# 3 Stress

Forces acting at the surfaces of components were considered in the previous chapter. The task now is to examine forces arising *inside* materials, **internal forces**. Internal forces are described using the idea of **stress**. There is a lot more to stress than the notion of "force over area", as will become clear in this chapter. First, the idea of surface (contact) stress distributions will be examined, together with their relationship to resultant forces and moments. Then internal stress and traction will be discussed. The means by which internal forces are described is through the **stress components**, for example $\sigma_{zx}$, $\sigma_{yy}$, and this "language" of sigmas and subscripts needs to be mastered in order to model sensibly the internal forces in real materials. **Stress analysis** involves representing the actual internal forces in a real physical component mathematically. Some of the limitations of this are discussed in §3.3.2.

Newton's laws are used to derive the **stress transformation equations**, and these are then used to derive expressions for the **principal stresses**, **stress invariants**, **principal directions** and **maximum shear stresses** acting at a material particle. The practical case of two dimensional **plane stress** is discussed.

# 3.1 Surface and Contact Stress

The concept of the force is fundamental to mechanics and many important problems can be cast in terms of forces only, for example the problems considered in Chapter 2. However, more sophisticated problems require that the action of forces be described in terms of *stress*, that is, force divided by area. For example, if one hangs an object from a rope, it is not the weight of the object which determines whether the rope will break, but the weight divided by the cross-sectional area of the rope, a fact noted by Galileo in 1638.

## 3.1.1 Stress Distributions

As an introduction to the idea of stress, consider the situation shown in Fig. 3.1.1a: a block of mass $m$ and cross sectional area $A$ sits on a bench. Following the methodology of Chapter 2, an analysis of a free-body of the block shows that a force equal to the weight $mg$ acts upward on the block, Fig. 3.1.1b. Allowing for more detail now, this force will actually be distributed over the surface of the block, as indicated in Fig. 3.1.1c. Defining the stress to be force divided by area, the stress acting on the block is

$$\sigma = \frac{mg}{A} \tag{3.1.1}$$

The unit of stress is the Pascal (Pa): 1Pa is equivalent to a force of 1 Newton acting over an area of 1 metre squared. Typical units used in engineering applications are the kilopascal, kPa ($10^3$ Pa), the megapascal, MPa ($10^6$ Pa) and the gigapascal, GPa ($10^9$ Pa).

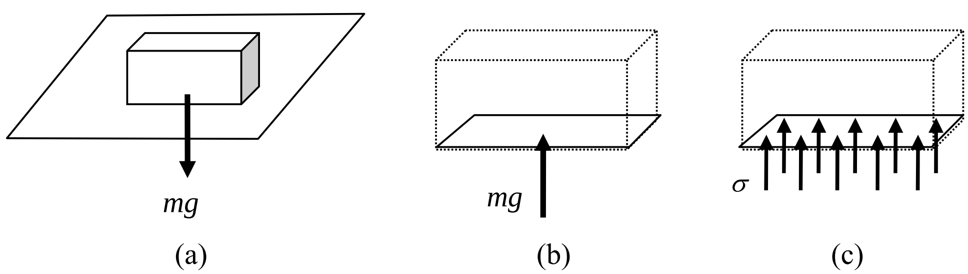



**Figure 3.1.1: a block resting on a bench; (a) weight of the block, (b) reaction of the bench on the block, (c) stress distribution acting on the block**

The stress distribution of Fig. 3.1.1c acts on the block. By Newton's third law, an equal and opposite stress distribution is exerted by the block on the bench; one says that the weight force of the block is *transmitted* to the underlying bench.

The stress distribution of Fig. 3.1.1 is **uniform**, i.e. constant everywhere over the surface. In more complex and interesting situations in which materials contact, one is more likely to obtain a *non-uniform* distribution of stress. For example, consider the case of a metal ball being pushed into a similarly stiff object by a force $F$, as illustrated in Fig. 3.1.2.[1] Again, an equal force $F$ acts on the underside of the ball, Fig. 3.1.2b. As with the block, the force will actually be distributed over a **contact region**. It will be shown in Part II that the ball (and the large object) will deform and a circular contact region will arise where the ball and object meet[2], and that the stress is largest at the centre of the contact surface, dying away to zero at the edges of contact, Fig. 3.1.2c ($\sigma_1 > \sigma_2$ in Fig. 3.1.2c). In this case, we can consider a small area of the contact region $dA$, Fig. 3.1.2d; the force on this region is $\sigma dA$. The total force is

$$F = \int_A dF = \int_A \sigma dA \tag{3.1.2}$$

The stress varies from point to point over the surface but the sum (or integral) of the stresses (times areas) equals the total force applied to the ball.

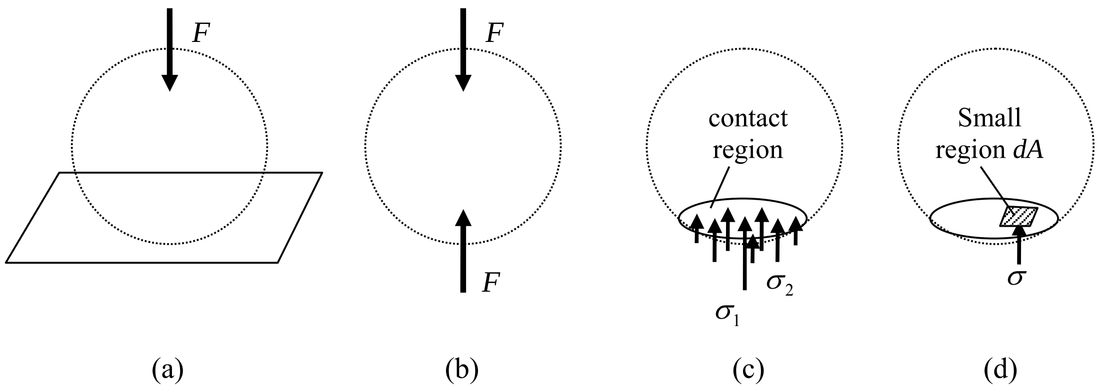



**Figure 3.1.2: a ball being forced into a large object, (a) force applied to ball, (b) reaction of object on ball, (c) a non-uniform stress distribution over the contacting surface, (d) the stress acting on a small (infinitesimal) area**

A given stress distribution gives rise to a resultant force, which is obtained by integration, Eqn. 3.1.2. It will also give rise to a resultant moment. This is examined in the following example.

## Example

Consider the surface shown in Fig. 3.1.3, of length 2m and depth 2m (into the page). The stress over the surface is given by $\sigma = x$ kPa, with $x$ measured in m from the left-hand side of the surface.

The force acting on an element of length $dx$ at position $x$ is (see Fig. 3.1.3b)

$$dF = \sigma\, dA = \left(x\,\mathrm{kPa}\right) \times \left(dx\,\mathrm{m} \times 2\,\mathrm{m}\right)$$

The resultant force is then, from Eqn. 3.1.2

[1] the weight of the ball is neglected here
[2] the radius of which depends on the force applied and the materials in contact

$$F = \int_A dF = 2\int_0^2 x dx \left(\text{kPa m}^2\right) = 4\,\text{kN}$$

The moment of the stress distribution is given by

$$M_0 = \int_A dM = \int_A \sigma \times l\, dA \tag{3.1.3}$$

where $l$ is the length of the moment-arm from the chosen axis.

Taking the axis to be at $x = 0$, the moment-arm is $l = x$, Fig. 3.1.3b, and

$$M_{x=0} = \int_A dM = 2\int_0^2 x \times x\, dx \left(\text{kPa m}^3\right) = \frac{16}{3}\,\text{kN m}$$

Taking moments about the right-hand end, $x = 2$, one has

$$M_{x=2} = \int_A dM = -2\int_0^2 x \times (2 - x) dx \left(\text{kPa m}^3\right) = -\frac{8}{3}\,\text{kN m}$$

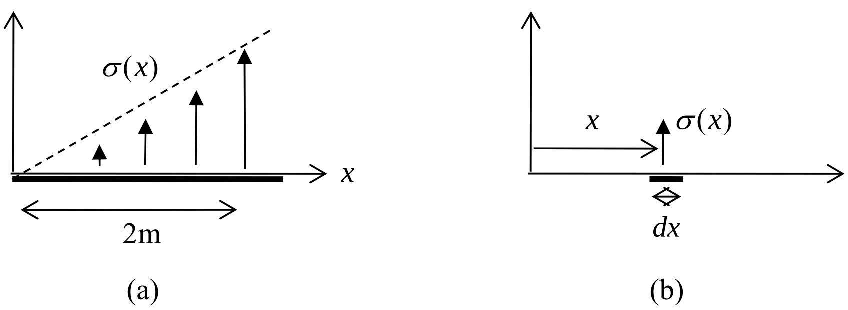



**Figure 3.1.3: a non-uniform stress acting over a surface; (a) the stress distribution, (b) stress acting on an element of size d*x***

■

## 3.1.2 Equivalent Forces and Moments

Sometimes it is useful to replace a stress distribution $\sigma$ with an **equivalent force** *F*, i.e. a force equal to the resultant force of the distribution and one which also gives the same moment about any axis as the distribution. Formulae for equivalent forces are derived in what follows for triangular and arbitrary linear stress distributions.

### Triangular Stress Distribution

Consider the triangular stress distribution shown in Fig. 3.1.4. The stress at the end is $\sigma_0$, the length of the distribution is *L* and the thickness "into the page" is *t*. With $\sigma(x) = \sigma_0 x / L$, the equivalent force is, from Eqn. 3.1.2,

$$F = t\sigma_0 \int_0^L \frac{x}{L} dx = \frac{1}{2}\sigma_0 Lt \tag{3.1.4}$$

which is just the average stress times the "area of the triangle". The point of action of this force should be such that the moment of the force is equivalent to the moment of the stress distribution. Taking moments about the left hand end, for the distribution one has, from 3.1.3,

$$M_o = t\int_0^L x\sigma(x)dx = \frac{1}{3}\sigma_0 L^2 t$$

Placing the force at position $x = x_c$, Fig. 3.1.4, the moment of the force is $M_o = (\sigma_0 Lt/2)x_c$. Equating these expressions leads to the position at which the equivalent force acts, two-thirds the way along the triangle:

$$x_c = \frac{2}{3}L. \tag{3.1.5}$$

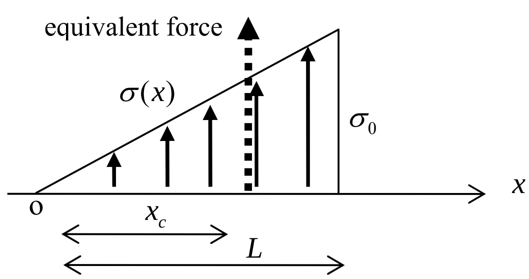



**Figure 3.1.4: triangular stress distribution and equivalent force**

Note that the moment about *any* axis is now the same for both the stress distribution and the equivalent force.

■

### Arbitrary Linear Stress Distribution

Consider the linear stress distribution shown in Fig. 3.1.5. The stress at the ends are $\sigma_1$ and $\sigma_2$ and this time the equivalent force is

$$F = t\int_0^L [\sigma_1 + (\sigma_2 - \sigma_1)(x/L)]dx = Lt(\sigma_1 + \sigma_2)/2 \tag{3.1.6}$$

Taking moments about the left hand end, for the distribution one has

$$M_o = t\int_0^L x\sigma(x)dx = L^2 t(\sigma_1 + 2\sigma_2)/6 .$$

The moment of the force is $M_o = Lt(\sigma_1 + \sigma_2)x_c / 2$. Equating these expressions leads to

$$x_c = \frac{L(\sigma_1 + 2\sigma_2)}{3(\sigma_1 + \sigma_2)} \tag{3.1.7}$$

Eqn. 3.1.5 follows from 3.1.7 by setting $\sigma_1 = 0$.

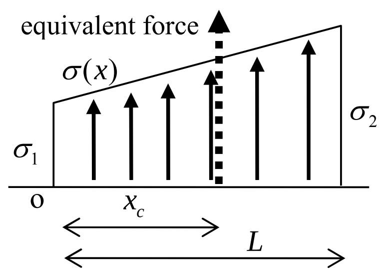



**Figure 3.1.5: a non-uniform stress distribution and equivalent force**

■

### The Centroid

Generalising the above cases, the line of action of the equivalent force for any arbitrary stress distribution $\sigma(x)$ is

$$\boxed{x_c = \frac{t\int x\sigma(x)dx}{t\int \sigma(x)dx} = \frac{\int xdF}{F}} \quad \textbf{Centroid} \tag{3.1.8}$$

This location is known as the **centroid** of the distribution.

Note that most of the discussion above is for two-dimensional cases, i.e. the stress is assumed constant "into the page". Three dimensional problems can be tackled in the same way, only now one must integrate two-dimensionally over a surface rather than one-dimensionally over a line.

Also, the forces considered thus far are *normal* forces, where the force acts perpendicular to a surface, and they give rise to **normal stresses**. Normal stresses are also called **pressures** when they are compressive as in Figs. 3.1.1-2.

### 3.1.3 Shear Stress

Consider now the case of **shear forces**, that is, forces which act tangentially to surfaces.

A normal force *F* acts on the block of Fig. 3.1.6a. The block does not move and, to maintain equilibrium, the force is resisted by a friction force $F = \mu mg$, where $\mu$ is the coefficient of friction. A free body diagram of the block is shown in Fig. 3.1.6b. Assuming a uniform distribution of stress, the stress and resultant force arising on the surfaces of the block and underlying object are as shown. The stresses are in this case called **shear stresses**.

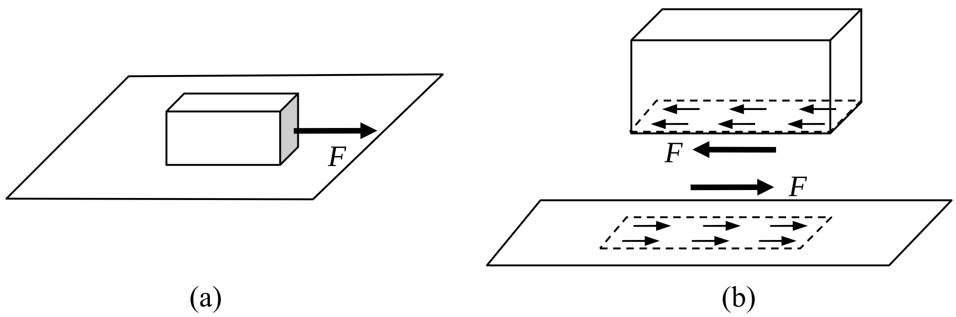



**Figure 3.1.6: shear stress; (a) a force acting on a block, (b) shear stresses arising on the contacting surfaces**

### 3.1.4 Combined Normal and Shear Stress

Forces acting inclined to a surface are most conveniently described by decomposing the force into components normal and tangential to the surface. Then one has both normal stress $\sigma_N$ and shear stress $\sigma_S$, as in Fig. 3.1.7.

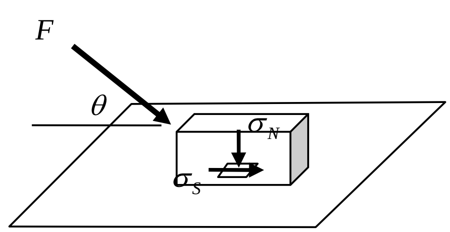
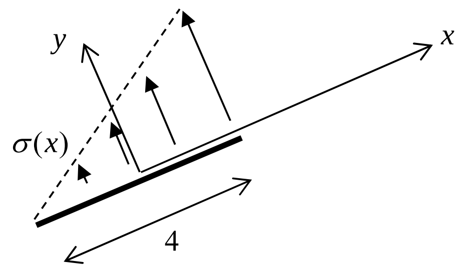



**Figure 3.1.7: a force F giving rise to normal and shear stress over the contacting surfaces**

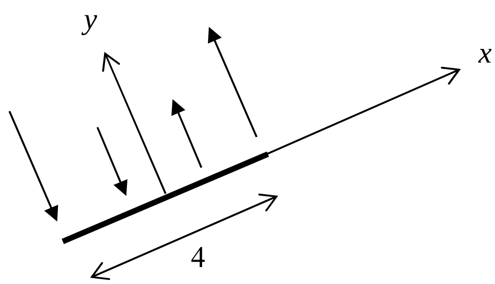
The stresses considered in this section are examples of **surface stresses** or **contact stresses**. They arise when materials meet at a common surface. Other examples would be sea-water pressurising a gas cylinder in deep water and the stress exerted by a train wheel on a train track.

### 3.1.5 Problems

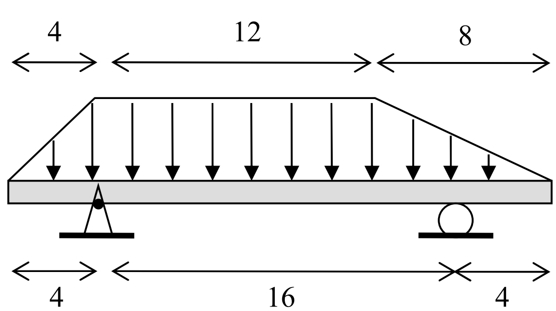
1. Consider the surface shown below, of length 4cm and unit depth (1cm into the page). The stress over the surface is given by $\sigma = 2 + x$ kPa, with $x$ measured in cm from the surface *centre*.
   (a) Evaluate the resultant force acting on the surface (in Newtons).
   (b) What is the moment about an axis (into the page) through the left-hand end of the surface?
   (c) What is the moment about an axis (into the page) through the centre of the surface?



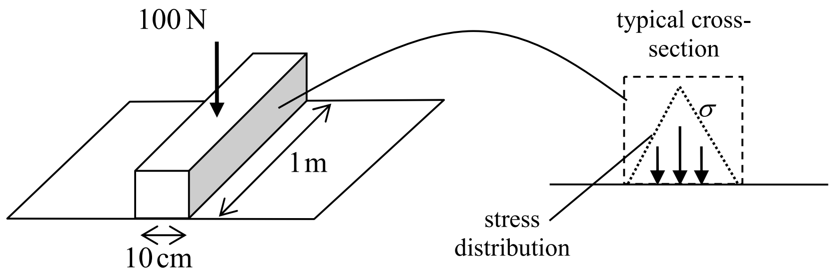

2. Consider the surface shown below, of length 4mm and unit depth (1mm into the page). The stress over the surface is given by $\sigma = x$ MPa, with $x$ measured from the surface centre, i.e. $x$ mm corresponds to $x$ MPa. What is the total force acting on the surface, and the moment acting about the centre of the surface?




3. Find the reaction forces (per unit length) at the pin and roller for the following beam, which is subjected to a varying pressure distribution, the maximum pressure being $\sigma(x) = 20\,\text{kPa}$ (all lengths are in cm – give answer in N/m)
   [Hint: first replace the stress distribution with three equivalent forces]



4. A block of material of width 10cm and length 1m is pushed into an underlying substrate by a normal force of 100 N. It is found that a uniform triangular normal stress distribution arises at the contacting surfaces, that is, the stress is maximum at the centre and dies off linearly to zero at the block edges, as sketched below right. What is the maximum pressure acting on the surface?



# 3.2 Body Forces

Surface forces act on surfaces. As discussed in the previous section, these are the forces which arise when bodies are in contact and which give rise to stress distributions. Surface forces also arise *inside* materials, acting on internal surfaces, Fig. 3.2.1a, as will be discussed in the following section.

To complete the description of forces acting on real materials, one needs to deal with forces which arise even when bodies are not in contact; one can think of these forces as *acting at a distance*, for example the force of gravity. To describe these forces, one can define the **body force**, which acts on volume elements of material. Fig. 3.2.1b shows a sketch of a volume element subjected to a magnetic body force and a gravitational body force $\mathbf{F}_g$.

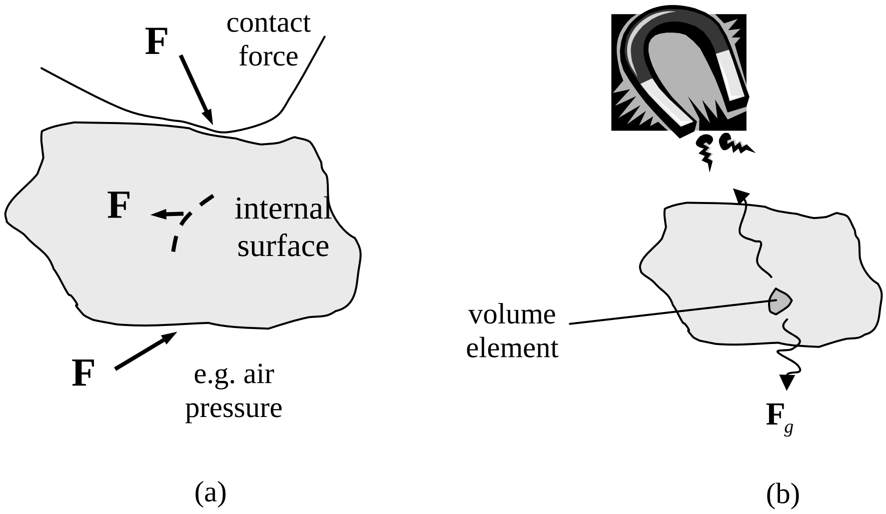



**Figure 3.2.1: forces acting on a body; (a) surface forces acting on surfaces, (b) body forces acting on a material volume element**

## 3.2.1 Weight

The most important body force is the force due to gravity, i.e. the weight force. In Chapter 2 there were examples involving the weight of components. In those cases it was simply stated that the weight could be taken to be a single force acting at the component centre (for example, Problem 3 in §2.2.3). This is true when the component is symmetrical, for example, in the shape of a circle or a square. However, it is not true in general for a component of arbitrary shape.

In what follows, the important case of a flat object of arbitrary shape will be examined.

The weight of a small volume element $\Delta V$ of material of density $\rho$ is $dF = \rho g \Delta V$ and the total weight is

$$F = \int_V \rho g dV \tag{3.2.1}$$

Consider the general two-dimensional case, Fig. 3.2.2, where material elements of area $\Delta A_i$ (and constant thickness $t$) are subjected to forces $\Delta F_i = t\rho g \Delta A_i$.

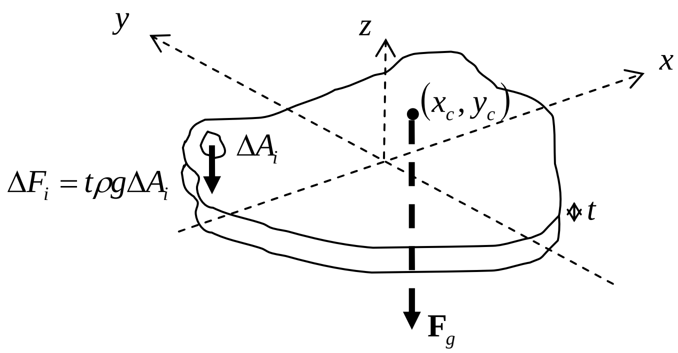



**Figure 3.2.2: Resultant Weight on a body**

The resultant, i.e. equivalent, weight force due to all elements, for a component with uniform density, is

$$F = \int dF = t\rho g \int dA = \rho g t A,$$

where $A$ is the cross-sectional area.

The resultant moments about the $x$ and $y$ axes, which can be positioned anywhere in the body, are $M_x = t\rho g \int y dA$ and $M_y = t\rho g \int x dA$ respectively; the moment $\Delta M_x$ is shown in Fig. 3.2.3. The equivalent weight force is thus positioned at $(x_c, y_c)$, Fig. 3.2.2, where

$$\boxed{x_c = \frac{\int x dA}{A}, \quad y_c = \frac{\int y dA}{A}} \qquad \textbf{Centroid of Area} \tag{3.2.2}$$

The position $(x_c, y_c)$ is called the **centroid of the area**. The quantities $\int x dA$, $\int y dA$, are called the **first moments of area** about, respectively, the $y$ and $x$ axes.

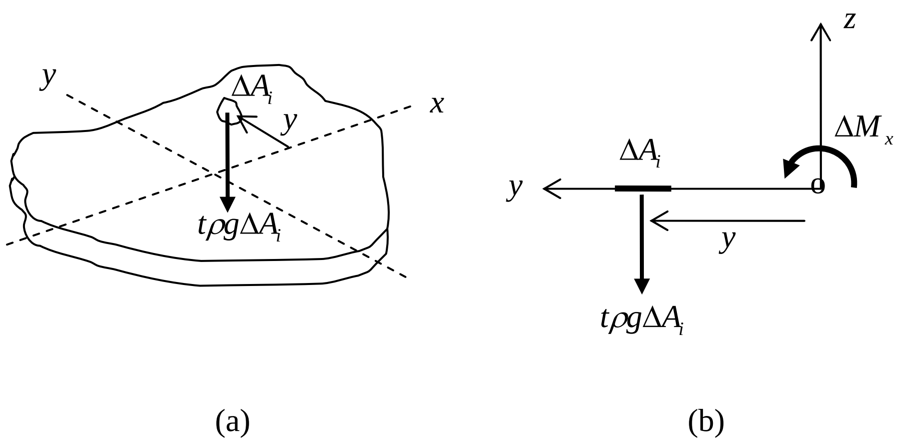
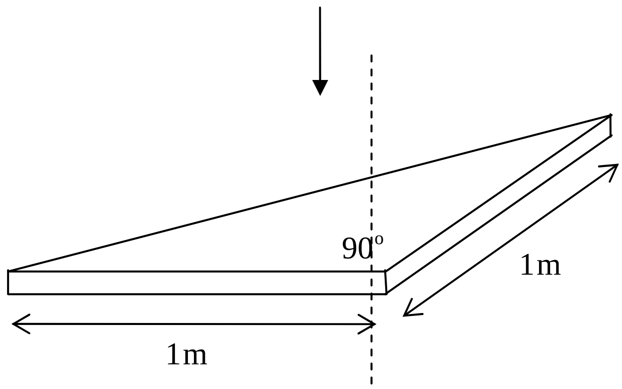



**Figure 3.2.3: The moment $M_x$; (a) full view, (b) plane view**

### 3.2.2 Problems

1. Where does the resultant force due to gravity act in the triangular component shown below? (Gravity acts downward in the direction of the arrow shown, perpendicular to the component's surface.)



# 3.3 Internal Stress

The idea of stress considered in §3.1 is not difficult to conceptualise since objects interacting with other objects are encountered all around us. A more difficult concept is the idea of forces and stresses acting *inside* a material, "within the interior where neither eye nor experiment can reach" as Euler put it. It took many great minds working for centuries on this question to arrive at the concept of stress we use today, an idea finally brought to us by Augustin Cauchy, who presented a paper on the subject to the Academy of Sciences in Paris, in 1822.

**Augustin Cauchy**

## 3.3.1 Cauchy's Concept of Stress

### Uniform Internal Stress

Consider first a long slender block of material subject to equilibrating forces $F$ at its ends, Fig. 3.3.1a. If the complete block is in equilibrium, then any sub-division of the block must be in equilibrium also. By imagining the block to be cut in two, and considering free-body diagrams of each half, as in Fig. 3.3.1b, one can see that forces $F$ must be acting *within* the block so that each half is in equilibrium. Thus *external loads create internal forces*; internal forces represent the action of one part of a material on another part of the same material across an internal surface. We can take it that a uniform stress $\sigma = F / A$ acts over this interior surface, Fig. 3.3.1b.

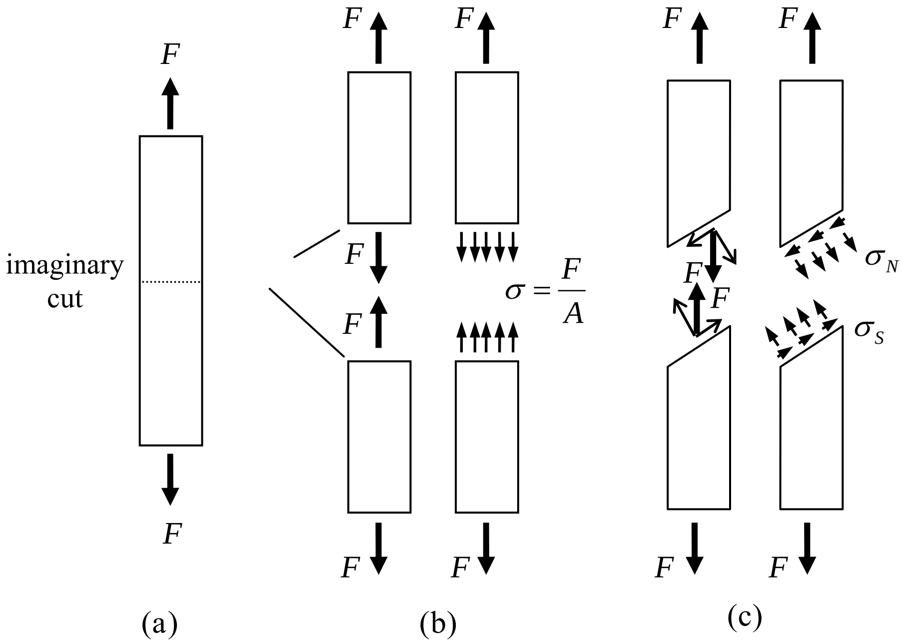



**Figure 3.3.1: a slender block of material; (a) under the action of external forces F, (b) internal normal stress σ, (c) internal normal and shear stress**

Note that, if the internal forces were not acting over the internal surfaces, the two half-blocks of Fig. 3.3.1b would fly apart; one can thus regard the internal forces as those required to maintain material in an un-cut state.

If the internal surface is at an incline, as in Fig. 3.3.1c, then the internal force required for equilibrium will not act normal to the surface. There will be components of the force normal and tangential to the surface, and thus both normal ($\sigma_N$) and shear ($\sigma_S$) stresses must arise. Thus, even though the material is subjected to a purely normal load, internal shear stresses develop.

From Fig. 3.3.2a, with the stress given by force divided by area, the normal and shear stresses arising on an interior surface inclined at angle $\theta$ to the horizontal are {▲Problem 1}

$$\sigma_N = \frac{F}{A}\cos^2\theta, \qquad \sigma_S = \frac{F}{A}\sin\theta\cos\theta \tag{3.3.1}$$

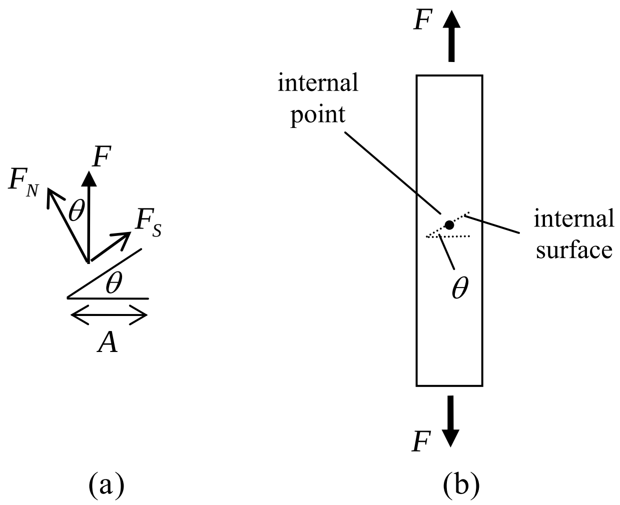



**Figure 3.3.2: stress on inclined surface; (a) decomposing the force into normal and shear forces, (b) stress at an internal point**

Although stress is associated with surfaces, one can speak of the stress "at a point". For example, consider some point interior to the block, Fig 3.3.2b. The stress there evidently depends on which surface through that point is under consideration. From Eqn. 3.3.1a, the normal stress at the point is a maximum $F/A$ when $\theta = 0$ and a minimum of zero when $\theta = 90^\circ$. The maximum normal stress arising at a point within a material is of special significance, for example it is this stress value which often determines whether a material will fail ("break") there. It has a special name: the **maximum principal stress**. From Eqn. 3.3.1b, the **maximum shear stress** at the point is $\pm F/2A$ and arises on surfaces inclined at $\pm 45^\circ$.

### Non-Uniform Internal Stress

Consider a more complex geometry under a more complex loading, as in Fig. 3.3.3. Again, using equilibrium arguments, there will be some stress distribution acting over any given internal surface. To evaluate these stresses is not a straightforward matter, suffice to say here that they will invariably be *non-uniform* over a surface, that is, the stress at some particle will differ from the stress at a neighbouring particle.

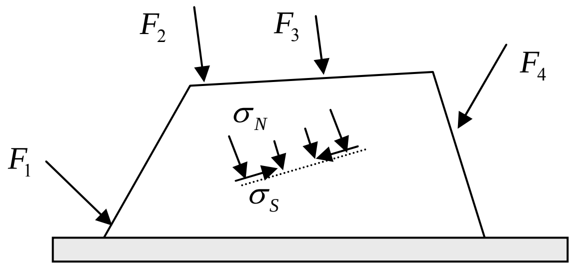



**Figure 3.3.3: a component subjected to a complex loading, giving rise to a non-uniform stress distribution over an internal surface**

## Traction and the Physical Meaning of Internal Stress

All materials have a complex molecular microstructure and each molecule exerts a force on each of its neighbours. The complex interaction of countless molecular forces maintains a body in equilibrium in its unstressed state. When the body is disturbed and deformed into a new equilibrium position, net forces act, Fig. 3.3.4a. An imaginary plane can be drawn through the material, Fig. 3.3.4b. Unlike some of his predecessors, who attempted the extremely difficult task of accounting for all the molecular forces, Cauchy discounted the molecular structure of matter and simply replaced the imagined molecular forces acting on the plane by a single force **F**, Fig 3.3.4c. This is the force exerted by the molecules above the plane *on* the material below the plane and can be attractive or repulsive. Different planes can be taken through the *same* portion of material and, in general, a *different* force will act on the plane, Fig 3.3.4d.

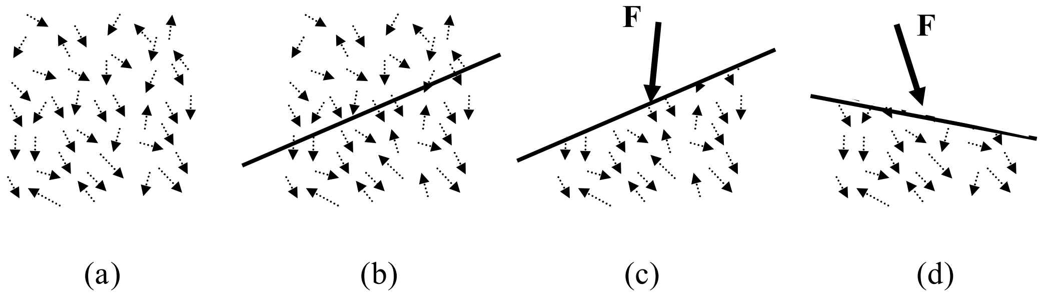



**Figure 3.3.4: a multitude of molecular forces represented by a single force; (a) molecular forces, a plane drawn through the material, replacing the molecular forces with an equivalent force F, a different equivalent force F acts on a different plane through the same material**

The definition of stress will now be made more precise. First, define the **traction** at some particular point in a material as follows: take a plane of surface area *S* through the point, on which acts a force *F*. Next shrink the plane – as it shrinks in size both *S* and *F* get smaller, and the direction in which the force acts may change, but eventually the ratio $F/S$ will remain constant and the force will act in a particular direction, Fig. 3.3.5. The limiting value of this ratio of force over surface area is defined as the **traction vector** (or **stress vector**) **t**:[1]

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \tag{3.3.2}$$

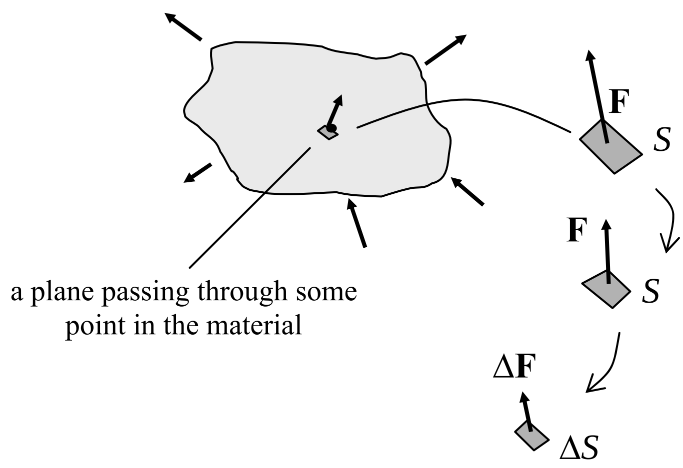



**Figure 3.3.5: the traction vector - the limiting value of force over area, as the surface area of the element on which the force acts is shrunk**

An infinite number of traction vectors act at any single point, since an infinite number of different planes pass through a point. Thus the notation $\lim_{\Delta S \to 0} \Delta F / \Delta S$ is ambiguous. For this reason the plane on which the traction vector acts must be specified; this can be done by specifying the normal **n** to the surface on which the traction acts, Fig 3.3.6. The traction is thus a special vector – associated with it is not only the direction in which it acts but also a second direction, the normal to the plane upon which it acts.

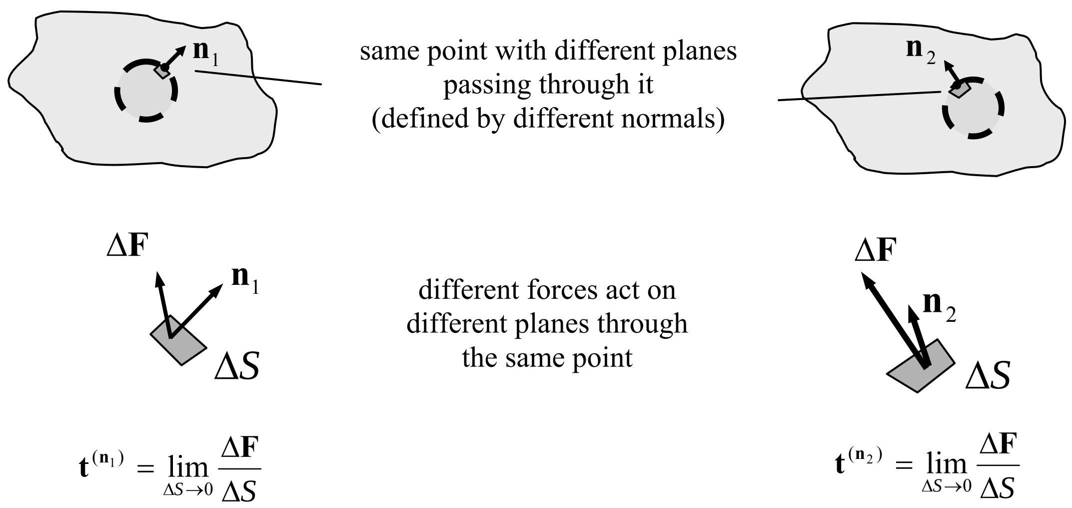



**Figure 3.3.6: two different traction vectors acting at the same point**

[1] this does not mean that the force is acting on a surface of zero area – the meaning of this limit is further examined in section 5.4, in the context of the continuum

## Stress Components

The traction vector can be decomposed into components which act normal and parallel to the surface upon which it acts. These components are called the **stress components**, or simply **stresses**, and are denoted by the symbol $\sigma$; subscripts are added to signify the surface on which the stresses act and the directions in which the stresses act.

Consider a particular traction vector acting on a surface element. Introduce a Cartesian coordinate system with base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ so that one of the base vectors is a normal to the surface, and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 3.3.7, the $\mathbf{k}$ direction is taken to be normal to the plane, and $\mathbf{t}^{(\mathbf{k})} = t_x \mathbf{i} + t_y \mathbf{j} + t_z \mathbf{k}$.

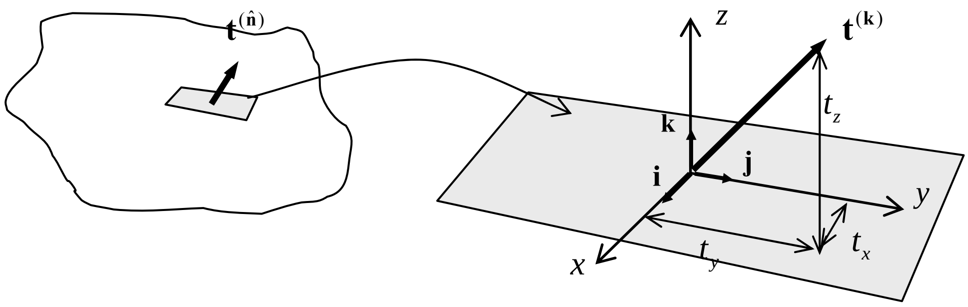



**Figure 3.3.7: the components of the traction vector**

Each of these components $t_i$ is represented by $\sigma_{ij}$ where the first subscript denotes the *direction of the normal* to the plane and the second denotes the *direction of the component*. Thus, re-drawing Fig. 3.3.7 as Fig. 3.3.8: $\mathbf{t}^{(\mathbf{k})} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$. The first two stresses, the components acting tangential to the surface, are shear stresses, whereas $\sigma_{zz}$, acting normal to the plane, is a normal stress[2].

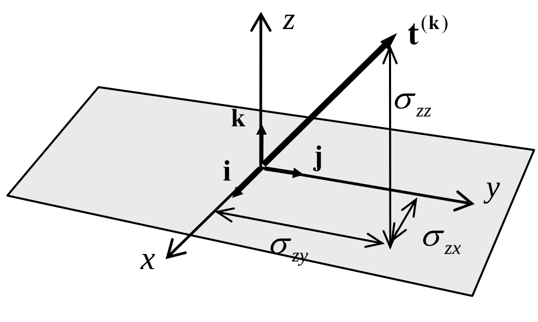



**Figure 3.3.8: stress components – the components of the traction vector**

[2] this convention for the subscripts is not universally followed. Many authors, particularly in the mathematical community, use the exact opposite convention, the first subscript to denote the direction and the second to denote the normal. It turns out that *both conventions are equivalent*, since, as will be shown later, $\sigma_{ij} = \sigma_{ji}$

The traction vector shown in Figs. 3.3.7, 3.3.8, represents the force (per unit area) exerted by the material above the surface *on* the material below the surface. By Newton's third law, an equal and opposite traction must be exerted by the material below the surface on the material above the surface, as shown in Fig. 3.3.9 (thick dotted line). If $\mathbf{t}^{(\mathbf{k})}$ has stress components $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$, then so should $\mathbf{t}^{(-\mathbf{k})}$: $\mathbf{t}^{(-\mathbf{k})} = \sigma_{zx}(-\mathbf{i}) + \sigma_{zy}(-\mathbf{j}) + \sigma_{zz}(-\mathbf{k}) = -\mathbf{t}^{(\mathbf{k})}$.

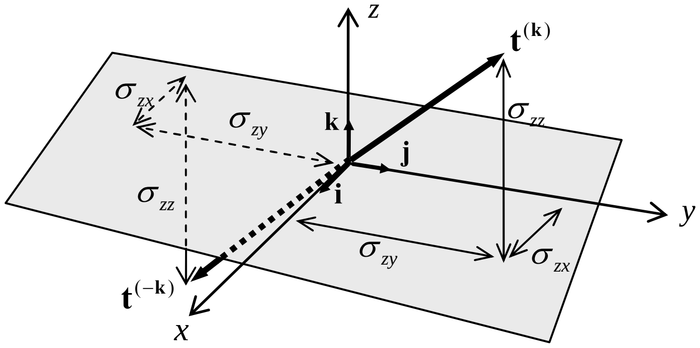



**Figure 3.3.9: equal and opposite traction vectors – each with the same stress components**

## Sign Convention for Stress Components

The following convention is used:

The stress is *positive* when the direction of the normal *and* the direction of the stress component are both positive *or* both negative
The stress is *negative* when one of the directions is positive and the other is negative

According to this convention, the three stresses in Figs. 3.3.7-9 are all positive.

Looking at the two-dimensional case for ease of visualisation, the (positive and negative) normal stresses and shear stresses on either side of a surface are as shown in Fig. 3.3.10. To clarify this, consider the $\sigma_{yy}$ stress in Fig. 3.310a: "above" the plane, the normal to the plane is in the positive *y* direction (up) and the component $\sigma_{yy}$ acts in the positive direction (up), so this stress is positive; "below" the plane, the normal to the plane is in the negative *y* direction (down) and the component $\sigma_{yy}$ acts in the negative direction (down), so this stress is positive. The simple consequence of this sign convention is that normal stresses which "pull" (tension) are positive and normal stresses which "push" (compression) are negative. Note that the shear stresses always go in opposite directions.

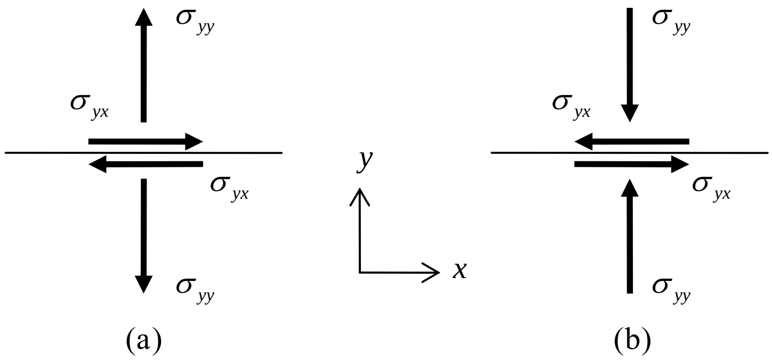



**Figure 3.3.10: stresses acting on either side of a material surface: (a) positive stresses, (b) negative stresses**

Examples of negative stresses are shown in Fig. 3.3.11 {▲Problem 4}.

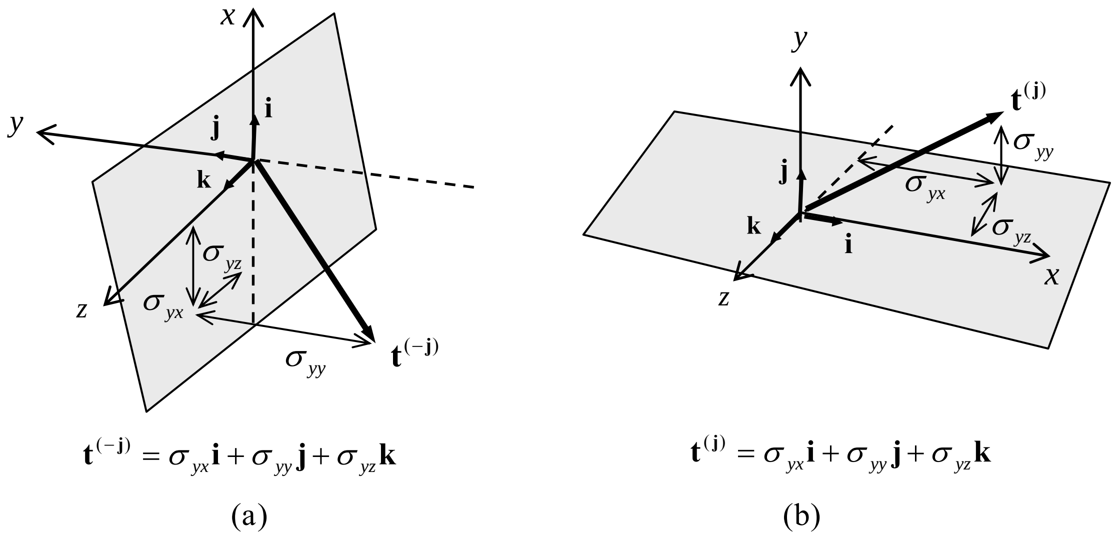



**Figure 3.3.11: examples of negative stress components**

## 3.3.2 Real Problems and Saint-Venant's Principle

Some examples have been given earlier of external forces acting on materials. In reality, an external force will be applied to a real material component in a complex way. For example, suppose that a block of material, welded to a large object at one end, is pulled at its other end by a rope attached to a metal hoop, which is itself attached to the block by a number of bolts, Fig. 3.3.12a. The block can be idealised as in Fig 3.3.12b; here, the precise details of the region in which the external force is applied are neglected.

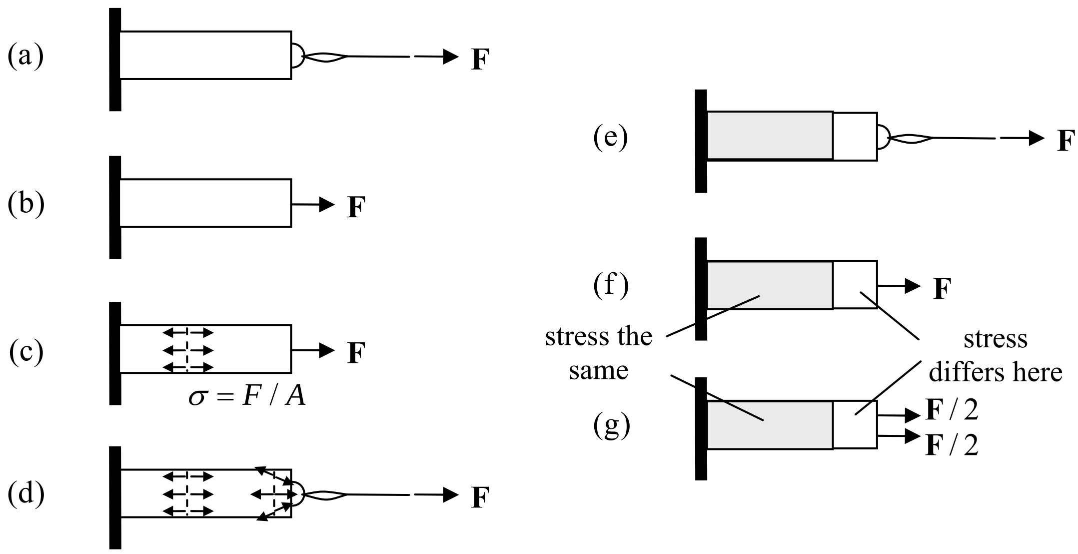



**Figure 3.3.12: a block subjected to an external force: (a) real case, (b) ideal model, (c) stress in ideal model, (d) stress in actual material, (e) the stress in the real material, away from the right hand end, is modelled well by either (f) or (g)**

According to the earlier discussion, the stress in the ideal model is as in Fig. 3.3.12c. One will find that, in the *real* material, the stress is indeed (approximately) as predicted, but only at an appreciable distance from the right hand end. Near where the rope is attached, the force will differ considerably, as sketched in Fig.3.3.12d.

Thus the ideal models of the type discussed in this section, and in much of this book, are useful only in predicting the stress field in real components in regions away from points of application of loads. This does not present too much of a problem, since the stresses internal to a structure in such regions are often of most interest. If one wants to know what happens near the bolted connection, then one will have to create a complex model incorporating all the details and the problem will be more difficult to solve.

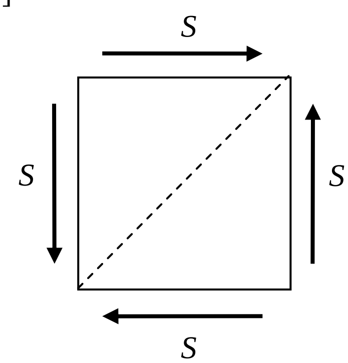
That said, it is an *experimental* fact that if two different force systems are applied to a material, but they are *equivalent force systems*, as in Fig. 3.3.12(f,g), then the stress fields in regions away from where the loads are applied will be the same. This is known as **Saint-Venant's Principle**. Typically, one needs to move a distance away from where the loads are applied roughly equal to the distance over which the loads are applied.

Saint-Venant's principle is extremely important in practical applications: we can replace a complicated problem by a simple model problem; the solution to this latter problem will often give us the information we require.

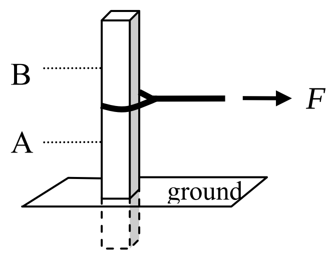
## 3.3.3 Problems

1. Derive Eqns. 3.3.1.

2. The four sides of a square block are subjected to equal *forces* $S$, as illustrated. The length of each side is $l$ and the block has unit depth (into the page). What normal and shear *stresses* act along the (dotted) diagonal? [Hint: draw a free body diagram of the upper left hand triangle.]

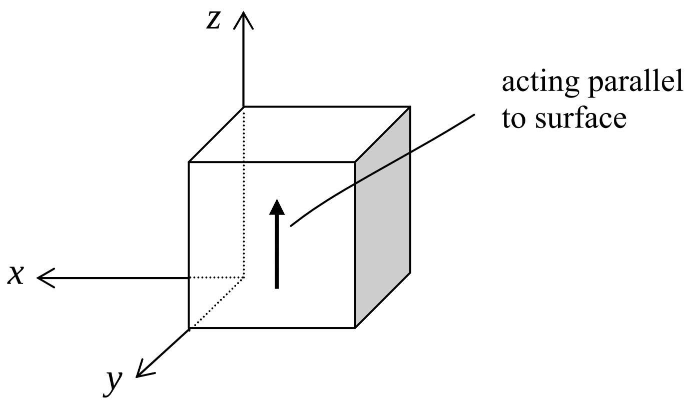



3. A shaft is concreted firmly into the ground. A thick steel rope is looped around the shaft and a force is applied normal to the shaft, as shown. The shaft is in static equilibrium. Draw a free body diagram of the shaft (from the top down to ground level) showing the forces/moments acting on the shaft (including the reaction forces at the ground-level; ignore the weight of the shaft). Draw a free body diagram of the section of shaft from the top down to the cross section at A. Draw a free body diagram of the section of shaft from the top to the cross section at B. Roughly sketch the stresses acting over the (horizontal) internal surfaces of the shaft at A and B.

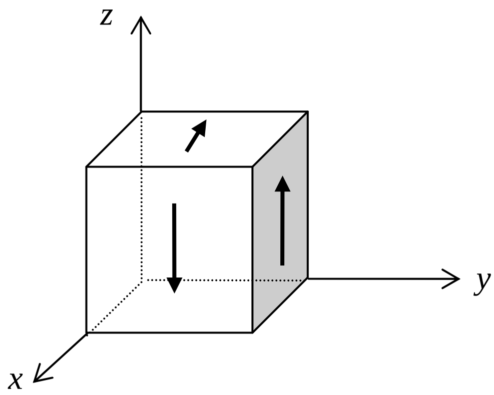



4. In Fig. 3.3.11, which of the stress components is/are negative?

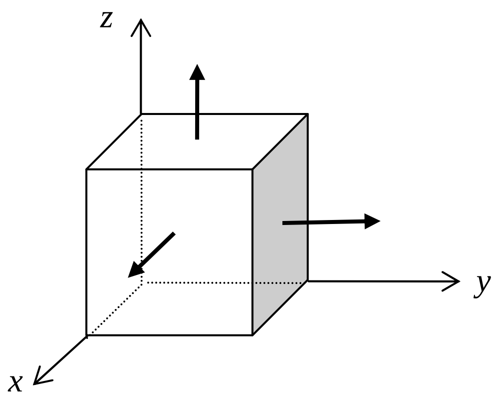
5. Label the following stress component acting on an internal material surface. Is it a positive or negative stress?



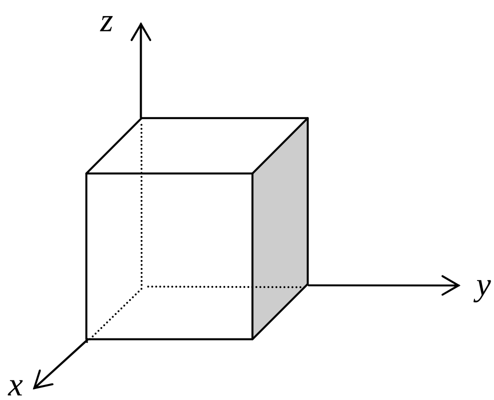
6. Label the following shear stresses. Are they positive or negative?




7. Label the following normal stresses. Are they positive or negative?




8. By the definition of the traction vector **t** which acts on the $x-z$ plane, $\mathbf{t}^{(\mathrm{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$ . Sketch these three stress components on the figure below.



# 3.4 Equilibrium of Stress

Consider two perpendicular planes passing through a point *p*. The stress components acting on these planes are as shown in Fig. 3.4.1a. These stresses are usually shown together acting on a small material element of finite size, Fig. 3.4.1b. It has been seen that the stress may vary from point to point in a material but, if the element is very small, the stresses on one side can be taken to be (more or less) equal to the stresses acting on the other side. By convention, in analyses of the type which will follow, all stress components shown are *positive*.

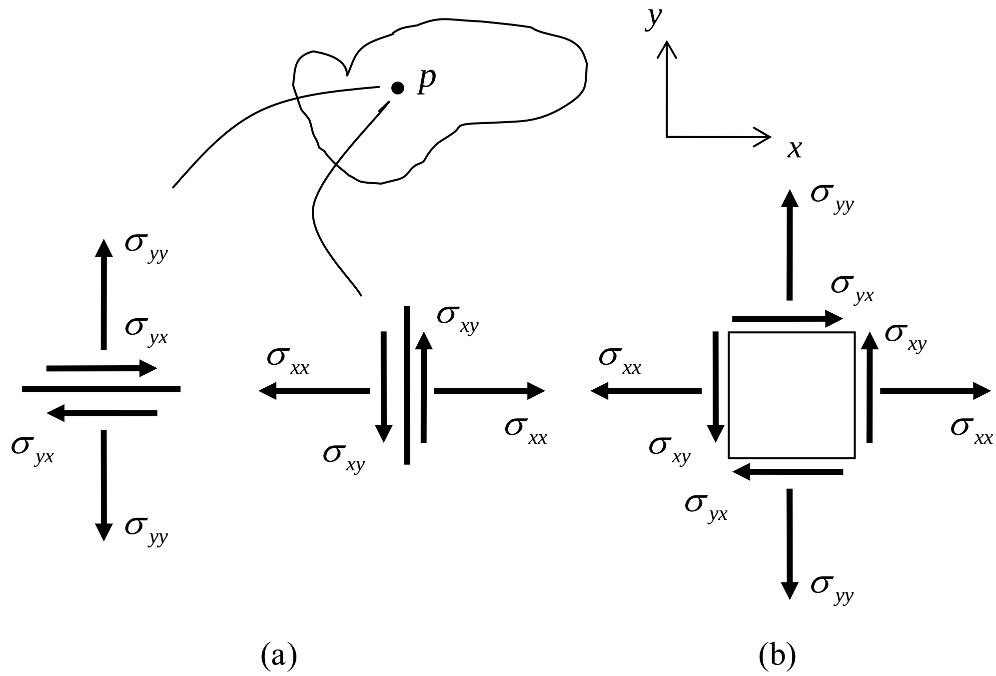



**Figure 3.4.1: stress components acting on two perpendicular planes through a point; (a) two perpendicular surfaces at a point, (b) small material element at the point**

The four stresses can conveniently be written in the form of a **stress matrix**:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \qquad (3.4.1)$$

It will be shown below that the stress components acting on *any* other plane through *p* can be evaluated from a knowledge of only these stress components.

## 3.4.1 Symmetry of the Shear Stress

Consider the material element shown in Fig. 3.4.1b, reproduced in Fig. 3.4.2a below. The element has dimensions is $\Delta x \times \Delta y$ and is subjected to uniform stresses over its sides. The resultant forces of the stresses acting on each side of the element act through the side-centres, and are shown in Fig. 3.4.2b. The stresses shown are positive, but note how *positive stresses can lead to negative forces*, depending on the definition of the $x-y$ axes used. The resultant force on the complete element is seen to be zero.

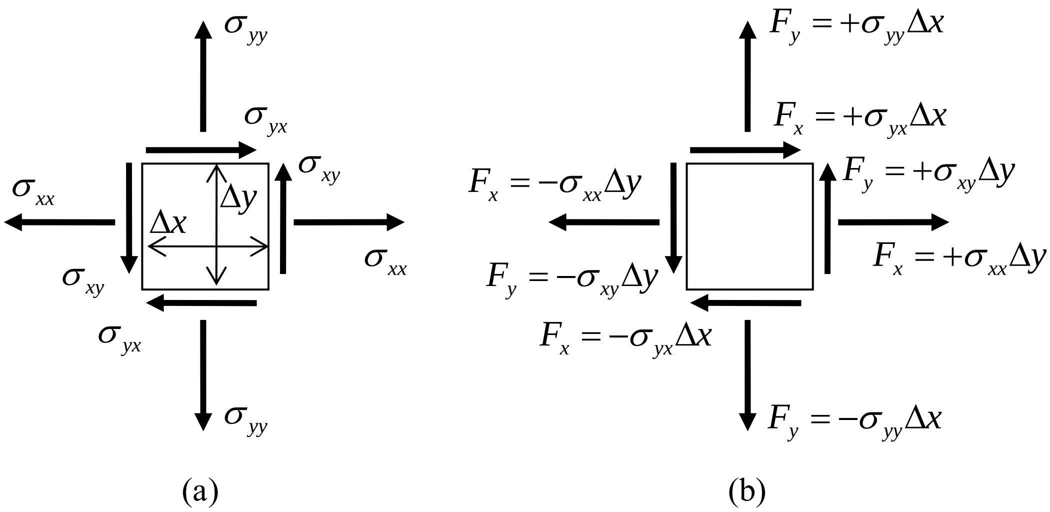



**Figure 3.4.2: stress components acting on a material element; (a) stresses, (b) resultant forces on each side**

By taking moments about any point in the block, one finds that {▲Problem 1}

$$\sigma_{xy} = \sigma_{yx} \tag{3.4.2}$$

Thus the shear stresses acting on the element are all equal, and for this reason the $\sigma_{yx}$ stresses are usually labelled $\sigma_{xy}$, Fig. 3.4.3a, or simply labelled $\tau$, Fig. 3.4.3b.

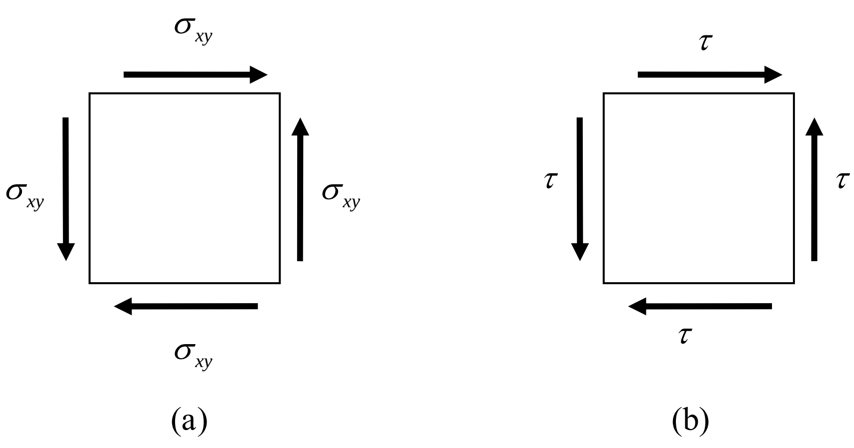



**Figure 3.4.3: shear stress acting on a material element**

In fact, in two-dimensional problems, the double-subscript notation is often dispensed with for simplicity, and the stress matrix can be expressed as

$$[\sigma]=\begin{bmatrix}\sigma_x & \tau \\ \tau & \sigma_y\end{bmatrix}, \tag{3.4.3}$$

to go along with the representation shown in Fig. 3.4.4.

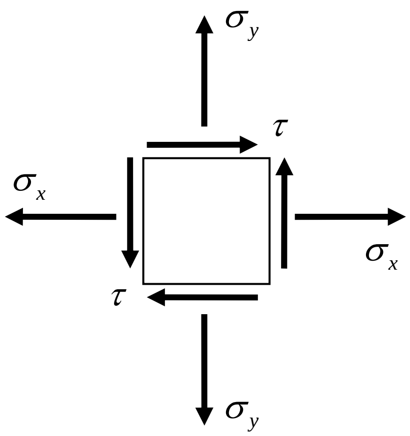



**Figure 3.4.4: a simpler notation for 2D stress components (without the double subscripts)**

## 3.4.2 Three Dimensional Stress

The three-dimensional counterpart to the two-dimensional element of Fig. 3.4.2 is shown in Fig. 3.4.5. Again, all stresses shown are positive.

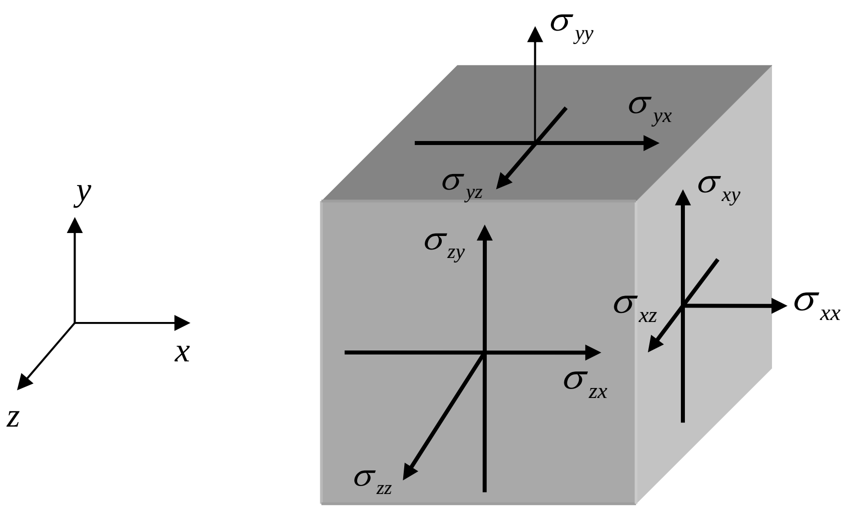



**Figure 3.4.5: a three dimensional material element**

Moment equilibrium in this case requires that

$$\sigma_{xy}=\sigma_{yx},\quad \sigma_{xz}=\sigma_{zx},\quad \sigma_{yz}=\sigma_{zy} \tag{3.4.4}$$

The nine stress components, six of which are independent, can be written in the matrix form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{3.4.5}$$

A vector **F** has one direction associated with it and is characterised by *three* components $(F_x, F_y, F_z)$. The stress is a quantity which has two directions associated with it (the direction of a force and the normal to the plane on which the force acts) and is characterised by the *nine* components of Eqn. 3.4.5. Such a mathematical object is called a **tensor**. Just as the three components of a vector change with a change of coordinate axes (for example, as in Fig. 2.2.1), so the nine components of the **stress tensor** change with a change of axes. This is discussed in the next section for the two-dimensional case.

## 3.4.3 Stress Transformation Equations

Consider the case where the nine stress components acting on three perpendicular planes through a material particle are known. These components are $\sigma_{xx}, \sigma_{xy}$, etc. when using $x, y, z$ axes, and can be represented by the cube shown in Fig. 3.4.6a. Rotate now the planes about the three axes – these new planes can be represented by the rotated cube shown in Fig. 3.4.6b; the axes normal to the planes are now labelled $x', y', z'$ and the corresponding stress components with respect to these new axes are $\sigma'_{xx}, \sigma'_{xy}$, etc.

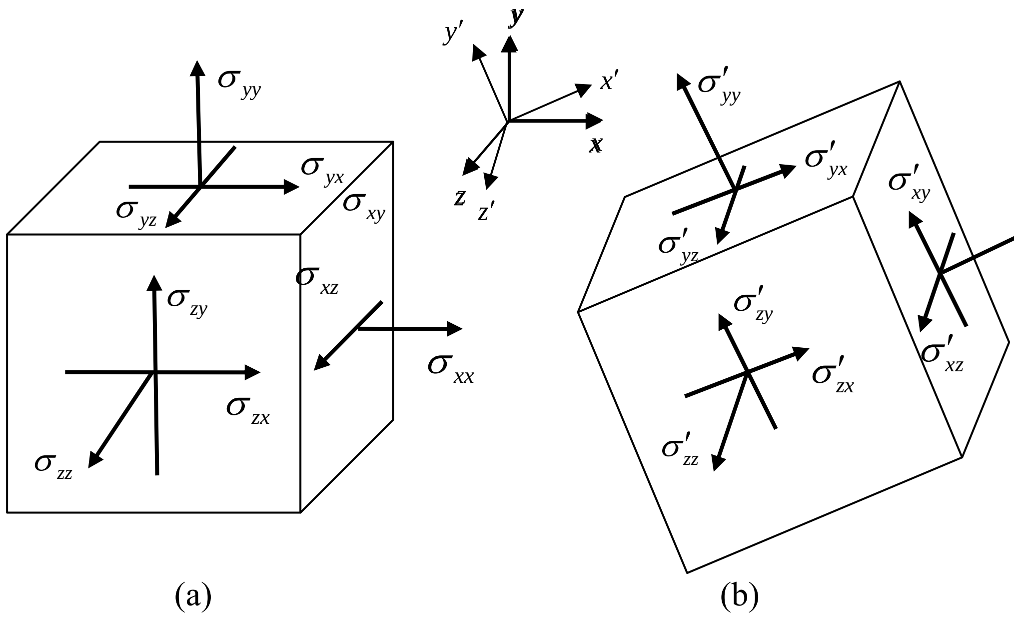



**Figure 3.4.6: a three dimensional material element; (a) original element, (b) rotated element**

There is a relationship between the stress components $\sigma_{xx}, \sigma_{xy}$, etc. and the stress components $\sigma'_{xx}, \sigma'_{xy}$, etc. The relationship can be derived using Newton's Laws. The equations describing the relationship in the fully three-dimensional case are very lengthy. Here, the relationship for the two-dimensional case will be derived – this 2D relationship will prove very useful in analysing many practical situations.

### Two-dimensional Stress Transformation Equations

Assume that the stress components of Fig. 3.4.7a are known. It is required to find the stresses arising on other planes through *p*. Consider the perpendicular planes shown in Fig. 3.4.7b, obtained by rotating the original element through a positive (counterclockwise) angle $\theta$. The new surfaces are defined by the axes $x' - y'$.

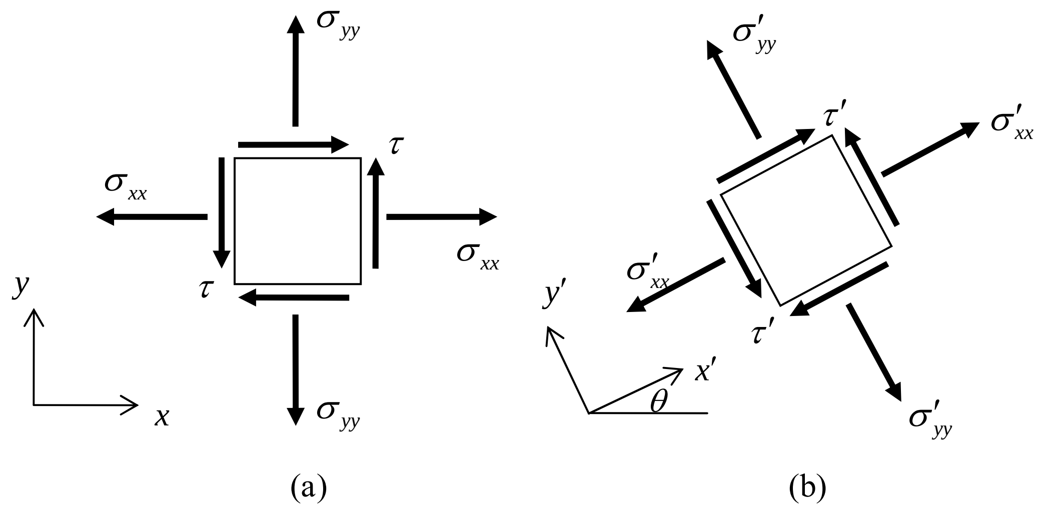



**Figure 3.4.7: stress components acting on two different sets of perpendicular surfaces, i.e. in two different coordinate systems; (a) original system, (b) rotated system**

To evaluate these new stress components, consider a triangular element of material at the point, Fig. 3.4.8. Carrying out force equilibrium in the direction $x'$, one has (with unit depth into the page)

$$\sum F_{x'}: \quad \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta = 0 \qquad (3.4.6)$$

Since $|OB| = |AB|\cos\theta,\ |OA| = |AB|\sin\theta$, and dividing through by $|AB|$,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \qquad (3.4.7)$$

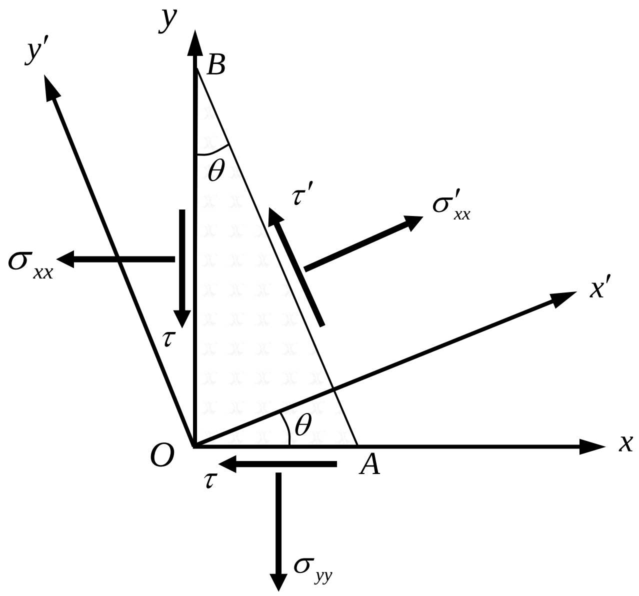



**Figure 3.4.8: a free body diagram of a triangular element of material**

The forces can also be resolved in the $y'$ direction and one obtains the relation

$$\tau' = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau\cos 2\theta \tag{3.4.8}$$

Finally, consideration of the element in Fig. 3.4.9 yields two further relations, one of which is the same as Eqn. 3.4.8.

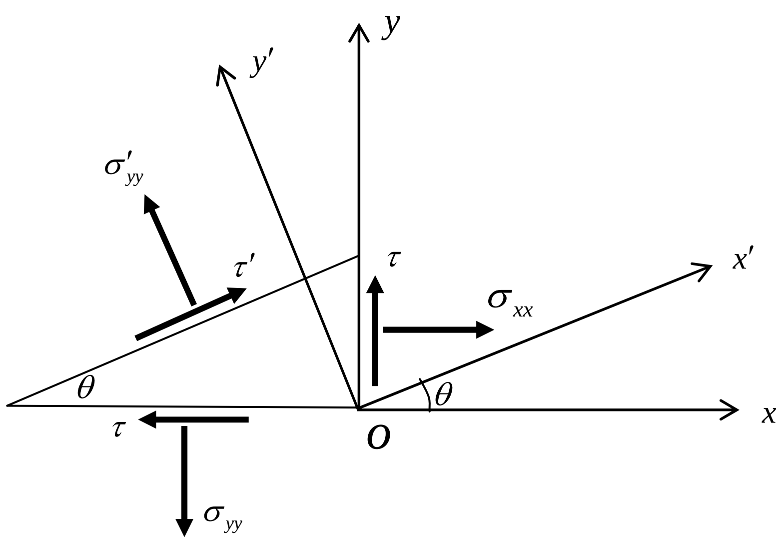



**Figure 3.4.9: a free body diagram of a triangular element of material**

In summary, one obtains the **stress transformation equations**:

$$\boxed{\begin{aligned}\sigma'_{xx} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\ \sigma'_{yy} &= \sin^2\theta\sigma_{xx} + \cos^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\ \sigma'_{xy} &= \sin\theta\cos\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy}\end{aligned}}$$

**2D Stress Transformation Equations** (3.4.9)

These equations have many uses, as will be seen in the next section.

In matrix form,

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (3.4.10)$$

These relations hold also in the case when there are body forces, when the material is accelerating and when there are non-uniform stress fields. (This is discussed in the next section.)

## 3.4.4 Problems

1. Derive Eqns. 3.4.2 by taking moments about the lower left corner of the block in Fig. 3.4.2.

2. Suppose that the stresses acting on two perpendicular planes through a point are

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Use the stress transformation formulae to evaluate the stresses acting on two new perpendicular planes through the point, obtained from the first set by a positive rotation of $30^{\circ}$. Use the conventional notation $x' - y'$ to represent the coordinate axes parallel to these new planes.

# 3.4b Stress Transformation: Further Aspects

Here, it will be shown that the Stress Transformation Equations are valid also when (i) there are body forces, (ii) the body is accelerating and (iii) the stress and other quantities are not uniform. We will also examine the fully three-dimensional stress subject to the transformation.

Suppose that a body force $\mathbf{F}_b = (\mathbf{F}_b)_x\,\mathbf{i} + (\mathbf{F}_b)_y\,\mathbf{j}$ acts on the material and that the material is accelerating with an acceleration $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$. The components of body force and acceleration are shown in Fig. 3.4.10 (a reproduction of Fig. 3.4.8).

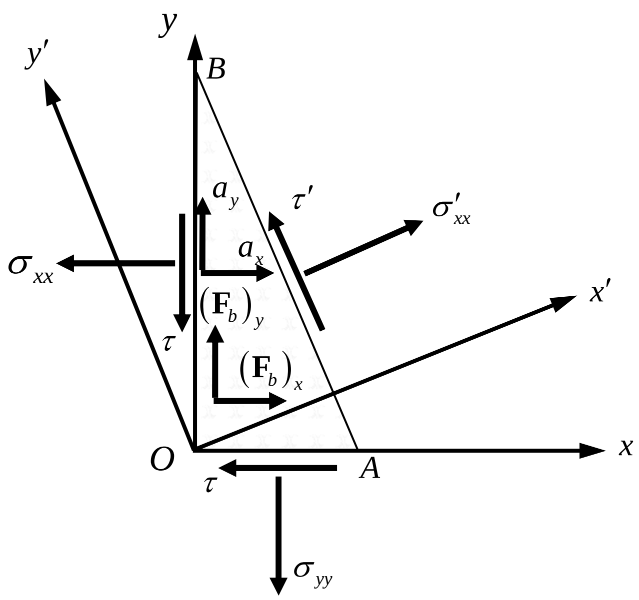



**Figure 3.4.10: a free body diagram of a triangular element of material, including a body force and acceleration**

The body force will vary depending on the size of the material under consideration, e.g. the force of gravity $\mathbf{F}_b = m\mathbf{g}$ will be larger for larger materials; therefore consider a quantity which is independent of the amount of material: the body force per unit mass, $\mathbf{F}_b / m$. Then, Eqn 3.4.6 now reads

$$\begin{aligned}\sum F_{x'}: \quad & \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta \\ & + (\mathbf{F}_b/m)_x\, m\cos\theta + (\mathbf{F}_b/m)m\sin\theta + ma_x\cos\theta + ma_y\sin\theta = 0\end{aligned} \tag{3.4.11}$$

where $m$ is the mass of the triangular portion of material. The volume of the triangle is $\frac{1}{2}|OA||OB| = |AB|^2/\sin 2\theta$ so that, this time, when 3.4.11 is divided through by $|AB|$, one is left with

$$\begin{aligned}\sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \\ &\quad -|AB|\rho\left\{(\mathbf{F}_b/m)_x/2\sin\theta + (\mathbf{F}_b/m)/2\cos\theta + a_x/2\sin\theta + a_y/2\cos\theta\right\}\end{aligned} \tag{3.4.12}$$

where $\rho$ is the density. Now, as the element is shrunk in size down to the vertex *O*, $|AB| \to 0$, and Eqn. 3.4.6 is recovered. Thus the Stress Transformation Equations are valid provided the element under consideration is very small; in the limit, they are valid "at the point" *O*.

Finally, consider the case where the stress is not uniform over the faces of the triangular portion of material. Intuitively, it can be seen that, if one again shrinks the portion of material down in size to the vertex *O*, the Stress Transformation Equations will again be valid, with the quantities $\sigma'_{xx}, \sigma_{xx}, \sigma_{yy}$ etc. being the values "at" the vertex. To be more precise, consider the $\sigma_{xx}$ stress acting over the face $|OB|$ in Fig. 3.4.11. No matter how the stress varies in the material, if the distance $|OB|$ is small, the stress can be approximated by a linear stress distribution, Fig. 3.4.11b. This linear distribution can itself be decomposed into two components, a uniform stress of magnitude $\sigma^o_{xx}$ (the value of $\sigma_{xx}$ at the vertex) and a triangular distribution with maximum value $\Delta\sigma_{xx}$. The resultant force on the face is then $|OB|\left(\sigma^o_{xx} + \Delta\sigma_{xx} / 2\right)$. This time, as the element is shrunk in size, $\Delta\sigma_{xx} \to 0$ and Eqn. 3.4.6 is again recovered. The same argument can be used to show that the Stress Transformation Equations are valid for any varying stress, body force or acceleration.

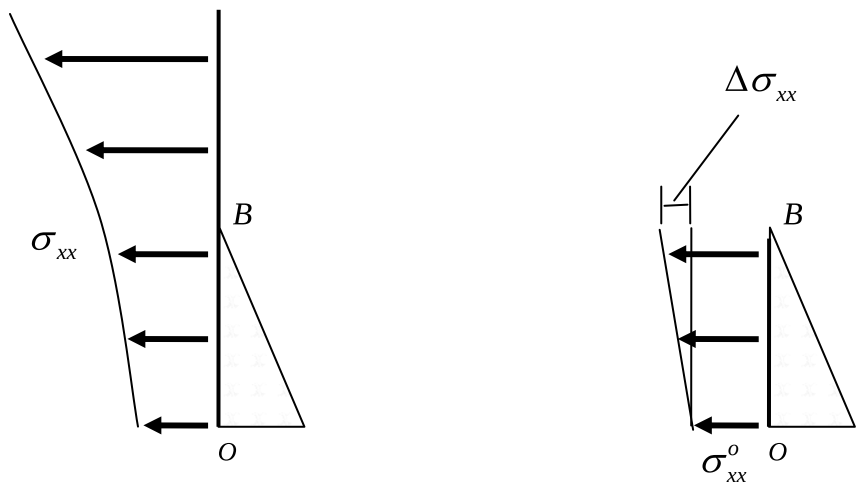



**Figure 3.4.11: stress varying over a face; (a) stress is linear over OB if OB is small, (b) linear distribution of stress as a uniform stress and a triangular stress**

## Three Dimensions Re-visited

As the planes were rotated in the two-dimensional analysis, no consideration was given to the stresses acting in the "third dimension". Considering again a three dimensional block, Fig. 3.4.12, there is only one traction vector acting *on* the $x-y$ plane at the material particle, $\mathbf{t}$. This traction vector can be described in terms of the $x, y, z$ axes as $\mathbf{t} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$, Fig 3.4.12a. Alternatively, it can be described in terms of the $x', y', z'$ axes as $\mathbf{t} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}' + \sigma'_{zz}\mathbf{k}'$, Fig 3.4.12b.

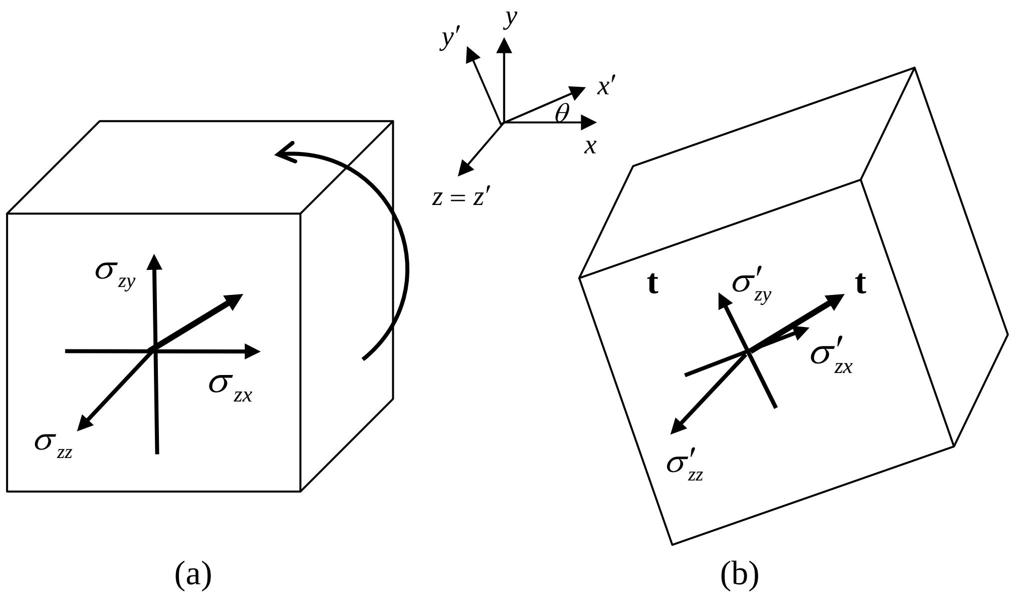



**Figure 3.4.12: a three dimensional material element; (a) original element, (b) rotated element (rotation about the *z* axis)**

With the rotation only happening *in* the $x-y$ plane, about the *z* axis, one has $\sigma_{zz} = \sigma'_{zz}$, $\mathbf{k} = \mathbf{k}'$. One can thus examine the two dimensional $x-y$ plane shown in Fig. 3.4.13, with

$$\sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}'. \qquad (3.4.13)$$

Using some trigonometry, one can see that

$$\begin{aligned} \sigma'_{zx} &= +\sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta \\ \sigma'_{zy} &= -\sigma_{zx}\sin\theta + \sigma_{zy}\cos\theta \end{aligned}. \qquad (3.4.14)$$

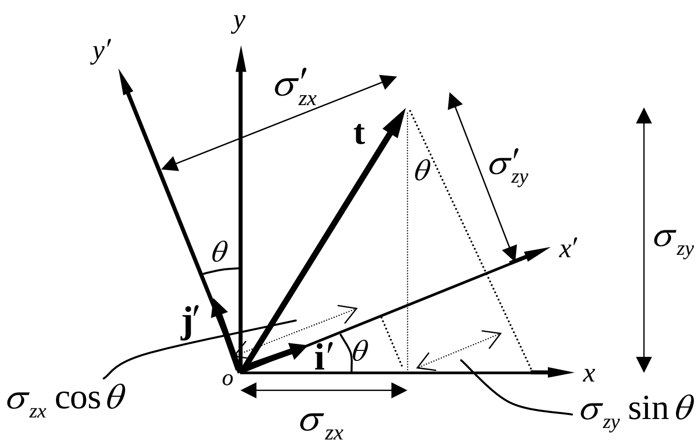



**Figure 3.4.12: the traction vector represented using two different coordinate systems**

# 3.5 Plane Stress

This section is concerned with a special two-dimensional state of stress called **plane stress**. It is important for two reasons: (1) it arises in real components (particularly in thin components loaded in certain ways), and (2) it is a two dimensional state of stress, and thus serves as an excellent introduction to more complicated three dimensional stress states.

## 3.5.1 Plane Stress

The state of plane stress is defined as follows:

**Plane Stress**:
If the stress state at a material particle is such that the only non-zero stress components act in one plane only, the particle is said to be in plane stress.

The axes are usually chosen such that the $x-y$ plane is the plane in which the stresses act, Fig. 3.5.1.

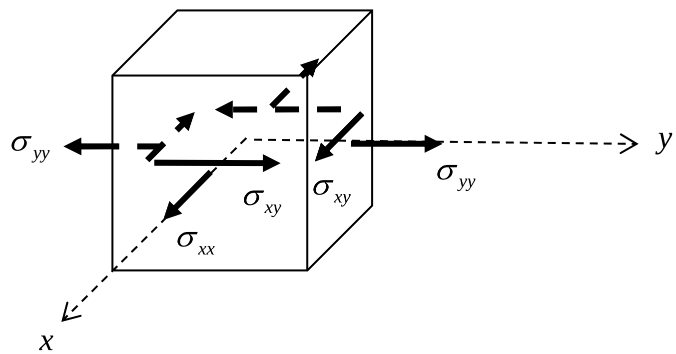



**Figure 3.5.1: non-zero stress components acting in the $x-y$ plane**

The stress can be expressed in the matrix form 3.4.1.

### Example

The thick block of uniform material shown in Fig. 3.5.2, loaded by a constant stress $\sigma_o$ in the $x$ direction, will have $\sigma_{xx} = \sigma_0$ and all other components zero everywhere. It is therefore in a state of plane stress.

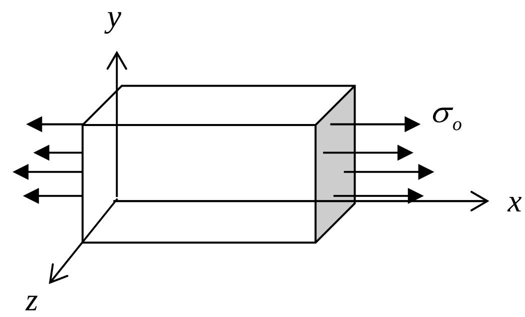



**Figure 3.5.2: a thick block of material in plane stress**

## 3.5.2 Analysis of Plane Stress

Next are discussed the **stress invariants**, **principal stresses** and **maximum shear stresses** for the two-dimensional plane state of stress, and tools for evaluating them. These quantities are useful because they tell us the complete state of stress at a point in simple terms. Further, these quantities are directly related to the strength and response of materials. For example, the way in which a material plastically (permanently) deforms is often related to the maximum shear stress, the directions in which flaws/cracks grow in materials are often related to the principal stresses, and the energy stored in materials is often a function of the stress invariants.

### Stress Invariants

A stress invariant is some function of the stress components which is independent of the coordinate system being used; in other words, they have the same value no matter where the $x-y$ axes are drawn through a point. In a two dimensional space there are two stress invariants, labelled $I_1$ and $I_2$. These are

$$\boxed{\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} \\ I_2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned}} \qquad \textbf{Stress Invariants} \qquad (3.5.1)$$

These quantities can be proved to be invariant directly from the stress transformation equations, Eqns. 3.4.9 {▲Problem 1}. Physically, invariance of $I_1$ and $I_2$ means that they are the same for any chosen perpendicular planes through a material particle.

Combinations of the stress invariants are also invariant, for example the important quantity

$$\tfrac{1}{2}I_1 \pm \sqrt{\tfrac{1}{4}I_1^2 - I_2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\tfrac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2} \qquad (3.5.2)$$

## Principal Stresses

Consider a material particle for which the stress, with respect to some $x-y$ coordinate system, is

$$\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} \tag{3.5.3}$$

The stress acting on different planes through the point can be evaluated using the Stress Transformation Equations, Eqns. 3.4.9, and the results are plotted in Fig. 3.5.3. The original planes are re-visited after rotating $180^{\circ}$.

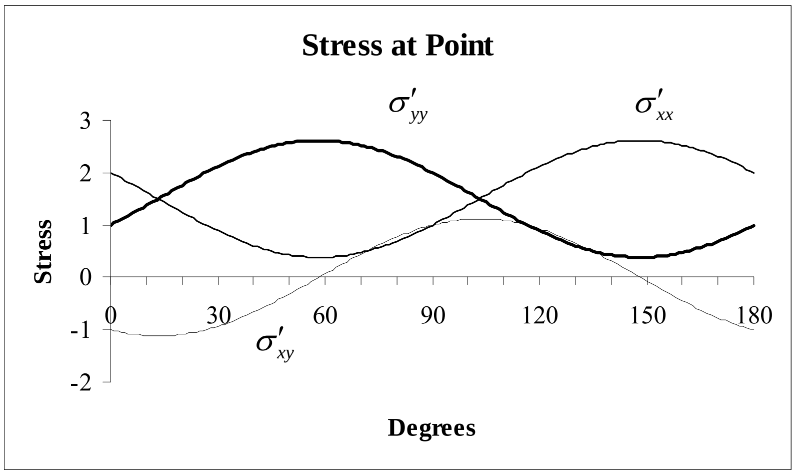



**Figure 3.5.3: stresses on different planes through a point**

It can be seen that there are two perpendicular planes for which the shear stress is zero, for $\theta \approx 58^{\circ}$ and $\theta \approx (58+90)^{\circ}$. In fact it can be proved that for every point in a material there are two (and only two) perpendicular planes on which the shear stress is zero (see below). These planes are called the **principal planes**. It will also be noted from the figure that the normal stresses acting on the planes of zero shear stress are either a maximum or minimum. Again, this can be proved (see below). These normal stresses are called principal stresses. The principal stresses are labelled $\sigma_1$ and $\sigma_2$, Fig. 3.5.4.

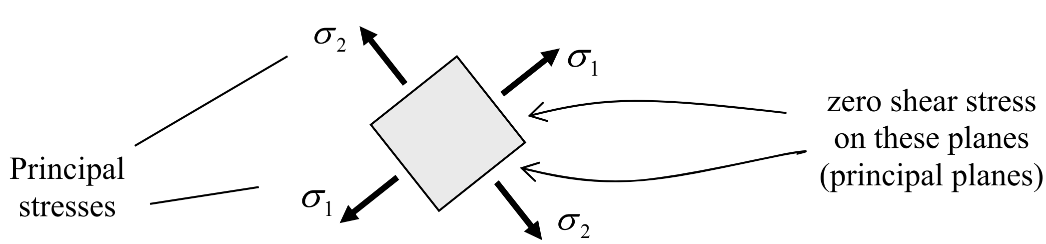



**Figure 3.5.4: principal stresses**

The principal stresses can be obtained by setting $\sigma'_{xy} = 0$ in the Stress Transformation Equations, Eqns. 3.4.9, which leads to the value of $\theta$ for which the planes have zero shear stress:

$$\boxed{\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}} \qquad \textbf{Location of Principal Planes} \qquad (3.5.4)$$

For the example stress state, Eqn. 3.5.3, this leads to

$$\theta = \frac{1}{2}\arctan(-2)$$

and so the perpendicular planes are at $\theta = -31.72° \;(148.28°)$ and $\theta = 58.3°$.

Explicit expressions for the principal stresses can be obtained by substituting the value of $\theta$ from Eqn. 3.5.4 into the Stress Transformation Equations, leading to (see the Appendix to this section, §3.5.7)

$$\boxed{\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma_2 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \end{aligned}} \qquad \textbf{Principal Stresses} \qquad (3.5.5)$$

For the example stress state Eqn.3.5.3, one has

$$\sigma_1 = \frac{3+\sqrt{5}}{2} \approx 2.62, \qquad \sigma_2 = \frac{3-\sqrt{5}}{2} \approx 0.38$$

Note here that one uses the symbol $\sigma_1$ to represent the maximum principal stress and $\sigma_2$ to represent the minimum principal stress. By maximum, it is meant the algebraically largest stress so that, for example, $+1 > -3$.

From Eqns. 3.5.2, 3.5.5, the principal stresses are invariant; they are intrinsic features of the stress state at a point and do not depend on the coordinate system used to describe the stress state.

The question now arises: why are the principal stresses so important? One part of the answer is that the maximum principal stress is the largest normal stress acting on any plane through a material particle. This can be proved by differentiating the stress transformation formulae with respect to θ,

$$\begin{aligned}\frac{d\sigma'_{xx}}{d\theta} &= -\sin 2\theta(\sigma_{xx} - \sigma_{yy}) + 2\cos 2\theta \sigma_{xy} \\ \frac{d\sigma'_{yy}}{d\theta} &= +\sin 2\theta(\sigma_{xx} - \sigma_{yy}) - 2\cos 2\theta \sigma_{xy} \\ \frac{d\sigma'_{xy}}{d\theta} &= -\cos 2\theta(\sigma_{xx} - \sigma_{yy}) - 2\sin 2\theta \sigma_{xy}\end{aligned} \tag{3.5.6}$$

The maximum/minimum values can now be obtained by setting these expressions to zero. One finds that the normal stresses are a maximum/minimum at the very value of $\theta$ in Eqn. 3.5.4 – the value of $\theta$ for which the shear stresses are zero – the principal planes.

Very often the only thing one knows about the stress state at a point are the principal stresses. In that case one can derive a very useful formula as follows: align the coordinate axes in the principal directions, so

$$\sigma_{xx} = \sigma_1, \quad \sigma_{yy} = \sigma_2, \quad \sigma_{xy} = 0 \tag{3.5.7}$$

Using the transformation formulae with the relations $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ then leads to

$$\boxed{\begin{aligned}\sigma'_{xx} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \sigma'_{yy} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \sigma'_{xy} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta\end{aligned}} \tag{3.5.8}$$

Here, $\theta$ is measured *from* the principal directions, as illustrated in Fig. 3.5.5.

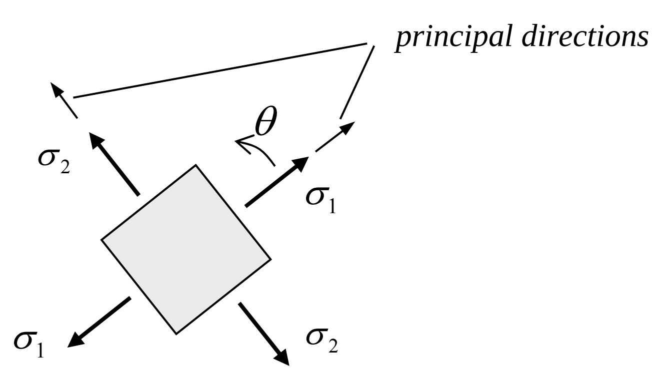



**Figure 3.5.5: principal stresses and principal directions**

### The Third Principal Stress

Although plane stress is essentially a two-dimensional stress-state, it is important to keep in mind that any real particle is three-dimensional. The stresses acting *on* the $x-y$ plane are the normal stress $\sigma_{zz}$ and the shear stresses $\sigma_{zx}$ and $\sigma_{zy}$, Fig. 3.5.6. These are all zero (in plane stress). It was discussed above how the principal stresses occur on planes of zero shear stress. Thus the $\sigma_{zz}$ stress is also a principal stress. Technically speaking, there are always three principal stresses in three dimensions, and (at least) one of these will be zero in plane stress. This fact will be used below in the context of maximum shear stress.

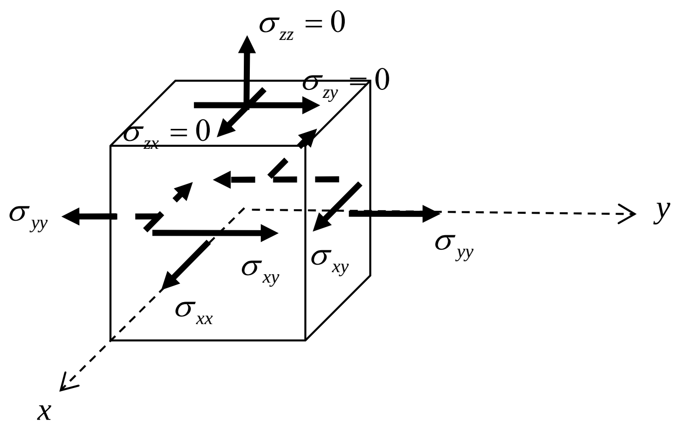



**Figure 3.5.6: stresses acting on the *x* – *y* plane**

### Maximum Shear Stress

Eqns. 3.5.8 can be used to derive an expression for the maximum shear stress. Differentiating the expression for shear stress with respect to $\theta$, setting to zero and solving, shows that the maximum/minimum occurs at $\theta = \pm 45$, in which case

$$\sigma_{xy}\big|_{\theta=+45} = -\frac{1}{2}(\sigma_1 - \sigma_2), \qquad \sigma_{xy}\big|_{\theta=-45} = +\frac{1}{2}(\sigma_1 - \sigma_2)$$

or

$$\boxed{\max(\sigma_{xy}) = \frac{1}{2}|\sigma_1 - \sigma_2|} \qquad \textbf{Maximum Shear Stress} \qquad (3.5.9)$$

Thus the shear stress reaches a maximum on planes which are oriented at $\pm 45°$ to the principal planes, and the value of the shear stress acting on these planes is as given above. Note that the formula Eqn. 3.5.9 does not let one know in which *direction* the shear stresses are acting but this is not usually an important issue. Many materials respond in certain ways when the maximum shear stress reaches a critical value, and the actual direction of shear stress is unimportant. The direction of the maximum principal stress is, on the other hand, important – a material will in general respond differently according to whether the normal stress is compressive or tensile.

The normal stress acting on the planes of maximum shear stress can be obtained by substituting $\theta = \pm 45$ back into the formulae for normal stress in Eqn. 3.5.8, and one sees that

$$\sigma'_{xx} = \sigma'_{yy} = (\sigma_1 + \sigma_2)/2 \tag{3.5.10}$$

The results of this section are summarised in Fig. 3.5.7.

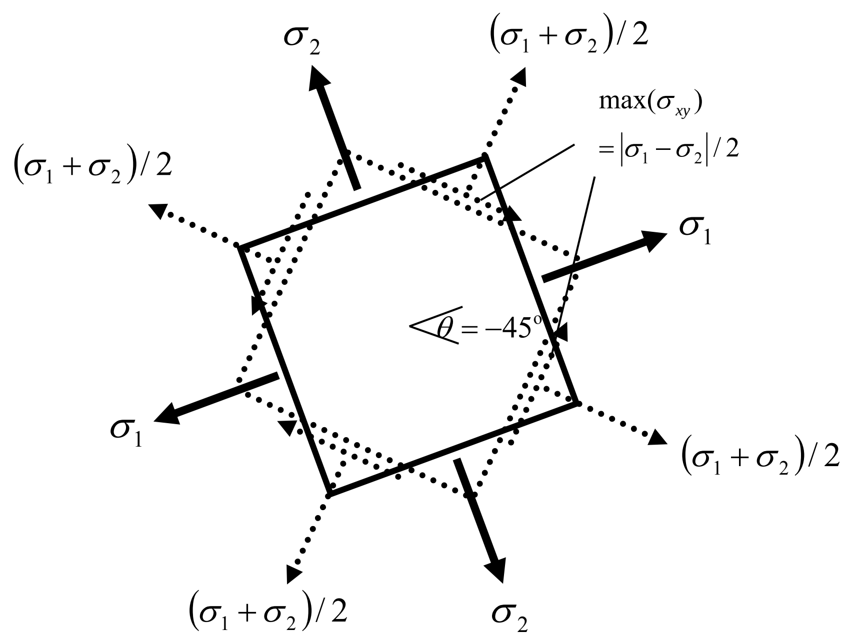



**Figure 3.5.7: principal stresses and maximum shear stresses acting in the $x - y$ plane**

The maximum shear stress in the $x - y$ plane was calculated above, Eqn. 3.5.9. This is not necessarily the maximum shear stress acting at the material particle. In general, it can be shown that the maximum shear stress is the maximum of the following three terms (see Part III, §3.4.3):

$$\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1 - \sigma_3|, \quad \frac{1}{2}|\sigma_2 - \sigma_3|$$

The first term is the maximum shear stress in the $1-2$ plane, i.e. the plane containing the $\sigma_1$ and $\sigma_2$ stresses (and given by Eqn. 3.5.9). The second term is the maximum shear stress in the $1-3$ plane and the third term is the maximum shear stress in the $2-3$ plane. These are sketched in Fig. 3.5.8 below.

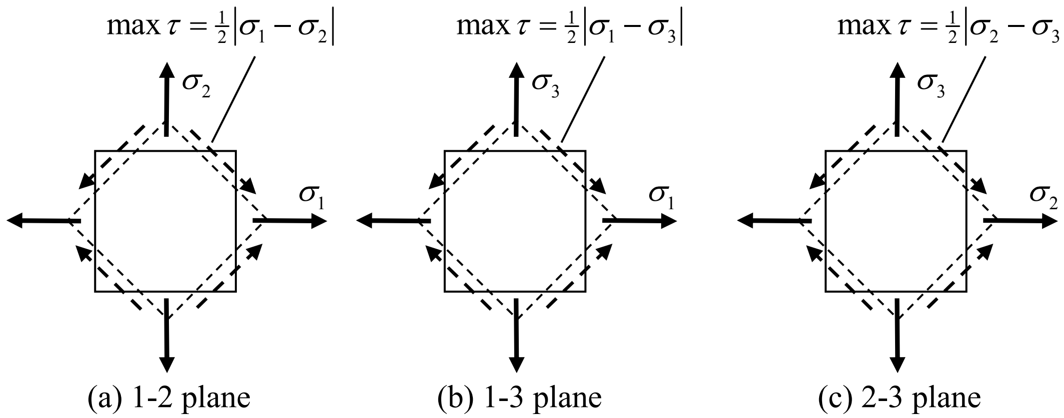



**Figure 3.5.8: principal stresses and maximum shear stresses**

In the case of plane stress, $\sigma_3 = \sigma_{zz} = 0$, and the maximum shear stress will be (see the Appendix to this section, §3.5.7)

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1|, \quad \frac{1}{2}|\sigma_2|\right\} \tag{3.5.11}$$

## 3.5.3 Stress Boundary Conditions

When solving problems, information is usually available on what is happening at the boundaries of materials. This information is called the **boundary conditions**. Information is usually not available on what is happening in the interior of the material – information there is obtained by solving the equations of mechanics.

A number of different conditions can be known at a boundary, for example it might be known that a certain part of the boundary is fixed so that the displacements there are zero. This is known as a **displacement boundary condition**. On the other hand the stresses over a certain part of the material boundary might be known. These are known as **stress boundary conditions** – this case will be examined here.

### General Stress Boundary Conditions

It has been seen already that, when one material contacts a second material, a force, or distribution of stress arises. This force $F$ will have arbitrary direction, Fig. 3.5.9a, and can be decomposed into the sum of a normal stress distribution $\sigma_N$ and a shear distribution $\sigma_S$, Fig. 3.5.9b. One can introduce a coordinate system to describe the applied stresses, for example the $x - y$ axes shown in Fig. 3.5.9c (the axes are most conveniently defined to be normal and tangential to the boundary).

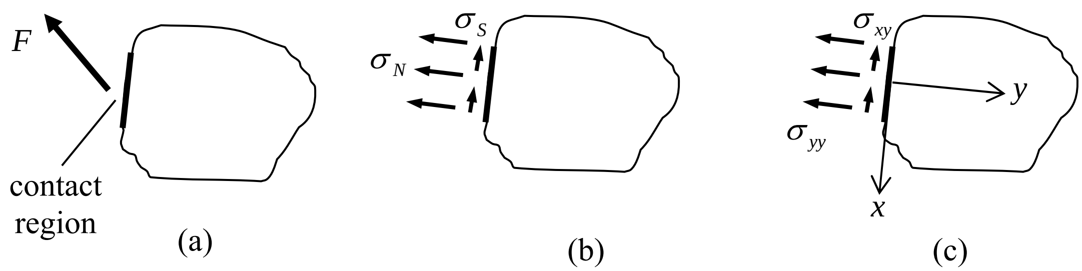



**Figure 3.5.9: Stress boundary conditions; (a) force acting on material due to contact with a second material, (b) the resulting normal and shear stress distributions, (c) applied stresses as stress components in a given coordinate system**

Figure 3.5.10 shows the same component as Fig. 3.5.9. Shown in detail is a small material element at the boundary. From equilibrium of the element, stresses $\sigma_{xy}, \sigma_{yy}$, equal to the applied stresses, must be acting inside the material, Fig. 3.5.10a. Note that the **tangential stresses**, which are the $\sigma_{xx}$ stresses in this example, can take on any value and the element will still be in equilibrium with the applied stresses, Fig. 3.5.10b.

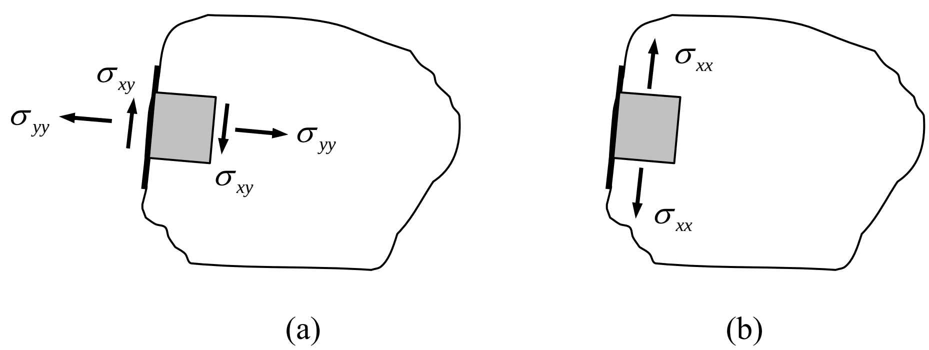



**Figure 3.5.10: Stresses acting on a material element at the boundary, (a) normal and shear stresses, (b) tangential stresses**

Thus, if the applied stresses are *known*, then so also are the normal and shear stresses acting at the boundary of the material.

## Stress Boundary Conditions at a Free Surface

A free surface is a surface that has "nothing" on one side and so there is nothing to provide reaction forces. Thus there must also be no normal or shear stress on the other side (the inside).

This leads to the following, Fig. 3.5.11:

**Stress boundary conditions at a free surface**:
the normal and shear stress at a free surface are zero

This simple fact is used again and again to solve practical problems.

Again, the stresses acting normal to any other plane at the surface do not have to be zero – they can be balanced as, for example, the tangential stresses $\sigma_T$ and the stress $\overline{\sigma}$ in Fig. 3.5.11.

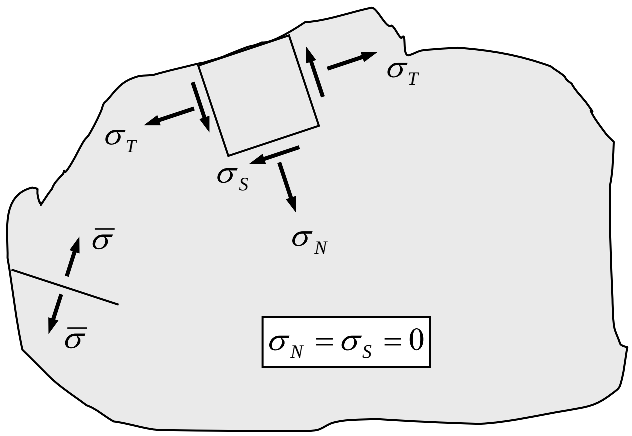



**Figure 3.5.11: A free surface - the normal and shear stresses there are zero**

## Atmospheric Pressure

There *is* something acting on the outside "free" surfaces of materials – the atmospheric pressure. This is a type of stress which is **hydrostatic**, that is, it acts normal at all points, as shown in Fig. 3.5.12. Also, it does not vary much. This pressure is present when one characterises a material, that is, when its material properties are determined from tests and so on, for example, its Young's Modulus (see Chapter 5). The atmospheric pressure is therefore a datum – stresses are really measured relative to this value, and so the atmospheric pressure is ignored.

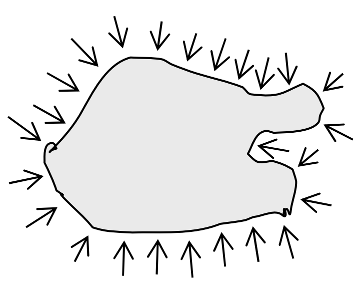


**Figure 3.5.12: a material subjected to atmospheric pressure**

### 3.5.4 Thin Components

Consider a thin component as shown in Fig. 3.5.13. With the coordinate axes aligned as shown, and with the large face free of loading, one has $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$. Strictly speaking, these stresses are zero *only* at the free surfaces of the material but, because it is thin, these stresses should not vary much from zero within. Taking the "z" stresses to be identically zero throughout the material, the component is in a state of plane stress[1]. On the other hand, were the sheet not so thin, the stress components that were zero at the free-surfaces might well deviate significantly from zero deep within the material and one could not safely argue that the component was in a state of plane stress.

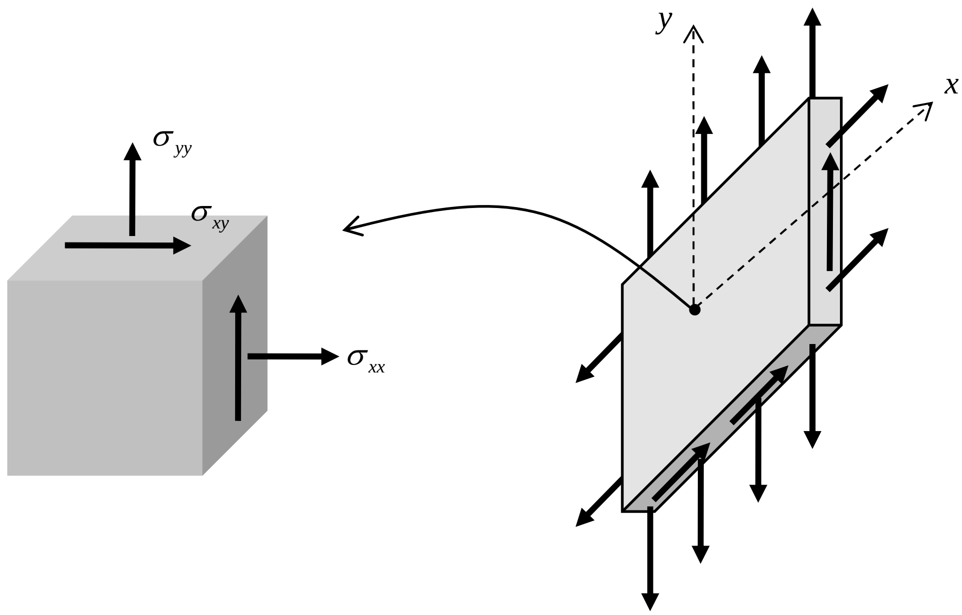



**Figure 3.5.13: a thin material loaded in-plane, leading to a state of plane stress**

When analysing plane stress states, only one cross section of the material need be considered. This is illustrated in Fig. 3.5.14.

[1] it can be shown that, when the applied stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ vary only *linearly* over the thickness of the component, the stresses $\sigma_{zz}, \sigma_{zx}, \sigma_{zy}$ are exactly zero throughout the component, otherwise they are only approximately zero

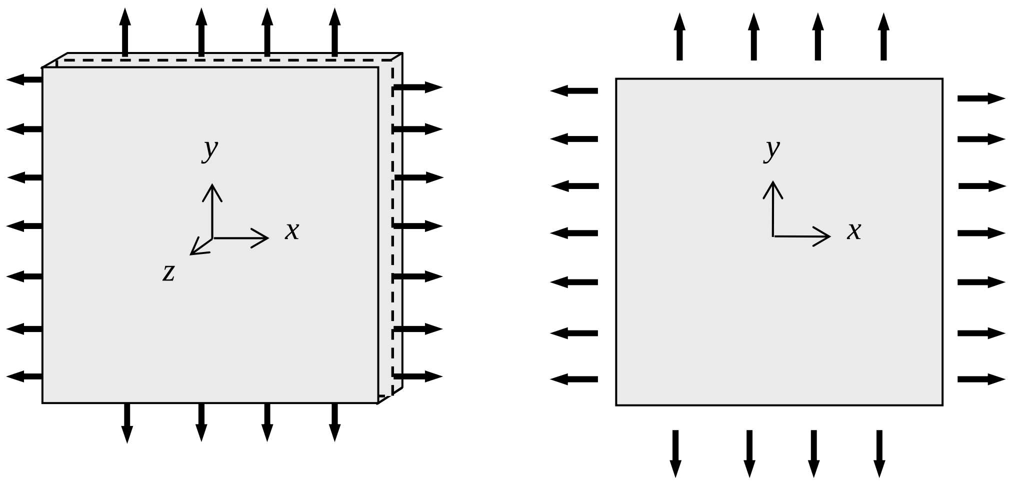



**Figure 3.5.14: one two-dimensional cross-section of material**

Note that, although the stress normal to the plane, $\sigma_{zz}$, is zero, the three dimensional sheet of material *is* deforming in this direction – it will obviously be getting thinner under the tensile loading shown in Fig. 3.5.14.

Note that plane stress arises in *all* thin materials (loaded in –plane), no matter what they are made of.

### 3.5.5 Mohr's Circle

Otto Mohr devised a way of describing the state of stress at a point using a single diagram, called the **Mohr's circle**.

To construct the Mohr circle, first introduce the **stress coordinates** $(\sigma, \tau)$, Fig. 3.5.15; the abscissae (horizontal) are the normal stresses $\sigma$ and the ordinates (vertical) are the shear stresses $\tau$. On the horizontal axis, locate the principal stresses $\sigma_1, \sigma_2$, with $\sigma_1 > \sigma_2$. Next, draw a circle, centred at the average principal stress $(\sigma, \tau) = ((\sigma_1 + \sigma_2)/2, 0)$, having radius $(\sigma_1 - \sigma_2)/2$.

The normal and shear stresses acting on a single plane are represented by a single point on the Mohr circle. The normal and shear stresses acting on two perpendicular planes are represented by two points, one at *each end of a diameter* on the Mohr circle. Two such diameters are shown in the figure. The first is horizontal. Here, the stresses acting on two perpendicular planes are $(\sigma, \tau) = (\sigma_1, 0)$ and $(\sigma, \tau) = (\sigma_2, 0)$ and so this diameter represents the principal planes/stresses.

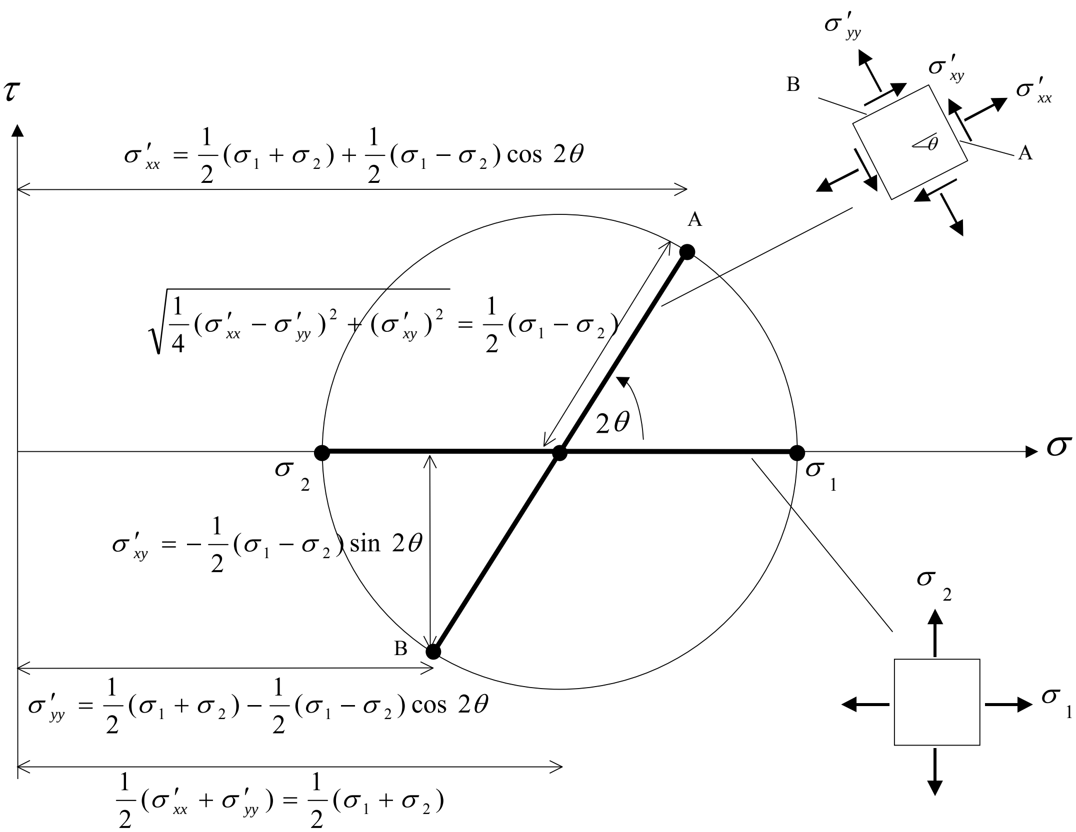



**Figure 3.5.15: Mohr's Circle**

The stresses on planes rotated by an amount $\theta$ from the principal planes are given by Eqn. 3.5.8. Using elementary trigonometry, these stresses are represented by the points A and B in Fig. 3.5.15. Note that a rotation of $\theta$ in the physical plane corresponds to a rotation of $2\theta$ in the Mohr diagram.

Note also that the conventional labeling of shear stress has to be altered when using the Mohr diagram. On the Mohr circle, a shear stress is positive if it yields a clockwise moment about the centre of the element, and is "negative" when it yields a negative moment. For example, at point A the shear stress is "positive" ($\tau > 0$), which means the direction of shear on face A of the element is actually opposite to that shown. This agrees with the formula $\sigma'_{xy} = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$, which is less than zero for $\sigma_1 > \sigma_2$ and $\theta \le 90^\circ$. At point B the shear stress is "negative" ($\tau < 0$), which again agrees with formula.

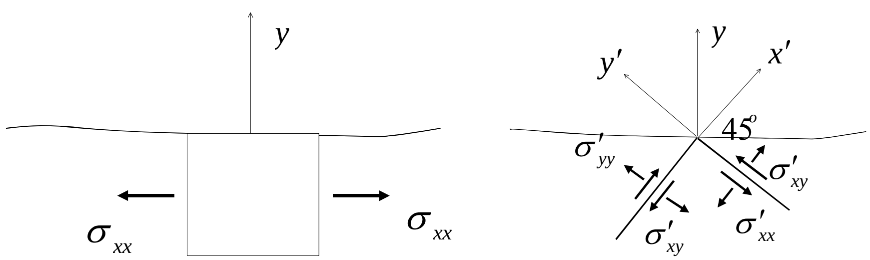
## 3.5.6 Problems

1. Prove that the function $\sigma_x + \sigma_y$, i.e. the sum of the normal stresses acting at a point, is a stress invariant. [Hint: add together the first two of Eqns. 3.4.9.]

2. Consider a material in plane stress conditions. An element at a free surface of this material is shown below left. Taking the coordinate axes to be orthogonal to the surface as shown (so that the tangential stress is $\sigma_{xx}$), one has

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{bmatrix}$$

   (a) what are the two in-plane principal stresses at the point? Which is the maximum and which is the minimum?
   (b) examine planes inclined at 45° to the free surface, as shown below right. What are the stresses acting on these planes and what have they got to do with maximum shear stress?




3. The stresses at a point in a state of plane stress are given by

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & 2 \end{bmatrix}$$

   (a) Draw a little box to represent the point and draw some arrows to indicate the magnitude and direction of the stresses acting at the point.
   (b) What relationship exists between $Oxy$ and a second coordinate set $Ox'y'$, such that the shear stresses are zero in $Ox'y'$?
   (c) Find the two in-plane principal stresses.
   (d) Draw another box whose sides are aligned to the principal directions and draw some arrows to indicate the magnitude and direction of the principal stresses acting at the point.
   (e) Check that the sum of the normal stresses at the point is an invariant.

4. A material particle is subjected to a state of stress given by

$$[\sigma_{ij}] = \begin{bmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Find the principal stresses (all three), maximum shear stresses (see Eqn. 3.5.11), and the direction of the planes on which these stresses act.

5. Consider the following state of stress (with respect to an $x, y, z$ coordinate system):

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   (a) Use the stress transformation equations to derive the stresses acting on planes obtained from the original planes by a counterclockwise rotation of 45° about $z$ axis.
   (b) What is the maximum normal stress acting at the point?
   (c) What is the maximum shear stress? On what plane(s) does it act? (See Eqn. 3.5.11.)

6. Consider the two dimensional stress state

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

   Show that this is an **isotropic state of stress**, that is, the stress components are the same on *all* planes through a material particle.

7. (a) Is a trampoline (the material you jump on) in a state of plane stress? When someone is actually jumping on it?
   (b) Is a picture hanging on a wall in a state of plane stress?
   (c) Is a glass window in a state of plane stress? On a very windy day?
   (d) A piece of rabbit skin is stretched in a testing machine – is it in a state of plane stress?

## 3.5.7 Appendix to §3.5

### A Note on the Formulae for Principal Stresses

To derive Eqns. 3.5.5, first rewrite the transformation equations in terms of $2\theta$ using $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to get

$$\begin{aligned}
\sigma'_{xx} &= \tfrac{1}{2}(1+\cos 2\theta)\sigma_{xx} + \tfrac{1}{2}(1-\cos 2\theta)\sigma_{yy} + \sin 2\theta\sigma_{xy} \\
\sigma'_{yy} &= \tfrac{1}{2}(1-\cos 2\theta)\sigma_{xx} + \tfrac{1}{2}(1+\cos 2\theta)\sigma_{yy} - \sin 2\theta\sigma_{xy} \\
\sigma'_{xy} &= \tfrac{1}{2}\sin 2\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy}
\end{aligned}$$

Next, from Eqn. 3.5.4,

$$\sin 2\theta = \frac{2\sigma_{xy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + 4\sigma_{xy}^2}}, \qquad \cos 2\theta = \frac{\sigma_{xx}-\sigma_{yy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + 4\sigma_{xy}^2}}$$

Substituting into the rewritten transformation formulae then leads to

$$\begin{aligned}\sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}\\ \sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}\\ \sigma'_{xy} &= 0\end{aligned}$$

Here $\sigma'_{xx} > \sigma'_{yy}$ so that the maximum principal stress is $\sigma_1 = \sigma'_{xx}$ and the minimum principal stress is $\sigma'_2 = \sigma'_{yy}$. Here it is implicitly assumed that $\tan 2\theta > 0$, i.e. that $0 < 2\theta < 90$ or $180 < 2\theta < 270$. On the other hand one could assume that $\tan 2\theta < 0$, i.e. that $90 < 2\theta < 180$ or $270 < 2\theta < 360$, in which case one arrives at the formulae

$$\begin{aligned}\sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}\\ \sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}\end{aligned}$$

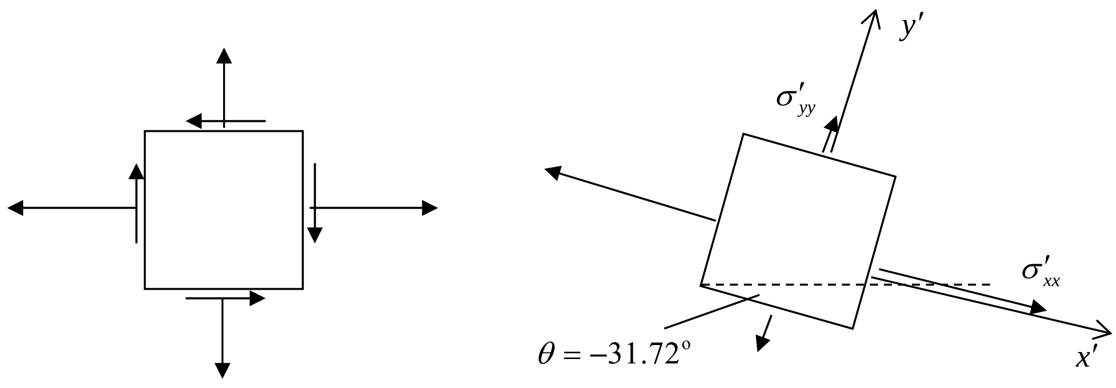
The results can be summarised as Eqn. 3.5.5,

$$\begin{aligned}\sigma_1 &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}\\ \sigma_2 &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}\end{aligned}$$

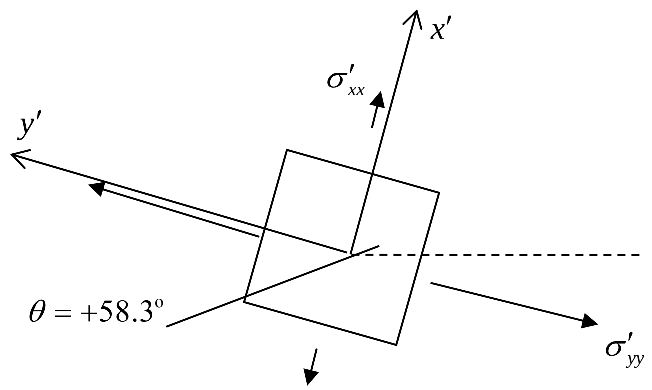
These formulae do not tell one on which of the two principal planes the maximum principal stress acts. This might not be an important issue, but if this information is required one needs to go directly to the stress transformation equations. In the example stress state, Eqn. 3.5.3, one has

$$\begin{aligned}\sigma'_{xx} &= \cos^2\theta(2) + \sin^2\theta(1) + \sin 2\theta(-1)\\ \sigma'_{yy} &= \sin^2\theta(2) + \cos^2\theta(1) - \sin 2\theta(-1)\end{aligned}$$

For $\theta = -31.72^\circ\ (148.28^\circ)$, $\sigma'_{xx} = 2.62$ and $\sigma'_{yy} = 0.38$. So one has the situation shown below.




If one takes the other angle, $\theta = 58.3°$, one has $\sigma'_{xx} = 0.38$ and $\sigma'_{yy} = 2.62$, and the situation below




### A Note on the Maximum Shear Stress

Shown below left is a box element with sides perpendicular to the $1, 2, z$ axes, i.e. aligned with the principal directions. The stresses in the new $x', y'$ axis system shown are given by Eqns. 3.5.8, with $\theta$ measured from the principal directions:

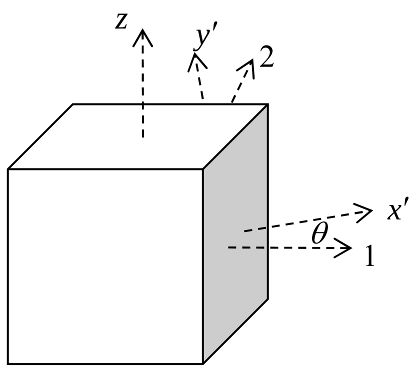
$$
\begin{aligned}
\sigma'_{xx} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\
\sigma'_{yy} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\
\sigma'_{xy} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta
\end{aligned}
$$

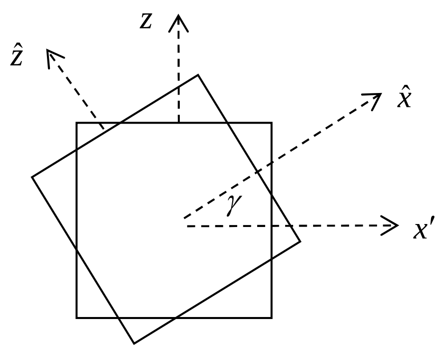
Now as well as rotating around in the $1-2$ plane through an angle $\theta$, rotate also in the $x', z$ plane through an angle $\gamma$ (see below right). This rotation leads to the new stresses

$$\hat{\sigma}_{xx} = \cos^2\gamma\sigma'_{xx} + \sin^2\gamma\sigma_{zz} + \sin 2\gamma\sigma_{x'z}$$
$$\hat{\sigma}_{zz} = \sin^2\gamma\sigma'_{xx} + \cos^2\gamma\sigma_{zz} - \sin 2\gamma\sigma_{x'z}$$
$$\hat{\sigma}_{xz} = \sin\gamma\cos\gamma(\sigma_{zz} - \sigma'_{xx}) + \cos 2\gamma\sigma_{x'z}$$

In plane stress, $\sigma_{zz} = \sigma_{x'z} = 0$, so one has the stresses

$$\hat{\sigma}_{xx} = \cos^2\gamma\sigma'_{xx}, \quad \hat{\sigma}_{yy} = \sin^2\gamma\sigma'_{xx}, \quad \hat{\sigma}_{xy} = -\tfrac{1}{2}\sin 2\gamma\sigma'_{xx}$$




The shear stress can be written out in full:

$$\hat{\sigma}_{xy}(\gamma,\theta) = -\tfrac{1}{2}\sin 2\gamma\left[\tfrac{1}{2}(\sigma_1 + \sigma_2) + \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta\right].$$

This is a function of two variables; its minimum value can be found by setting the partial derivatives with respect to these variables to zero. Differentiating,

$$\partial\hat{\sigma}_{xy} / \partial\gamma = -\cos 2\gamma\left[\tfrac{1}{2}(\sigma_1 + \sigma_2) + \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta\right]$$
$$\partial\hat{\sigma}_{xy} / \partial\theta = -\tfrac{1}{2}\sin 2\gamma\left[-(\sigma_1 - \sigma_2)\sin 2\theta\right]$$

Setting to zero gives the solutions $\sin 2\theta = 0$, $\cos 2\gamma = 0$, i.e. $\theta = 0$, $\gamma = 45^{\circ}$. Thus the maximum shear stress occurs at $45^{\circ}$ to the $1-2$ plane, and in the $1-z$, i.e. $1-3$ plane (as in Fig. 3.5.8b). The value of the maximum shear stress here is then $\left|\hat{\sigma}_{xy}\right| = \left|\tfrac{1}{2}\sigma_1\right|$, which is the expression in Eqn. 3.5.11.

# 3.5b Stress Boundary Conditions: Continued

Consider now in more detail a surface between two different materials, Fig. 3.5.16. One says that the normal and shear stresses are **continuous** across the surface, as illustrated.

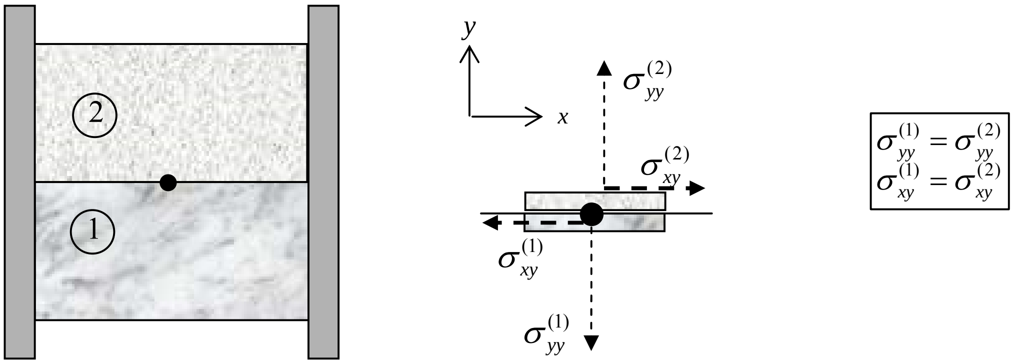



**Figure 3.5.16: normal and shear stress continuous across an interface between two different materials, material '1' and material '2'**

Note also that, since the shear stress $\sigma_{xy}$ is the same on both sides of the surface, the shear stresses acting on both sides of a perpendicular plane passing *through* the interface between the materials, by the symmetry of stress, must also be the same, Fig. 3.5.17a.

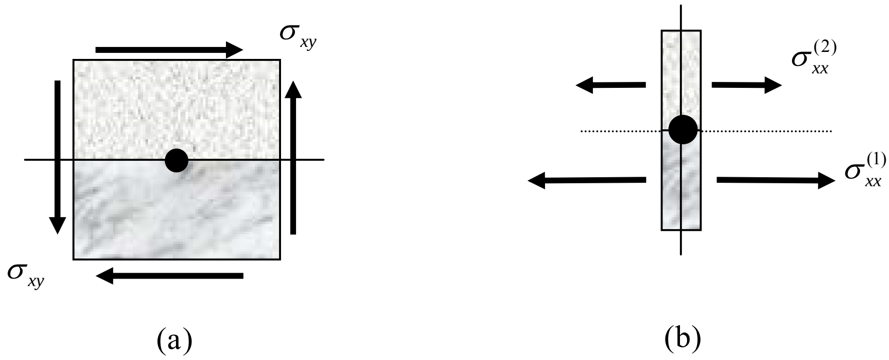



**Figure 3.5.17: stresses at an interface; (a) shear stresses continuous across the interface, (b) tangential stresses not necessarily continuous**

However, again, the tangential stresses, those acting parallel to the interface, do *not* have to be equal. For example, shown in Fig. 3.5.17b are the tangential stresses acting in the upper material, $\sigma_{xx}^{(2)}$ - they balance no matter what the magnitude of the stresses $\sigma_{xx}^{(1)}$.

## Description of Boundary Conditions

The following example brings together the notions of stress boundary conditions, stress components, equilibrium and equivalent forces.

### Example

Consider the plate shown in Fig. 3.5.18. It is of width $2a$, height *b* and depth *t*. It is subjected to a tensile stress *r*, pressure *p* and shear stresses *s*. The applied stresses are uniform through the thickness of the plate. It is welded to a rigid base.

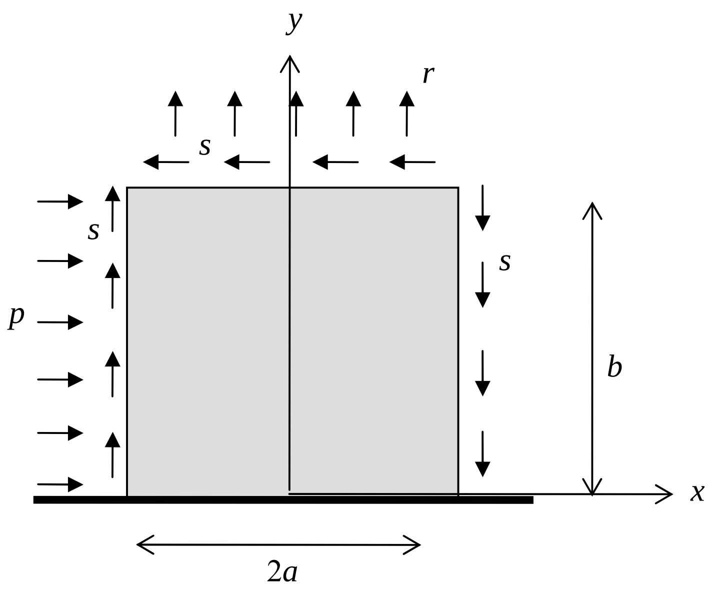



**Figure 3.5.18: a plate subjected to stress distributions**

Using the $x-y$ axes shown, the stress boundary conditions can be expressed as:

Left-hand surface: $\begin{cases} \sigma_{xx}(-a,y) = -p \\ \sigma_{xy}(-a,y) = -s \end{cases}, \quad 0 < y < b$

Top surface: $\begin{cases} \sigma_{yy}(x,b) = +r \\ \sigma_{xy}(x,b) = -s \end{cases}, \quad -a < x < +a$

Right-hand surface: $\begin{cases} \sigma_{xx}(+a,y) = 0 \\ \sigma_{xy}(+a,y) = -s \end{cases}, \quad 0 < y < b$

Note carefully the description of the normal and shear stresses over each side and the signs of the stress components.

The stresses at the lower edge are unknown (there is a displacement boundary condition there: zero displacement). They will in general not be uniform. Using the given $x-y$ axes, these unknown reaction stresses, exerted by the base on the plate, are (see Fig 3.5.19)

Lower surface: $$\begin{cases} \sigma_{yy}(x,0) \\ \sigma_{xy}(x,0) \end{cases}, \qquad -a < x < +a$$

Note the directions of the arrows in Fig. 3.5.19, they have been drawn in the direction of positive $\sigma_{yy}(x,0)$, $\sigma_{xy}(x,0)$.

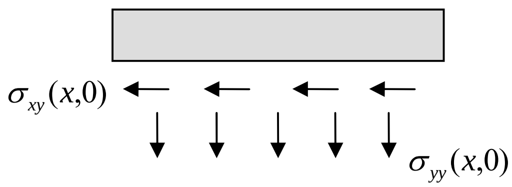



**Figure 3.5.19: unknown reaction stresses acting on the lower edge**

For force equilibrium of the complete plate, consider the free-body diagram 3.5.20; shown are the resultant forces of the stress distributions. Force equilibrium requires that

$$\sum F_x = bpt - 2ast - t\int_{-a}^{+a} \sigma_{xy}(x,0)dx = 0$$

$$\sum F_y = 2art - t\int_{-a}^{+a} \sigma_{yy}(x,0)dx = 0$$

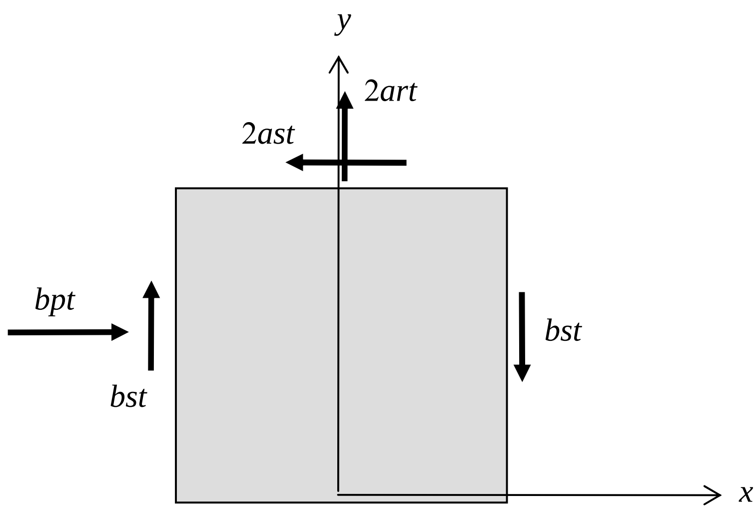



**Figure 3.5.20: a free-body diagram of the plate in Fig. 3.5.18 showing the known resultant forces (forces on the lower boundary are not shown)**

For moment equilibrium, consider the moments about, for example, the lower left-hand corner. One has

$$\sum M_0 = -bpt(b/2) + 2ast(b) + 2art(a) - bst(2a) - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x)dx = 0$$

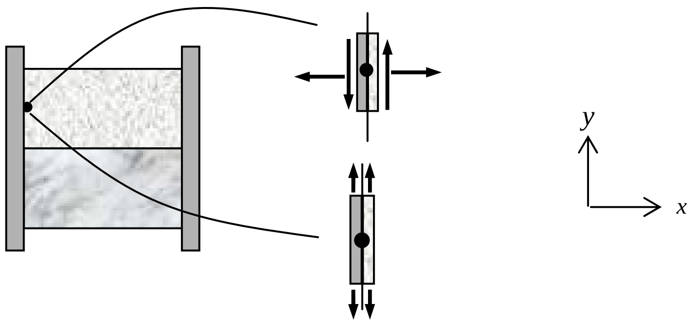
If one had taken moments about the top-left corner, the equation would read

$$\sum M_0 = +bpt(b/2) + 2art(a) - bst(2a)$$
$$- t\int_{-a}^{+a} \sigma_{xy}(x,0) \times b\,dx - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x)dx = 0$$

■

### Problems

8. Consider the point shown below, at the boundary between a wall and a dissimilar material. Label the stress components displayed using the coordinate system shown. Which stress components are continuous across the wall/material boundary? (Add a superscript 'w' for the stresses in the wall.)




9. A thin metal plate of width $2b$, height $h$ and depth $t$ is loaded by a pressure distribution $p(x)$ along $-a < x < +a$ and welded at its base to the ground, as shown in the figure below. Write down expressions for the stress boundary conditions (two on each of the three edges). Write down expressions for the force equilibrium of the plate and moment equilibrium of the plate about the corner A.