last two being equal because of the isotropy of the material. With stress proportional to strain, one can write

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} \tag{6.1.1}$$

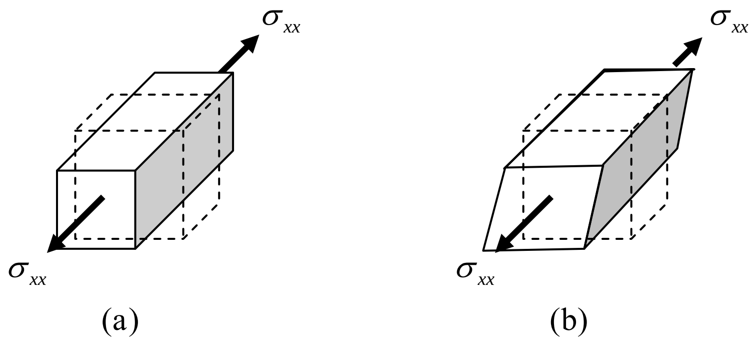


Figure 6.1.2: an element of material subjected to a uniaxial stress; (a) normal strain, (b) shear strain

The constant of proportionality between the normal stress and strain is the Young's Modulus, Eqn. 5.2.5, the measure of the stiffness of the material. The material parameter ν is the Poisson's ratio, Eqn. 5.2.6. Since $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$, it is a measure of the contraction relative to the normal extension.

Because of the isotropy/symmetry of the material, the shear strains are zero, and so the deformation of Fig. 6.1.2b, which shows a non-zero ε_{xy}, is not possible – shear strain can arise if the material is not isotropic.

One can write down similar expressions for the strains which result from a uniaxial tensile σ_{yy} stress and a uniaxial σ_{zz} stress:

$$\begin{aligned} \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy}, \quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{zz} &= \frac{1}{E}\sigma_{zz}, \quad \varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{zz} \end{aligned} \tag{6.1.2}$$

Similar arguments can be used to write down the shear strains which result from the application of a shear stress:

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz} \tag{6.1.3}$$

The constant of proportionality here is the Shear Modulus μ, Eqn. 5.2.8, the measure of the resistance to shear deformation (the letter *G* was used in Eqn. 5.2.8 – both *G* and μ are used to denote the Shear Modulus, the latter in more "mathematical" and "advanced" discussions) .

The strain which results from a combination of all six stresses is simply the sum of the strains which result from each[2]:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \quad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz}, \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right], \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz} \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]
\end{aligned} \tag{6.1.4}$$

These equations involve three material parameters. It will be proved in §6.3 that an isotropic linear elastic material can have only *two* independent material parameters and that, in fact,

$$\mu = \frac{E}{2(1+\nu)}. \tag{6.1.5}$$

This relation will be verified in the following example.

Example: Verification of Eqn. 6.1.5

Consider the simple shear deformation shown in Fig. 6.1.3, with $\varepsilon_{xy} > 0$ and all other strains zero. With the material linear elastic, the only non-zero stress is $\sigma_{xy} = 2\mu\varepsilon_{xy}$.

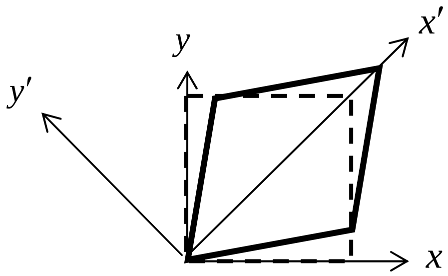


Figure 6.1.3: a simple shear deformation

[2] this is called the **principle of linear superposition**: the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause". For a linear relation, e.g. $\sigma = E\varepsilon$, the effects of two causes σ_1, σ_2 are $E\varepsilon_1$ and $E\varepsilon_2$, and the effect of the sum of the causes $\sigma_1 + \sigma_2$ is indeed equal to the sum of the individual effects: $E(\varepsilon_1 + \varepsilon_2) = E\varepsilon_1 + E\varepsilon_2$. This is not true of a non-linear relation, e.g. $\sigma = E\varepsilon^2$, since $E(\varepsilon_1 + \varepsilon_2)^2 \neq E\varepsilon_1^2 + E\varepsilon_2^2$

Using the strain transformation equations, Eqns. 4.2.2, the only non-zero strains in a second coordinate system $x' - y'$, with x' at $\theta = 45^{\circ}$ from the *x* axis (see Fig. 6.1.3), are $\varepsilon'_{xx} = +\varepsilon_{xy}$ and $\varepsilon'_{yy} = -\varepsilon_{xy}$. Because the material is isotropic, Eqns 6.1.4 hold also in this second coordinate system and so the stresses in the new coordinate system can be determined by solving the equations

$$
\begin{aligned}
\varepsilon'_{xx} &= +\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{xx} - \nu\left(\sigma'_{yy} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{xy} = 0 = \frac{1}{2\mu}\sigma'_{xy}, \quad \varepsilon'_{xz} = 0 = \frac{1}{2\mu}\sigma'_{xz}, \\
\varepsilon'_{yy} &= -\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{yy} - \nu\left(\sigma'_{xx} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{yz} = 0 = \frac{1}{2\mu}\sigma'_{yz} \\
\varepsilon'_{zz} &= 0 \quad = \frac{1}{E}\left[\sigma'_{zz} - \nu\left(\sigma'_{xx} + \sigma'_{yy}\right)\right]
\end{aligned}
\qquad (6.1.6)
$$

which results in

$$
\sigma'_{xx} = +\frac{E}{1+\nu}\varepsilon_{xy}, \quad \sigma'_{yy} = -\frac{E}{1+\nu}\varepsilon_{xy} \qquad (6.1.7)
$$

But the stress transformation equations, Eqns. 3.4.8, with $\sigma_{xy} = 2\mu\varepsilon_{xy}$, give $\sigma'_{xx} = +2\mu\varepsilon_{xy}$ and $\sigma'_{yy} = -2\mu\varepsilon_{xy}$ and so Eqn. 6.1.5 is verified.

■

Relation 6.1.5 allows the Linear Elastic Solid stress-strain law, Eqn. 6.1.4, to be written as

$$
\boxed{
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned}
}
\qquad \textbf{Stress-Strain Relations} \qquad (6.1.8)
$$

This is known as **Hooke's Law**. These equations can be solved for the stresses to get

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \\
\sigma_{xz} &= \frac{E}{1+\nu}\varepsilon_{xz} \\
\sigma_{yz} &= \frac{E}{1+\nu}\varepsilon_{yz}
\end{aligned}$$

Stress-Strain Relations (6.1.9)

Values of E and ν for a number of materials are given in Table 6.1.1 below (see also Table 5.2.2).

Material	*E* (GPa)	ν
Grey Cast Iron	100	0.29
A316 Stainless Steel	196	0.3
A5 Aluminium	68	0.33
Bronze	130	0.34
Plexiglass	2.9	0.4
Rubber	0.001-2	0.4-0.49
Concrete	23-30	0.2
Granite	53-60	0.27
Wood (pinewood) fibre direction transverse direction	 17 1	 0.45 0.79

Table 6.1.1: Young's Modulus *E* and Poisson's Ratio ν for a selection of materials at 20°C

Volume Change

Recall that the volume change in a material undergoing small strains is given by the sum of the normal strains (see Section 4.3). From Hooke's law, normal stresses cause normal strain and shear stresses cause shear strain. It follows that *normal stresses produce volume changes* and *shear stresses produce distortion* (change in shape), but no volume change.

6.1.3 Two Dimensional Elasticity

The above three-dimensional stress-strain relations reduce in the case of a two-dimensional stress state or a two-dimensional strain state.

Plane Stress

In plane stress (see Section 3.5), $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, Fig. 6.1.5, so the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left[\nu\varepsilon_{xx} + \varepsilon_{yy}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Stress) (6.1.10)

with

$$\begin{aligned}
&\varepsilon_{zz} = -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right], \quad \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
&\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \qquad (6.1.11)$$

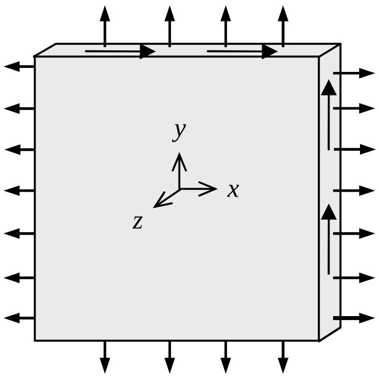


Figure 6.1.5: Plane stress

Note that the ε_{zz} strain is *not* zero. Physically, ε_{zz} corresponds to a change in thickness of the material perpendicular to the direction of loading.

Plane Strain

In plane strain (see Section 4.2), $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$, Fig. 6.1.6, and the stress-strain relations reduce to

$$
\boxed{
\begin{aligned}
\varepsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}
}
$$

Stress-Strain Relations (Plane Strain) (6.1.12)

with

$$
\begin{gathered}
\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
\sigma_{zz} = \nu\left[\sigma_{xx} + \sigma_{yy}\right], \quad \sigma_{xz} = \sigma_{yz} = 0
\end{gathered}
\tag{6.1.13}
$$

Again, note here that the stress component σ_{zz} is *not* zero. Physically, this stress corresponds to the forces preventing movement in the *z* direction.

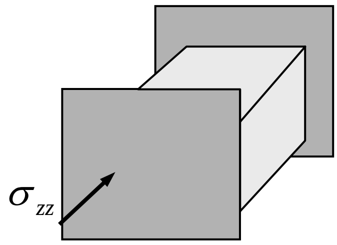


Figure 6.1.6 Plane strain - a thick component constrained in one direction

Similar Solutions

The expressions for plane stress and plane strain are very similar. For example, the plane strain constitutive law 6.1.12 can be derived from the corresponding plane stress expressions 6.1.10 by making the substitutions

$$
E = \frac{E'}{1-\nu'^2}, \quad \nu = \frac{\nu'}{1-\nu'} \tag{6.1.14}
$$

in 6.1.10 and then dropping the primes. The plane stress expressions can be derived from the plane strain expressions by making the substitutions

$$
E = E'\frac{1+2\nu'}{(1+\nu')^2}, \quad \nu = \frac{\nu'}{1+\nu'} \tag{6.1.15}
$$

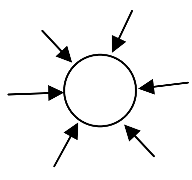
in 6.1.12 and then dropping the primes. Thus, if one solves a plane stress problem, one has automatically solved the corresponding plane strain problem, and *vice versa*.

6.1.4 Problems

1. A strain gauge at a certain point on the surface of a thin A5 Aluminium component (loaded in-plane) records strains of $\varepsilon_{xx} = 60\,\mu\text{m}$, $\varepsilon_{yy} = 30\,\mu\text{m}$, $\varepsilon_{xy} = 15\,\mu\text{m}$. Determine the principal stresses. (See Table 6.1.1 for the material properties.)

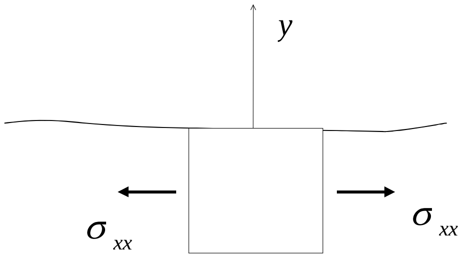
2. Use the stress-strain relations to prove that, for a linear elastic solid,
$$\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}$$
and, indeed,
$$\frac{2\sigma_{xz}}{\sigma_{xx} - \sigma_{zz}} = \frac{2\varepsilon_{xz}}{\varepsilon_{xx} - \varepsilon_{zz}}, \quad \frac{2\sigma_{yz}}{\sigma_{yy} - \sigma_{zz}} = \frac{2\varepsilon_{yz}}{\varepsilon_{yy} - \varepsilon_{zz}}$$
Note: from Eqns. 3.5.4 and 4.2.4, these show that the principal axes of stress and strain coincide for an *isotropic* elastic material

3. Consider the case of hydrostatic pressure in a linearly elastic solid:
$$[\sigma_{ij}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$
as might occur, for example, when a spherical component is surrounded by a fluid under high pressure, as illustrated in the figure below. Show that the volumetric strain (Eqn. 4.3.5) is equal to
$$-p\frac{3(1-2\nu)}{E},$$
so that the Bulk Modulus, Eqn. 5.2.9, is
$$K = \frac{E}{3(1-2\nu)}$$



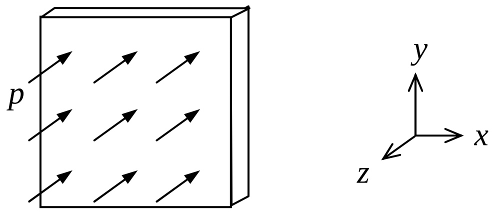
4. Consider again Problem 2 from §3.5.7.
   (a) Assuming the material to be linearly elastic, what are the strains? Draw a second material element (superimposed on the one shown below) to show the deformed shape of the square element – assume the displacement of the box-centre to be zero and that there is no rotation. Note how the free surface moves, even though there is no stress acting on it.

(b) What are the principal strains ε_1 and ε_2? You will see that the principal directions of stress and strain coincide (see Problem 2) – the largest normal stress and strain occur in the same direction.



5. Consider a very thin sheet of material subjected to a normal pressure *p* on one of its large surfaces. It is fixed along its edges. This is an example of a **plate** problem, an important branch of elasticity with applications to boat hulls, aircraft fuselage, etc.
   (a) write out the complete three dimensional stress-strain relations (both cases, strain in terms of stress, Eqns. 6.1.8, stress in terms of strain, Eqns. 6.1.9). Following the discussion on thin plates in section 3.5.4, the shear stresses σ_{xz}, σ_{yz}, can be taken to be zero throughout the plate. Simplify the relations using this fact, the pressure boundary condition on the large face and the coordinate system shown.
   (b) assuming that the through thickness change in the sheet can be neglected, show that

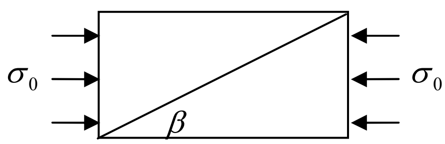
$$p = -\nu(\sigma_{xx} + \sigma_{yy})$$



6. A thin linear elastic rectangular plate with width *a* and height *b* is subjected to a uniform compressive stress σ_0 as shown below. Show that the slope of the plate diagonal shown after deformation is given by

$$\tan(\beta + \delta\beta) = \frac{b}{a}\left(\frac{1 + \nu\sigma_0 / E}{1 - \sigma_0 / E}\right)$$

What is the magnitude of $\delta\beta$ for a steel plate ($E = 210\text{GPa}$, $\nu = 0.3$) of dimensions $20 \times 20\ \text{cm}^2$ with $\sigma_0 = 1\text{MPa}$?



6.2 Homogeneous Problems in Linear Elasticity

A **homogeneous** stress (strain) field is one where the stress (strain) is the same at all points in the material. Homogeneous conditions will arise when the geometry is simple and the loading is simple.

6.2.1 Elastic Rectangular Cuboids

Hooke's Law, Eqns. 6.1.8 or 6.1.9, can be used to solve problems involving homogeneous stress and deformation. Hoooke's law is 6 equations in 12 unknowns (6 stresses and 6 strains). If some of these unknowns are given, the rest can be found from the relations.

Example

Consider the block of linear elastic material shown in Fig. 6.2.1. It is subjected to an equi-biaxial stress of $\sigma_{xx} = \sigma_{yy} = \bar{\sigma} > 0$.

Since this is an isotropic elastic material, the shears stresses and strains will be all zero for such a loading. One thus need only consider the three normal stresses and strains.

There are now 3 equations (the first 3 of Eqns. 6.1.8 or 6.1.9) in 6 unknowns. One thus needs to know *three* of the normal stresses and/or strains to find a solution. From the loading, one knows that $\sigma_{xx} = \bar{\sigma}$ and $\sigma_{yy} = \bar{\sigma}$. The third piece of information comes from noting that the surfaces parallel to the $x - y$ plane are free surfaces (no forces acting on them) and so $\sigma_{zz} = 0$.

From Eqn. 6.1.8 then, the strains are

$$\varepsilon_{xx} = \varepsilon_{yy} = (1-\nu)\frac{\bar{\sigma}}{E}, \quad \varepsilon_{zz} = -2\nu\frac{\bar{\sigma}}{E}, \quad \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

As expected, $\varepsilon_{xx} = \varepsilon_{yy}$ and $\varepsilon_{zz} < 0$.

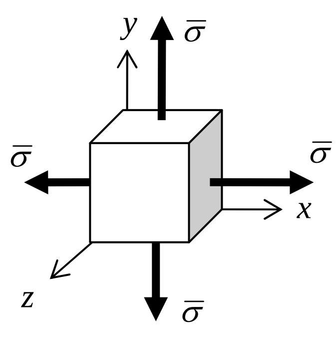


Figure 6.2.1: A block of linear elastic material subjected to an equi-biaxial stress

■

6.2.2 Problems

1. A block of isotropic linear elastic material is subjected to a compressive normal stress σ_o over two opposing faces. The material is constrained (prevented from moving) in one of the direction normal to these faces. The other faces are free.
   (a) What are the stresses and strains in the block, in terms of σ_o, ν, E ?
   (b) Calculate three maximum shear stresses, one for each plane (parallel to the faces of the block). Which of these is the overall maximum shear stress acting in the block?

2. Repeat problem 1a, only with the free faces now fixed also.

6.3 Anisotropic Elasticity

There are many materials which, although well modelled using the linear elastic model, are not nearly isotropic. Examples are wood, composite materials and many biological materials. The mechanical properties of these materials differ in different directions. Materials with this direction dependence are called anisotropic (see Section 5.2.7).

6.3.1 Material Constants

The most general form of Hooke's law, the **generalised Hooke's Law**, for a linear elastic material is

$$\begin{bmatrix} \sigma_1 = \sigma_{xx} \\ \sigma_2 = \sigma_{yy} \\ \sigma_3 = \sigma_{zz} \\ \sigma_4 = \sigma_{yz} \\ \sigma_5 = \sigma_{xz} \\ \sigma_6 = \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 = \varepsilon_{xx} \\ \varepsilon_2 = \varepsilon_{yy} \\ \varepsilon_3 = \varepsilon_{zz} \\ \varepsilon_4 = \varepsilon_{yz} \\ \varepsilon_5 = \varepsilon_{xz} \\ \varepsilon_6 = \varepsilon_{xy} \end{bmatrix} \tag{6.3.1}$$

where each stress component depends (linearly) on all strain components. This new notation, with only one subscript for the stress and strain, numbered from 1…6, is helpful as it allows the equations of anisotropic elasticity to be written in matrix form. The 36 C_{ij}'s are material constants called the **stiffnesses**, and in principle are to be obtained from experiment. The matrix of stiffnesses is called the **stiffness matrix**. Note that these equations imply that a normal stress σ_{xx} will induce a material element to not only stretch in the *x* direction and contract laterally, but to undergo shear strain too, as illustrated schematically in Fig. 6.3.1.

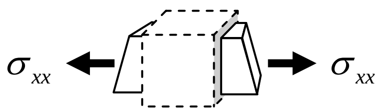


Figure 6.3.1: an element undergoing shear strain when subjected to a normal stress only

In section 8.4.3, when discussing the strain energy in an elastic material, it will be shown that it is necessary for the stiffness matrix to be *symmetric* and so there are only 21 independent elastic constants in the most general case of anisotropic elasticity.

Eqns. 6.3.1 can be inverted so that the strains are given explicitly in terms of the stresses:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad (6.3.2)$$

The S_{ij}'s here are called **compliances**, and the matrix of compliances is called the **compliance matrix**. The bottom half of the compliance matrix has been omitted since it too is symmetric.

It is difficult to model fully anisotropic materials due to the great number of elastic constants. Fortunately many materials which are not fully isotropic still have certain **material symmetries** which simplify the above equations. These material types are considered next.

6.3.2 Orthotropic Linear Elasticity

An **orthotropic** material is one which has three orthogonal planes of microstructural symmetry. An example is shown in Fig. 6.3.2a, which shows a glass-fibre composite material. The material consists of thousands of very slender, long, glass fibres bound together in bundles with oval cross-sections. These bundles are then surrounded by a plastic binder material. The continuum model of this composite material is shown in Fig. 6.3.2b wherein the fine microstructural details of the bundles and surrounding matrix are “smeared out” and averaged. Three mutually perpendicular planes of symmetry can be passed through each point in the continuum model. The x, y, z axes forming these planes are called the **material directions**.

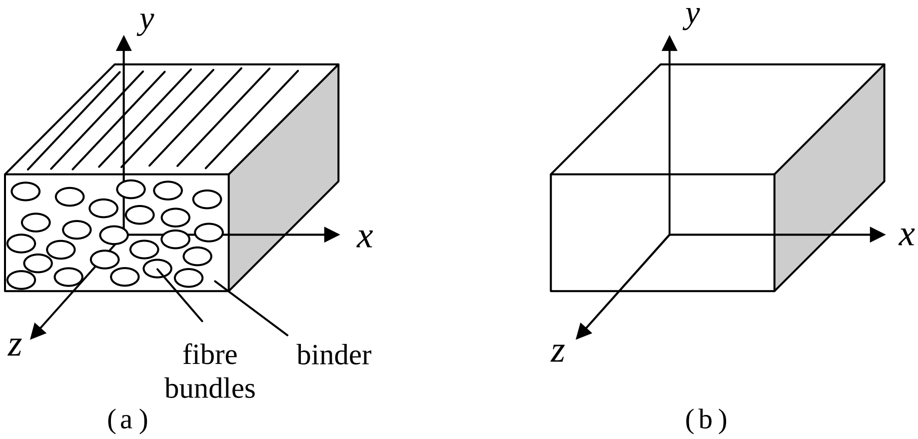


Figure 6.3.2: an orthotropic material; (a) microstructural detail, (b) continuum model

The material symmetry inherent in the orthotropic material reduces the number of independent elastic constants. To see this, consider an element of orthotropic material subjected to a shear strain $\varepsilon_6 \left(= \varepsilon_{xy}\right)$ and also a strain $-\varepsilon_6 \left(= -\varepsilon_{xy}\right)$, as in Fig. 6.3.3.

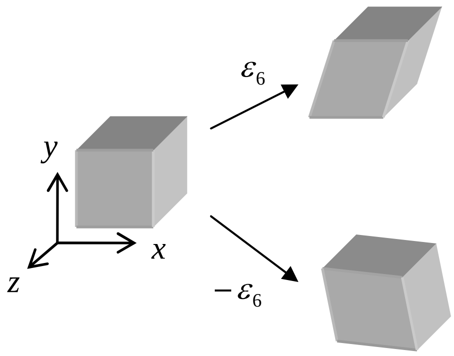


Figure 6.3.3: an element of orthotropic material undergoing shear strain

From Eqns. 6.3.1, the stresses induced by a strain ε_6 only are

$$\begin{aligned} \sigma_1 &= C_{16}\varepsilon_6, \quad \sigma_2 = C_{26}\varepsilon_6, \quad \sigma_3 = C_{36}\varepsilon_6 \\ \sigma_4 &= C_{46}\varepsilon_6, \quad \sigma_5 = C_{56}\varepsilon_6, \quad \sigma_6 = C_{66}\varepsilon_6 \end{aligned} \tag{6.3.3}$$

The stresses induced by a strain $-\varepsilon_6$ only are (the prime is added to distinguish these stresses from those of Eqn. 6.3.3)

$$\begin{aligned} \sigma_1' &= -C_{16}\varepsilon_6, \quad \sigma_2' = -C_{26}\varepsilon_6, \quad \sigma_3' = -C_{36}\varepsilon_6 \\ \sigma_4' &= -C_{46}\varepsilon_6, \quad \sigma_5' = -C_{56}\varepsilon_6, \quad \sigma_6' = -C_{66}\varepsilon_6 \end{aligned} \tag{6.3.4}$$

These stresses, together with the strain, are shown in Fig. 6.3.4 (the microstructure is also indicated)

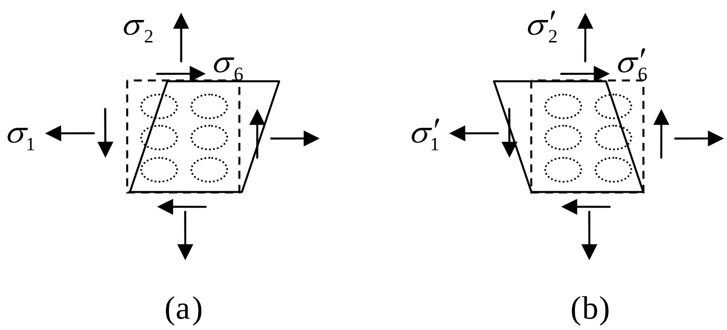


Figure 6.3.4: an element of orthotropic material undergoing shear strain; (a) positive strain, (b) negative strain

Because of the symmetry of the material (print this page out, turn it over, and Fig. 6.3.4a viewed from the "other side" of the page is the same as Fig. 6.3.4b on "this side" of the page), one would expect the normal stresses in Fig. 6.3.4 to be the same, $\sigma_1 = \sigma_1'$,

$\sigma_2 = \sigma'_2$, but the shear stresses to be of opposite sign, $\sigma_6 = -\sigma'_6$. Eqns. 6.3.3-4 then imply that

$$C_{16} = C_{26} = C_{36} = C_{46} = C_{56} = 0 \tag{6.3.5}$$

Similar conclusions follow from considering shear strains in the other two planes:

$$\begin{aligned} \varepsilon_5 &: C_{15} = C_{25} = C_{35} = C_{45} = 0 \\ \varepsilon_4 &: C_{14} = C_{24} = C_{34} = 0 \end{aligned} \tag{6.3.6}$$

The stiffness matrix is thus reduced, and there are only *nine* independent elastic constants:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.7}$$

These equations can be inverted to get, introducing elastic constants E, ν and G in place of the S_{ij}'s:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.8}$$

The nine independent constants here have the following meanings:

E_i is the Young's modulus (stiffness) of the material in direction $i = 1,2,3$; for example, $\sigma_1 = E_1\varepsilon_1$ for uniaxial tension in the direction 1.

ν_{ij} is the Poisson's ratio representing the ratio of a transverse strain to the applied strain in *uniaxial tension*; for example, $\nu_{12} = -\varepsilon_2 / \varepsilon_1$ for uniaxial tension in the direction 1.

G_{ij} are the shear moduli representing the shear stiffness in the corresponding plane; for example, G_{12} is the shear stiffness for shearing in the 1-2 plane.

If the 1-axis has long fibres along that direction, it is usual to call G_{12} and G_{13} the **axial shear moduli** and G_{23} the **transverse** (**out-of-plane**) **shear modulus**.

Note that, from symmetry of the stiffness matrix,

$$\nu_{23}E_3 = \nu_{32}E_2, \quad \nu_{13}E_3 = \nu_{31}E_1, \quad \nu_{12}E_2 = \nu_{21}E_1 \tag{6.3.9}$$

An important feature of the orthotropic material is that there is no **shear coupling** with respect to the material axes. In other words, normal stresses result in normal strains only and shear stresses result in shear strains only.

Note that there will in general be shear coupling when the reference axes used, x, y, z, are not aligned with the material directions $1, 2, 3$. For example, suppose that the $x-y$ axes were oriented to the material axes as shown in Fig. 6.3.5. Assuming that the material constants were known, the stresses and strains in the constitutive equations 6.3.8 can be transformed into $\varepsilon_{xx}, \varepsilon_{xy}$, etc. and σ_{xx}, σ_{xy}, etc. using the strain and stress transformation equations. The resulting matrix equations relating the strains $\varepsilon_{xx}, \varepsilon_{xy}$ to the stresses σ_{xx}, σ_{xy} will then not contain zero entries in the stiffness matrix, and normal stresses, e.g. σ_{xx}, will induce shear strain, e.g. ε_{xy}, and shear stress will induce normal strain.

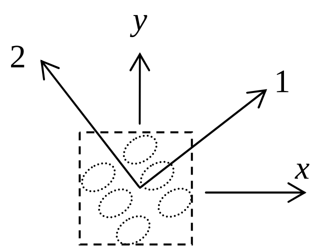


Figure 6.3.5: reference axes not aligned with the material directions

6.3.3 Transversely Isotropic Linear Elasticity

A **transversely isotropic** material is one which has a single material direction and whose response in the plane orthogonal to this direction is isotropic. An example is shown in Fig. 6.3.6, which again shows a glass-fibre composite material with aligned fibres, only now the cross-sectional shapes of the fibres are circular. The characteristic material direction is z and the material is isotropic in any plane parallel to the $x-y$ plane. The material properties are the same in all directions transverse to the fibre direction.

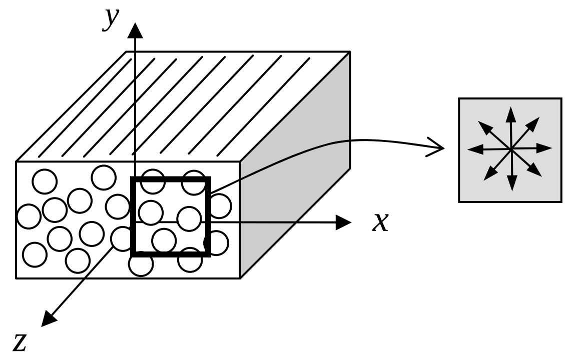


Figure 6.3.6: a transversely isotropic material

This extra symmetry over that inherent in the orthotropic material reduces the number of independent elastic constants further. To see this, consider an element of transversely isotropic material subjected to a normal strain $\varepsilon_1 \left(= \varepsilon_{xx}\right)$ only of magnitude ε, Fig. 6.3.7a, and also a normal strain $\varepsilon_2 \left(= \varepsilon_{yy}\right)$ of the same magnitude, ε, Fig. 6.3.7b. The $x - y$ plane is the plane of isotropy.

Figure 6.3.7: elements of a transversely isotropic material undergoing normal strain in the plane of isotropy

From Eqns. 6.3.7, the stresses induced by a strain $\varepsilon_1 = \varepsilon$ only are

$$\begin{gathered}\sigma_1 = C_{11}\varepsilon, \quad \sigma_2 = C_{21}\varepsilon, \quad \sigma_3 = C_{31}\varepsilon \\ \sigma_4 = 0, \quad \sigma_5 = 0, \quad \sigma_6 = 0\end{gathered} \tag{6.3.10}$$

The stresses induced by the strain $\varepsilon_2 = \varepsilon$ only are (the prime is added to distinguish these stresses from those of Eqn. 6.3.10)

$$\begin{gathered}\sigma_1' = C_{12}\varepsilon, \quad \sigma_2' = C_{22}\varepsilon, \quad \sigma_3' = C_{32}\varepsilon \\ \sigma_4' = 0, \quad \sigma_5' = 0, \quad \sigma_6' = 0\end{gathered} \tag{6.3.11}$$

Because of the isotropy, the $\sigma_1 (= \sigma_{xx})$ due to the ε_1 should be the same as the $\sigma_2 = (\sigma_{yy})$ due to the ε_2, and it follows that $C_{11} = C_{22}$. Further, the $\sigma_3 (= \sigma_{zz})$ should be the same for both, and so $C_{31} = C_{32}$.

Further simplifications arise from consideration of shear deformations, and rotations about the material axis, and one finds that $C_{44} = C_{55}$ and $C_{66} = C_{11} - C_{12}$.

The stiffness matrix is thus reduced, and there are only *five* independent elastic constants:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \qquad (6.3.12)$$

with '3' being the material direction. These equations can be inverted to get, introducing elastic constants $E,\ \nu$ and G in place of the $S_{ij}\,'s$. One again gets Eqn. 6.3.8, but now

$$E_1 = E_2, \quad \nu_{12} = \nu_{21}, \quad \nu_{13} = \nu_{23}, \quad \nu_{31} = \nu_{32}, \quad G_{13} = G_{23} \qquad (6.3.13)$$

so

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad (6.3.14)$$

with, due to symmetry,

$$\nu_{13} / E_1 = \nu_{31} / E_3 \qquad (6.3.15)$$

Eqns. 6.3.13-15 seem to imply that there are 6 independent constants; however, the transverse modulus G_{12} is related to the transverse Poisson ratio and the transverse stiffness through (see Eqn. 6.1.5, and 6.3.20 below, for the isotropic version of this relation)

$$G_{12} = \frac{E_1}{2(1 + \nu_{12})} \qquad (6.3.16)$$

These equations are often expressed in terms of "*a*" for fibre (or "*a*" for axial) and "*t*" for transverse:

$$\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}=\begin{bmatrix}\frac{1}{E_t} & -\frac{\nu_t}{E_t} & -\frac{\nu_f}{E_f} & 0 & 0 & 0\\ -\frac{\nu_t}{E_t} & \frac{1}{E_t} & -\frac{\nu_f}{E_f} & 0 & 0 & 0\\ -\frac{\nu_f}{E_f} & -\frac{\nu_f}{E_f} & \frac{1}{E_f} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2G_f} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2G_f} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_t}\end{bmatrix}\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix} \qquad (6.3.17)$$

6.3.4 Isotropic Linear Elasticity

An isotropic material is one for which the material response is independent of orientation. The symmetry here further reduces the number of elastic constants to *two*, and the stiffness matrix reads

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ & C_{11} & C_{12} & 0 & 0 & 0\\ & & C_{11} & 0 & 0 & 0\\ & & & C_{11}-C_{12} & 0 & 0\\ & & & & C_{11}-C_{12} & 0\\ & & & & & C_{11}-C_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix} \qquad (6.3.18)$$

These equations can be inverted to get, introducing elastic constants E, ν and G,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.19}$$

with

$$\frac{1}{2G} = \frac{1+\nu}{E} \tag{6.3.20}$$

which are Eqns. 6.1.8 and 6.1.5.

Eqns. 6.3.18 can also be written concisely in terms of the engineering constants E, ν and G with the help of the **Lamé constants**, λ and μ:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.21}$$

with

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad (=G) \tag{6.3.22}$$

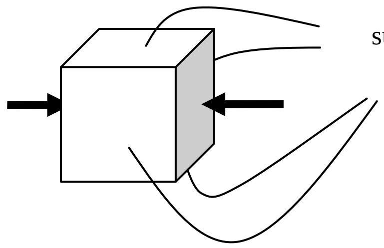
6.3.5 Problems

1. A piece of orthotropic material is loaded by a uniaxial stress σ_1 (aligned with the material direction '1'). What are the strains in the material, in terms of the engineering constants?

2. A specimen of bone in the shape of a cube is fixed and loaded by a compressive stress $\sigma = 1\text{MPa}$ as shown below. The bone can be considered to be orthotropic, with material properties

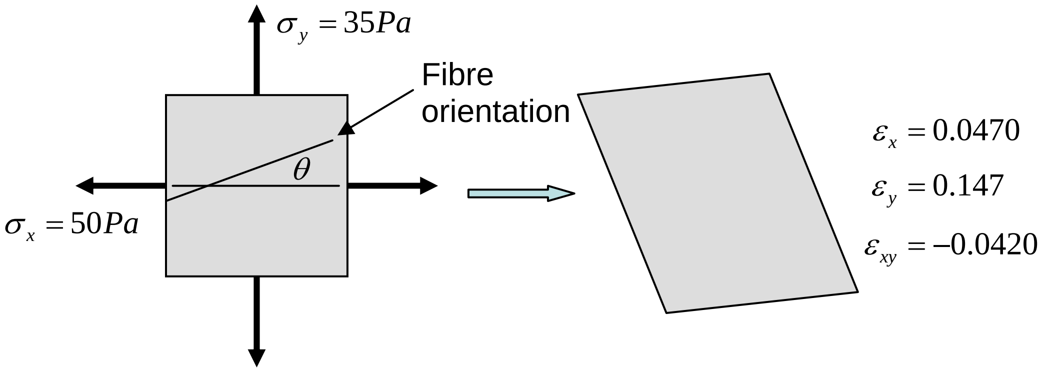
$$E_1 = 6.91\text{GPa}, \quad E_2 = 8.51\text{GPa}, \quad E_3 = 18.4\text{GPa}$$
$$G_{12} = 2.41\text{GPa}, \quad G_{13} = 3.56\text{GPa}, \quad G_{23} = 4.91\text{GPa}$$
$$\nu_{21} = 0.62, \quad \nu_{31} = 0.32, \quad \nu_{32} = 0.31$$

What are the stresses and strains which arise from the test according to this model (the bone is compressed along the '1' direction)?



3. Consider a block of transversely isotropic material subjected to a compressive stress $\sigma_1 = -p$ (perpendicular to the material direction) and constrained from moving in the other two perpendicular directions (as in Problem 2). Evaluate the stresses σ_2 and σ_3 in terms of the engineering constants E_t, E_f and ν_t, ν_f.

4. A strip of skin is tested in biaxial tension as shown below. The measured stresses and strains are as given in the figure. The orientation of the fibres in the material is later measured to be $\theta = 20^o$.



(a) Calculate the normal stresses along and transverse to the fibres, and the corresponding shear stress. (Hint: use the stress transformation equations.)
(b) Calculate the normal strains along and transverse to the fibres, and the corresponding shear strain. (Hint: use the strain transformation equations.)
(c) Assuming the material to be orthotropic, determine the elastic constants of the material (assume the stiffness in the fibre direction to be five times greater than the stiffness in the transverse direction). Note: because the material is thin, one can take $\sigma_3 = \sigma_4 = \sigma_5 = 0$.
(d) Calculate the magnitude and orientations of the principal normal stresses and strains. (Hint: the principal directions of stress are where there is zero shear stress.)
(e) Do the principal directions of stress and strain coincide?

5. A biaxial test is performed on a roughly planar section of skin (thickness 1mm) from the back of a test-animal. The test axes (*x* and *y*) are aligned such that deformation is induced in the skin along the spinal direction and transverse to this direction, under the assumption that the fibres are oriented principally in these directions. However, it is found during the experiment that shear stresses are necessary to maintain a biaxial deformation state. Measured stresses are

$$\sigma_{xx} = 5\text{kPa}, \quad \sigma_{yy} = 2\text{kPa}, \quad \sigma_{xy} = 1\text{kPa}$$

Determine the in-plane orientation of the fibres given the data $E_1 = 1000\,\text{kPa}$, $E_2 = 500\,\text{kPa}$, $G_6 = 500\,\text{kPa}$, $\nu_{21} = 0.2$.

[Hint: derive an expression for ε_{xy} involving θ only, where θ is the inclination of the material axes from the $x-y$ axes]