4.2 Plane Strain

A state of plane strain is defined as follows:

Plane Strain:
If the strain state at a material particle is such that the only non-zero strain components act in one plane only, the particle is said to be in plane strain.

The axes are usually chosen such that the $x-y$ plane is the plane in which the strains are non-zero, Fig. 4.2.1.

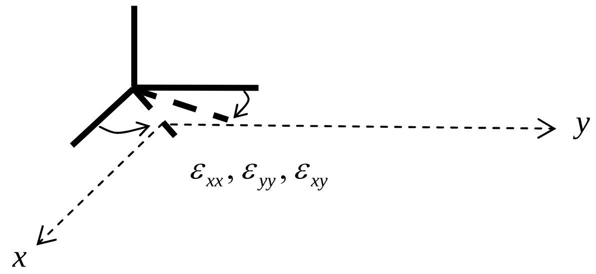


Figure 4.2.1: non-zero strain components acting in the $x-y$ plane

Then $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$. The fully three dimensional strain matrix reduces to a two dimensional one:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad \to \quad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.2.1}$$

4.2.1 Analysis of Plane Strain

Stress transformation formulae, principal stresses, stress invariants and formulae for maximum shear stress were presented in §4.4-§4.5. The strain is very similar to the stress. They are both mathematical objects called tensors, having nine components, and all the formulae for stress hold also for the strain. All the equations in section 3.5.2 are valid again in the case of plane strain, with σ replaced with ε. This will be seen in what follows.

Strain Transformation Formula

Consider two perpendicular line-elements lying in the coordinate directions x and y, and suppose that it is known that the strains are $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$, Fig. 4.2.2. Consider now a second coordinate system, with axes x', y', oriented at angle θ to the first system, and consider line-elements lying along these axes. Using some trigonometry, it can be shown that the line-elements in the second system undergo strains according to the following (two dimensional) **strain transformation equations** (see the Appendix to this section, §4.2.5, for their derivation):

$$\boxed{\begin{aligned} \varepsilon'_{xx} &= \cos^2\theta\varepsilon_{xx} + \sin^2\theta\varepsilon_{yy} + \sin 2\theta\varepsilon_{xy} \\ \varepsilon'_{yy} &= \sin^2\theta\varepsilon_{xx} + \cos^2\theta\varepsilon_{yy} - \sin 2\theta\varepsilon_{xy} \\ \varepsilon'_{xy} &= \sin\theta\cos\theta(\varepsilon_{yy} - \varepsilon_{xx}) + \cos 2\theta\varepsilon_{xy} \end{aligned}}$$

Strain Transformation Formulae (4.2.2)

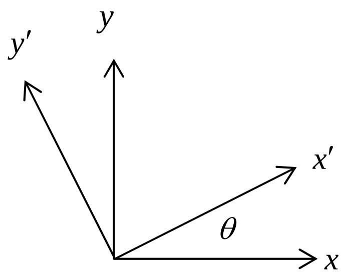


Figure 4.2.2: A rotated coordinate system

Note the similarity between these equations and the stress transformation formulae, Eqns. 3.4.9. Although they have the same structure, the stress transformation equations were derived using Newton's laws, whereas no physical law is used to derive the strain transformation equations 4.2.2, just geometry.

Eqns. 4.2.2 are valid only when the strains are small (as can be seen from their derivation in the Appendix to this section), and the engineering/small strains are assumed in all of which follows. The exact strains, Eqns. 4.1.7, do not satisfy Eqn. 4.2.2 and for this reason they are rarely used – when the strains are large, other strain measures, such as those in Eqns. 4.1.4, are used.

Principal Strains

Using exactly the same arguments as used to derive the expressions for principal stress, there is always at least one set of perpendicular line elements which stretch and/or contract, but which do not undergo angle changes. The strains in this special coordinate system are called **principal strains**, and are given by (compare with Eqns. 3.5.5)

$$\boxed{\begin{aligned} \varepsilon_1 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2} \\ \varepsilon_2 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) - \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2} \end{aligned}}$$

Principal Strains (4.2.3)

Further, it can be shown that ε_1 is the maximum normal strain occurring at the point, and that ε_2 is the minimum normal strain occurring at the point.

The **principal directions**, that is, the directions of the line elements which undergo the principal strains, can be obtained from (compare with Eqns. 3.5.4)

$$\tan 2\theta = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{4.2.4}$$

Here, θ is the angle at which the principal directions are oriented with respect to the x axis, Fig. 4.2.2.

Maximum Shear Strain

Analogous to Eqn. 3.5.9, the maximum shear strain occurring at a point is

$$\varepsilon_{xy}\Big|_{\max} = \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right) \tag{4.2.5}$$

and the perpendicular line elements undergoing this maximum angle change are oriented at 45° to the principal directions.

Example (of Strain Transformation)

Consider the block of material in Fig. 4.2.3a. Two sets of perpendicular lines are etched on its surface. The block is then stretched, Fig. 4.2.3b.

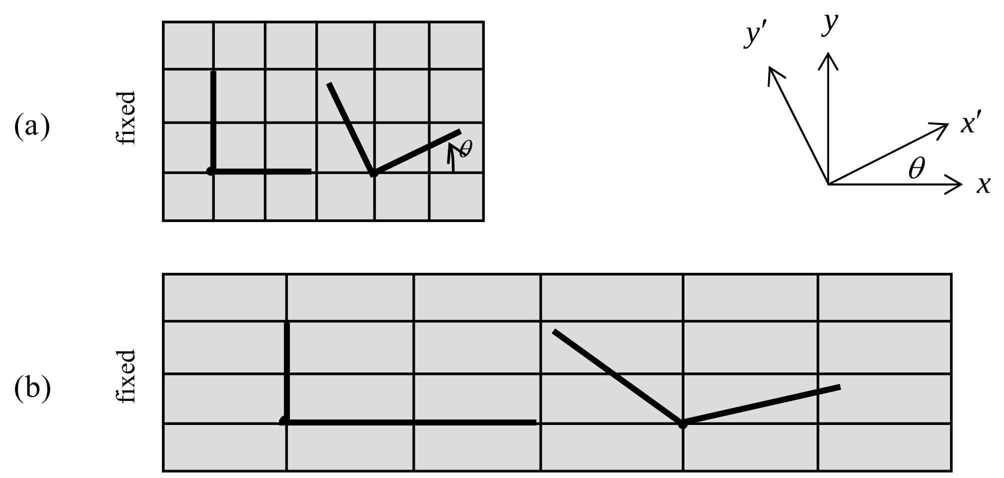


Figure 4.2.3: A block with strain measured in two different coordinate systems

This is a homogeneous deformation, that is, *the strain is the same at all points*. However, in the $x - y$ description, $\varepsilon_{xx} > 0$ and $\varepsilon_{yy} = \varepsilon_{xy} = 0$, but in the $x' - y'$ description, none of the strains is zero. The two sets of strains are related through the strain transformation equations.

■

Example (of Strain Transformation)

As another example, consider a square material element which undergoes a pure shear, as illustrated in Fig. 4.2.4, with

$$\varepsilon_{xx} = \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = 0.01$$

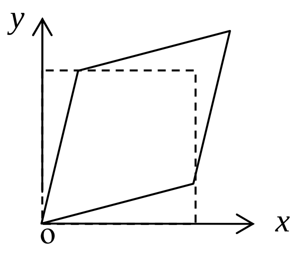


Figure 4.2.4: A block under pure shear

From Eqn. 4.2.3, the principal strains are $\varepsilon_1 = +0.01, \varepsilon_2 = -0.01$ and the principal directions are obtained from Eqn. 4.2.4 as $\theta = \pm 45^\circ$. To find the direction in which the maximum normal strain occurs, put $\theta = +45^\circ$ in the strain transformation formulae to find that $\varepsilon_1 = \varepsilon'_{xx} = +0.01$, so the deformation occurring in a piece of material whose sides are aligned in these principal directions is as shown in Fig. 4.2.5.

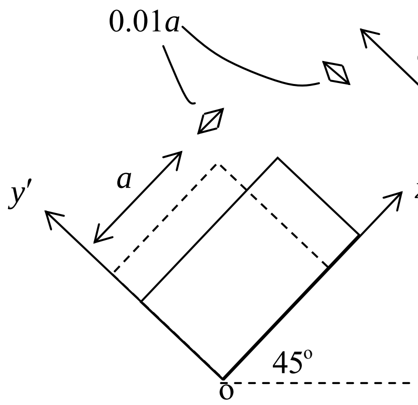


Figure 4.2.5: Principal strains for the block in pure shear

The strain as viewed along the principal directions, and also using the $x - y$ system, are as shown in Fig. 4.2.6.

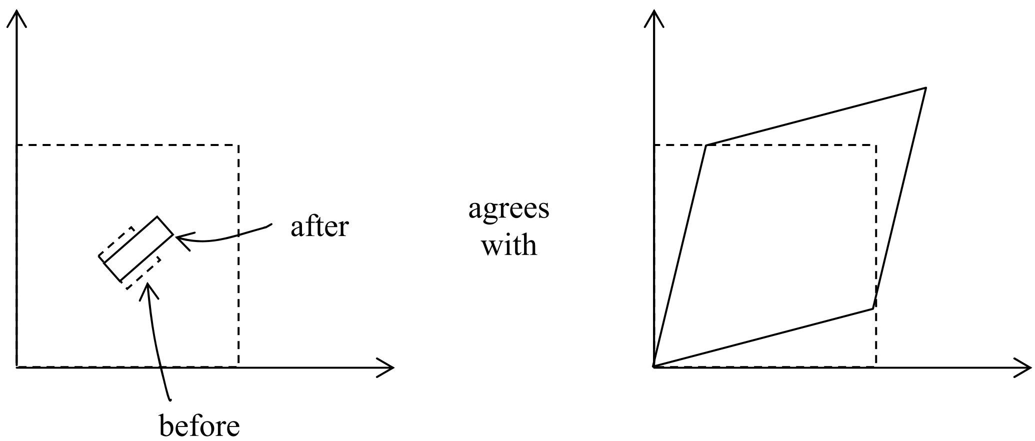


Figure 4.2.6: Strain viewed from two different coordinate systems

The two deformations, square into diamond and square into rectangle, look very different, but they are actually the same thing. For example, the square which deforms into the diamond can be considered to be made up of an infinite number of small rotated squares, Fig. 4.2.7. These then deform into rectangles, which then form the diamond.

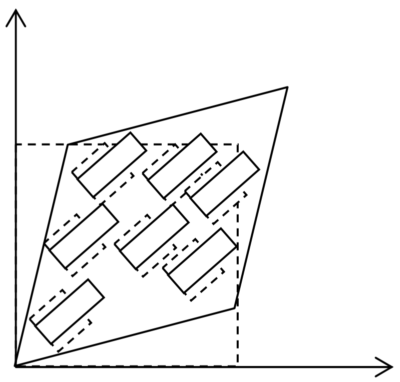


Figure 4.2.7: Alternative viewpoint of the strains in Fig. 4.2.6

Note also that, since the original $x-y$ axes were oriented at $\pm 45°$ to the principal directions, these axes are those of maximum shear strain – the original $\varepsilon_{xy} = 0.01$ is the maximum shear strain occurring at the material particle.

■

4.2.2 Thick Components

It turns out that, just as the state of plane stress often arises in thin components, a state of plane strain often arises in very thick components.

Consider the three dimensional block of material in Fig. 4.2.7. The material is constrained from undergoing normal strain in the z direction, for example by preventing movement with rigid immovable walls – and so $\varepsilon_{zz} = 0$.

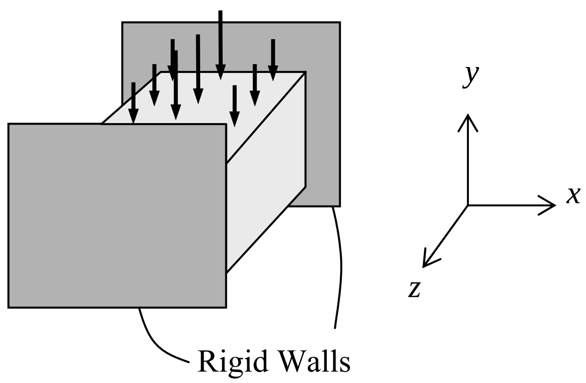


Figure 4.2.7: A block of material constrained by rigid walls

If, in addition, the loading is as shown in Fig. 4.2.7, i.e. it is the same on all cross sections parallel to the $y-z$ plane (or $x-z$ plane) – then the line elements shown in Fig. 4.2.8 will remain perpendicular (although they might move out of plane).

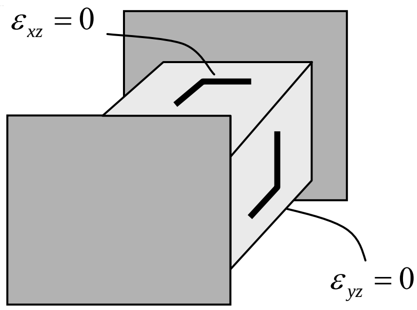


Figure 4.2.8: Line elements etched in a block of material – they remain perpendicular in a state of plane strain

Then $\varepsilon_{xz} = \varepsilon_{yz} = 0$. Thus a state of plane strain will arise.

The problem can now be analysed using the three independent strains, which simplifies matters considerable. Once a solution is found for the deformation of one plane, the solution has been found for the deformation of the whole body, Fig. 4.2.9.

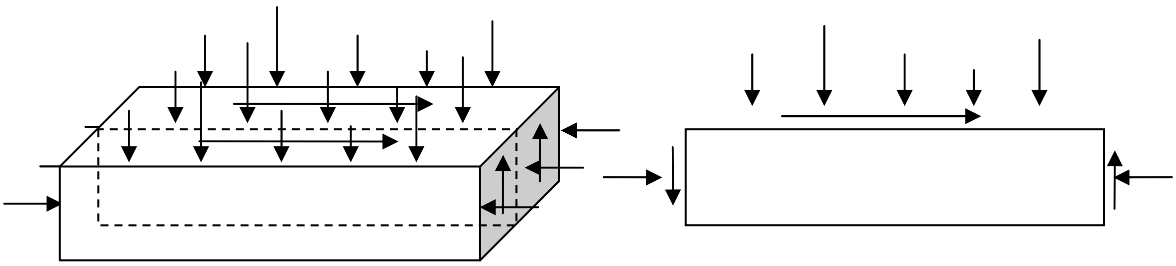


Figure 4.2.9: three dimensional problem reduces to a two dimensional one for the case of plane strain

Note that reaction stresses σ_{zz} act over the ends of the large mass of material, to prevent any movement in the z direction, i.e. ε_{zz} strains, Fig. 4.2.10.

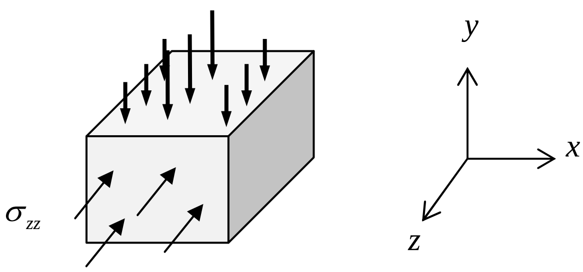


Figure 4.2.10: end-stresses required to prevent material moving in the *z* direction

A state of plane strain will also exist in thick structures without end walls. Material towards the centre is constrained by the mass of material on either side and will be (approximately) in a state of plane strain, Fig. 4.2.10.

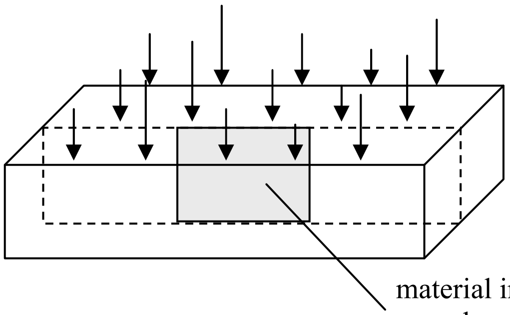


Figure 4.2.10: material in an approximate state of plane strain

Plane Strain is useful when solving many types of problem involving thick components, even when the ends of the mass of material are allowed to move (as in Fig. 4.2.10), using a concept known as **generalised plane strain** (see more advanced mechanics material).

4.2.3 Mohr's Circle for Strain

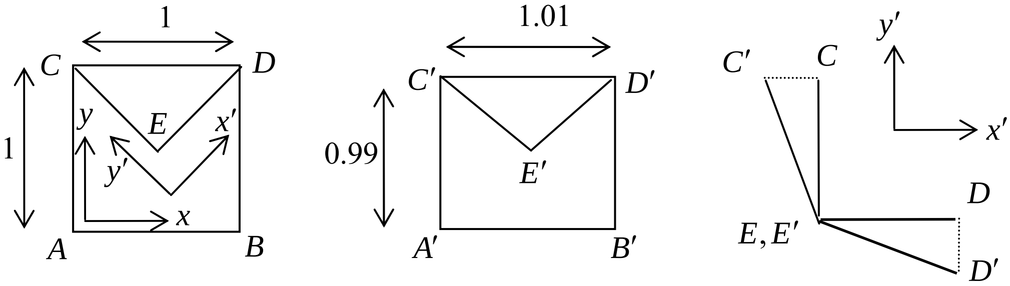
Because of the similarity between the stress transformation equations 3.4.9 and the strain transformation equations 4.2.2, Mohr's Circle for strain is identical to Mohr's Cirlce for stress, section 3.5.5, with σ replaced by ε (and τ replaced by ε_{xy}).

4.2.4 Problems

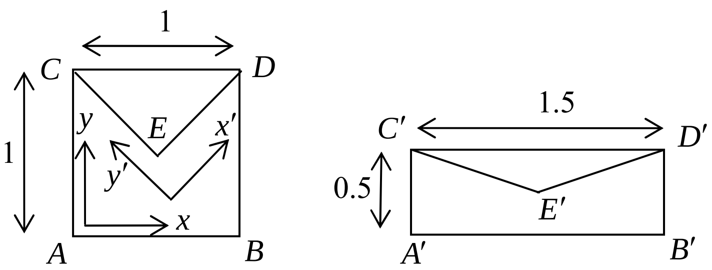
1. In Fig. 4.2.3, take $\theta = 30^{\circ}$ and $\varepsilon_{xx} = 0.02$.
   (a) Calculate the strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$.
   (b) What are the principal strains?
   (c) What is the maximum shear strain?
   (d) Of all the line elements which could be etched in the block, at what angle θ to the *x* axis are the perpendicular line elements which undergo the largest angle change from the initial right angle?

2. Consider the undeformed rectangular element below left which undergoes a uniform strain as shown centre.
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$. Hint: use the two half-diagonals *EC* and *ED* sketched; by superimposing points E, E' (to remove the rigid body motion of *E*), it will be seen that point *D* moves straight down and *C* moves left, when viewed along the x', y' axes, as shown below right.
   (c) Use the strain transformation formulae 4.2.2 and your results from (a) to check your results from (b). Are they the same?

(d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



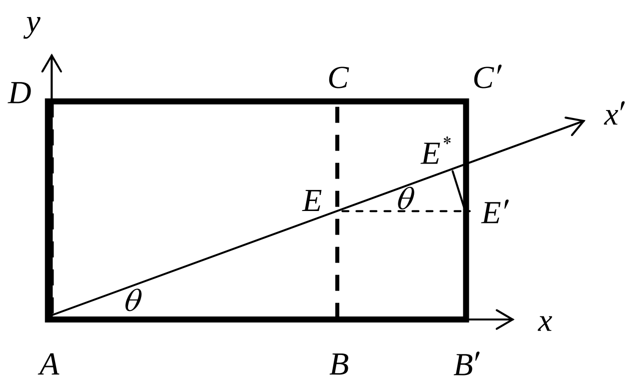
3. Repeat problem 2 only now consider the larger deformation shown below:
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$
   (c) Use the strain transformation formulae 2.4.2 and your results from (a) to check your results from (b). Are they accurate?
   (d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



4.2.5 Appendix to §4.2

Derivation of the Strain Transformation Formulae

Consider an element $ABCD$ undergoing a strain ε_{xx} with $\varepsilon_{yy} = \varepsilon_{xy} = 0$ to $AB'C'D$ as shown in the figure below.



In the $x - y$ coordinate system, by definition, $\varepsilon_{xx} = BB' / AB$. In the $x' - y'$ system, AE moves to AE', and one has

$$\varepsilon'_{xx} = \frac{EE^*}{AE} = \frac{\cos\theta\, EE'}{AB/\cos\theta} = \cos^2\theta \frac{BB'}{AB}$$

which is the first term of Eqn. 4.2.2a. Also,

$$\varepsilon'_{xy} = -\frac{E'E^*}{AE} = \frac{\sin\theta\, EE'}{AB/\cos\theta} = -\sin\theta\cos\theta \frac{BB'}{AB}$$

which is in Eqn. 4.2.2c. The remainder of the transformation formulae can be derived in a similar manner.