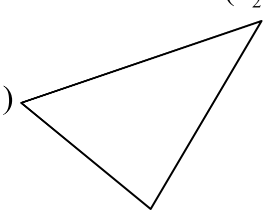
7 Triangular Elements

Here a number of triangular elements are discussed. Two linear elements, the standard **linear element** and the **nonconforming linear element** are discussed in quite some detail. The **quadratic triangular element** is briefly introduced towards the end.

7.1 The Linear Triangular Element

The most basic type of triangular element is the linear element, with three nodes at the vertices, for which the shape functions vary linearly. The shape functions for this element can be constructed as follows: consider a triangle with vertices

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$$



The first shape function, $N_1(x, y)$, is a linear function with value 1 at the first node and zero at the other two, so

$$N_1(x,y) = a_1 x + b_1 y + c_1 \qquad \begin{aligned} a_1 x_1 + b_1 y_1 + c_1 &= 1 \\ a_1 x_2 + b_1 y_2 + c_1 &= 0 \\ a_1 x_3 + b_1 y_3 + c_1 &= 0 \end{aligned} \tag{7.1}$$

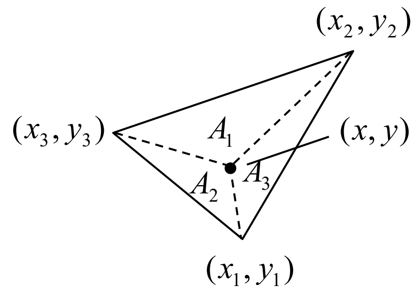
which can be solved to get

$$N_1(x,y) = \frac{(y_2 - y_3)x + (x_3 - x_2)y + (x_2 y_3 - x_3 y_2)}{2\Delta} \tag{7.2}$$

where Δ is the area of the triangle,

$$\begin{aligned} 2\Delta &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &= (x_1 - x_2)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_2) \\ &= (x_2 - x_3)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_3) \\ &= (x_3 - x_1)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_1) \end{aligned}, \tag{7.3}$$

and the other two shape functions can be determined similarly. Note that the nodes should be numbered counterclockwise around the triangle, as is done here, so that the area Δ is positive.

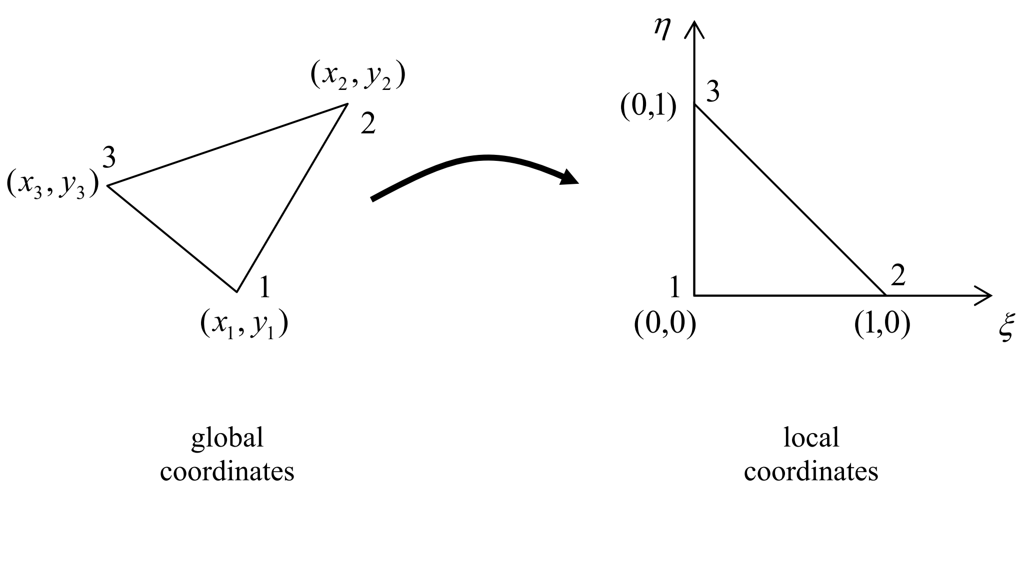


These shape functions are also called *area coordinates*, because they are the ratios of the areas shown in the figure here to the total area of the triangle: for example, $N_1 = A_1 / \Delta$ (and clearly $N_1 + N_2 + N_3 = 1$).

7.1.1 Local Coordinates

The local coordinates appropriate to this triangular element are

$$
\begin{aligned}
\xi &= \frac{(x - x_1)(y_3 - y_1) - (y - y_1)(x_3 - x_1)}{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)} = \frac{(x - x_1)(y_3 - y_1) - (y - y_1)(x_3 - x_1)}{2\Delta} \\
\eta &= \frac{(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)}{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)} = \frac{(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)}{2\Delta}
\end{aligned}
\tag{7.4}
$$



With these definitions, the local coordinates of the three nodes $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are $(0,0)$, $(1,0)$ and $(0,1)$.

The inverse of these relations are

$$\begin{aligned} x(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta)x_i \\ y(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta)y_i \end{aligned} \tag{7.5}$$

and the shape functions in terms of the local coordinates are

Shape Functions for the Linear Triangular Element:

$$\begin{aligned} N_1 &= 1-\xi-\eta \\ N_2 &= \xi \\ N_3 &= \eta \end{aligned} \tag{7.6}$$

7.1.2 Transformation into Local Coordinates

For the transformation from the physical coordinate system into the local coordinate system, one again uses the transformation matrices (6.7). Again, as in (6.9), one can write

$$\mathbf{J} = \begin{bmatrix} \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix} = \mathbf{N}_d \mathbf{x} \tag{7.7}$$

with the sum now over three nodes, and with

$$\mathbf{N}_d = \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -1 & +1 & 0 \\ -1 & 0 & +1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \tag{7.8}$$

One also has the Jacobian of the transformation $J = |\mathbf{J}| = |\mathbf{N}_d \mathbf{x}|$ and the relation between the derivatives in local and physical coordinates (as in 6.11):

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \mathbf{J}^{-1} \mathbf{N}_d$$

Because of the simplicity of the element, these quantities can be evaluated in closed form:

$$\mathbf{J} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{1}{2\Delta} \begin{bmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{bmatrix}, \qquad J = 2\Delta \tag{7.9}$$

and

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \tag{7.10}$$

7.1.3 Derivatives of the Shape Functions

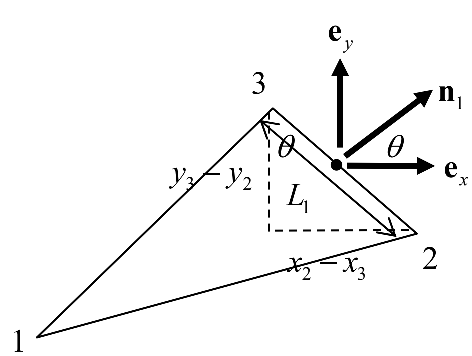
An interesting property of the linear triangular element is that

$$\nabla N_i = -\frac{L_i}{2\Delta} \mathbf{n}_i \tag{7.11}$$

where L_i is the length of the element edge opposite node/vertex i and $\mathbf{n}_i$ is the unit normal to that edge. This is a constant vector over the element.

For example, from the figure, with

$$\begin{aligned} \mathbf{e}_x &= \mathbf{n}_1 \cos\theta - \bar{\mathbf{n}} \sin\theta \\ \mathbf{e}_y &= \mathbf{n}_1 \sin\theta + \bar{\mathbf{n}} \cos\theta \end{aligned}$$



where $\overline{\mathbf{n}}$ is perpendicular to $\mathbf{n}_1$, then

$$\begin{aligned}\nabla N_1 &= \frac{\partial N_1}{\partial x}\mathbf{e}_x + \frac{\partial N_1}{\partial y}\mathbf{e}_y \\ &= \frac{1}{2\Delta}\left[(y_2 - y_3)\mathbf{e}_x + (x_3 - x_2)\mathbf{e}_y\right] \\ &= -\frac{L_1}{2\Delta}\mathbf{n}_1\end{aligned}$$

Once the values of p have been obtained at the three nodes, the gradient can be evaluated through

$$\begin{aligned}\nabla p &= \sum_{i=1}^{n} p_i \nabla N_i \\ &= \frac{1}{2\Delta}\left\{ p_1 \begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \end{bmatrix} + p_2 \begin{bmatrix} y_3 - y_1 \\ x_1 - x_3 \end{bmatrix} + p_3 \begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \end{bmatrix} \right\}\end{aligned} \tag{7.12}$$

or

$$\nabla p = -\frac{1}{2\Delta}\left\{ p_1 L_1 \mathbf{n}_1 + p_2 L_2 \mathbf{n}_2 + p_3 L_3 \mathbf{n}_3 \right\} \tag{7.13}$$

which is a constant vector.

The gradients normal to the element edges are then $\nabla p \cdot \mathbf{n}_i$, where

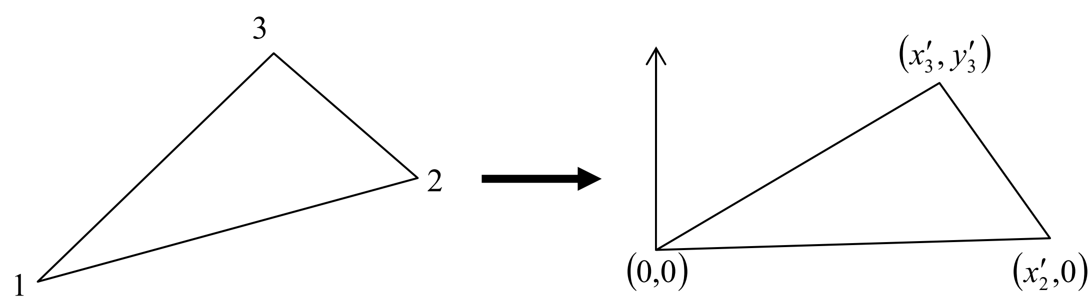
$$\begin{aligned}\mathbf{n}_1 &= \frac{1}{L_1}\left[(y_3 - y_2)\mathbf{e}_x + (x_2 - x_3)\mathbf{e}_y\right] \\ \mathbf{n}_2 &= \frac{1}{L_2}\left[(y_1 - y_3)\mathbf{e}_x + (x_3 - x_1)\mathbf{e}_y\right] \\ \mathbf{n}_3 &= \frac{1}{L_3}\left[(y_2 - y_1)\mathbf{e}_x + (x_1 - x_2)\mathbf{e}_y\right]\end{aligned} \tag{7.14}$$

The gradient can also be conveniently written in terms of a shifted global coordinate system, where the vertex 1 is position at position (x'_1, y'_1). In terms of these coordinates,

$$\Delta = \frac{1}{2} x'_2 y'_3$$

and

$$\nabla p = \begin{bmatrix} -\dfrac{1}{x'_2} & +\dfrac{1}{x'_2} & 0 \\ -\dfrac{1}{y'_3} + \dfrac{x'_3}{x'_2 y'_3} & -\dfrac{x'_3}{x'_2 y'_3} & +\dfrac{1}{y'_3} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{7.15}$$



7.1.4 Some Integrals

The integral of a function over an element is given by

$$\int_E f(x,y) dS = J \int_0^1 \int_0^{1-\xi} f(\xi,\eta) d\eta d\xi \tag{7.16}$$

Some integrals involving the shape functions are {▲Problem 1}

$$\int_E N_j dS = \frac{\Delta}{3}, \quad j = 1, 2, 3, \qquad \int_E N_i N_j dS = \frac{\Delta}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{7.17}$$

7.1.5 The Boundary Integrals

Looking now at the boundary integral, one has

$$\int_{1-2} dC_1 = L_1 \int_0^1 d\xi, \quad \int_{2-3} dC_2 = L_2 \int_0^1 d\eta = L_2 \int_0^1 d\xi, \quad \int_{3-1} dC_3 = L_3 \int_0^1 d\eta \tag{7.18}$$

where L_1 is the length of the line joining nodes 1 and 2, etc.

The shape functions along the three edges are given in the following table:

	N_1	N_2	N_3
C_1	$1-\xi$	ξ	0
C_2	0	ξ (or $1-\eta$)	$1-\xi$ (or η)
C_3	$1-\eta$	0	η

Table 7.1: Shape Functions for the Linear Element along Element Edges

For natural boundary conditions of the type

$$\frac{\partial p}{\partial n} = A, \tag{7.19}$$

were A is a constant, one obtains the boundary vectors

$$\int_{C_1} N_j dC_1 = \frac{L_1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \int_{C_2} N_j dC_2 = \frac{L_2}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \int_{C_3} N_j dC_3 = \frac{L_3}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} \tag{7.20}$$

7.2 Numerical Integration

The numerical integration rule for triangular regions takes the form

$$\int_0^1 \int_0^{1-\xi} f(\xi,\eta)\,d\eta\, d\xi = \sum_{i=1}^{N} W_i f(\xi_i,\eta_i) \tag{7.21}$$

where the weights W_i and integration points s_i are given in the table below. The rule integrates polynomials of the order r exactly using an $N-$ point rule.

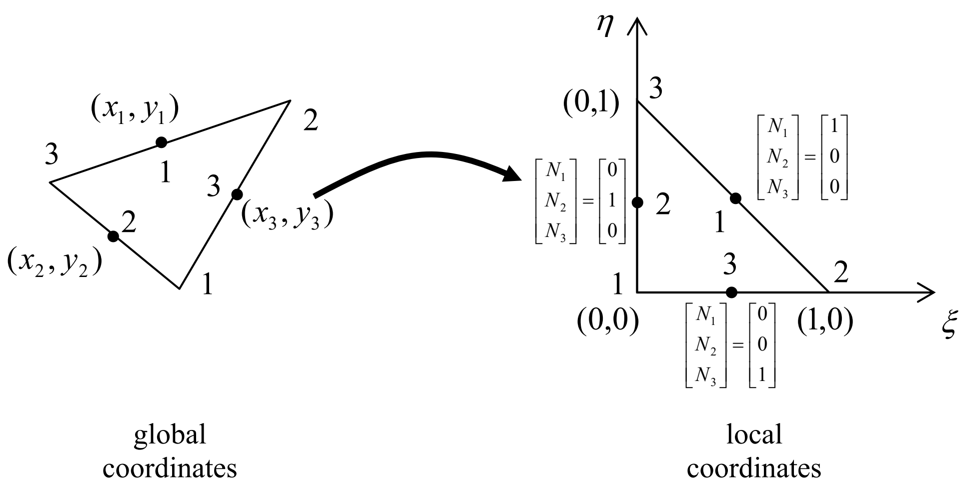
N	ξ_i	η_i	W_i	r
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	1
3	$\frac{1}{2}, \frac{1}{2}, 0$	$\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	2

Table 7.2: Quadrature for triangles

7.3 The Nonconforming Linear Triangle

The non-conforming triangular element has the following shape functions:

$$\begin{aligned} N_1 &= -1 + 2\xi + 2\eta \\ N_2 &= 1 - 2\xi \\ N_3 &= 1 - 2\eta \end{aligned} \tag{7.22}$$



If the shape functions of the standard, conforming, element are denoted by $\overline{N}_i$, then

$$N_i = 1 - 2\overline{N}_i \tag{7.23}$$

The equations

$$\begin{aligned} x(\xi,\eta) &= \sum_{i=1}^{3} N_i(\xi,\eta)x_i \\ y(\xi,\eta) &= \sum_{i=1}^{3} N_i(\xi,\eta)y_i \end{aligned} \tag{7.24}$$

still hold only the coordinates x_i, y_i refer to the nodes which are now mid-way along element edges, not at the vertices.

For this element one has

$$\mathbf{N}_d = \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = -2\begin{bmatrix} -1 & +1 & 0 \\ -1 & 0 & +1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \tag{7.25}$$

so that $\mathbf{N}_d$ is -2 times the $\mathbf{N}_d$ of the standard (conforming) triangular element. Also,

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{3} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix} = \mathbf{N}_d \mathbf{x} = -2\begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix} \tag{7.26}$$

which is -2 times the **J** matrix of the standard (conforming) triangular element.

The inverse Jacobian matrix is

$$\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = -\frac{1}{2}\left\{ \frac{1}{2\Delta} \begin{bmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{bmatrix} \right\} \tag{7.27}$$

and the Jacobian determinant is

$$J = |\mathbf{J}| = |\mathbf{N}_d \mathbf{x}| = 8\Delta_s = 2\Delta \tag{7.28}$$

where Δ_s is the area of the triangle with vertices (x_i, y_i) and Δ is the area of the complete triangular element.

Also, using (6.11),

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{2\Delta_s} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \tag{7.29}$$

which is a similar expression to that of the standard (conforming) triangular element.

7.3.1 Derivatives of the Shape Functions

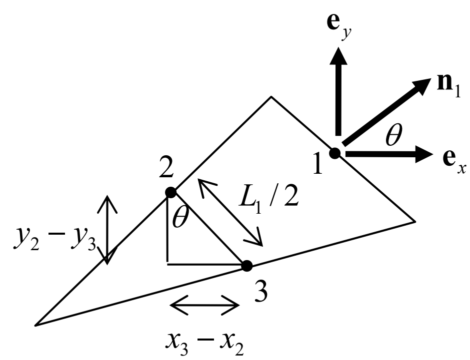
In this case,

$$\nabla N_i = \frac{L_i}{\Delta} \mathbf{n}_i \tag{7.30}$$

where L_i is the length of he element edge associated with node i and $\mathbf{n}_i$ is the unit normal to the edge.

For example, from the figure, with

$$\mathbf{e}_x = \mathbf{n}_1 \cos\theta - \overline{\mathbf{n}} \sin\theta$$
$$\mathbf{e}_y = \mathbf{n}_1 \sin\theta + \overline{\mathbf{n}} \cos\theta$$



where $\overline{\mathbf{n}}$ is perpendicular to $\mathbf{n}_1$, then

$$\begin{aligned}\nabla N_1 &= \frac{\partial N_1}{\partial x}\mathbf{e}_x + \frac{\partial N_2}{\partial y}\mathbf{e}_y \\ &= \frac{1}{2\Delta_s}\left[(y_2 - y_3)\mathbf{e}_x + (x_3 - x_2)\mathbf{e}_y\right] \\ &= \frac{L_1}{\Delta}\mathbf{n}_1\end{aligned}$$

Once the values of p have been obtained, the gradient can be evaluated through

$$\begin{aligned}\nabla p &= \sum_{i=1}^{n} p_i \nabla N_i \\ &= \frac{1}{2\Delta_s}\left\{p_1\begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \end{bmatrix} + p_2\begin{bmatrix} y_3 - y_1 \\ x_1 - x_3 \end{bmatrix} + p_3\begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \end{bmatrix}\right\}\end{aligned} \tag{7.31}$$

or

$$\nabla p = \frac{1}{\Delta}\left\{p_1 L_1 \mathbf{n}_1 + p_2 L_2 \mathbf{n}_2 + p_3 L_3 \mathbf{n}_3\right\} \tag{7.32}$$

which is a constant vector.

The gradients normal to the element edges are then $\nabla p \cdot \mathbf{n}_i$, where

$$\begin{aligned}\mathbf{n}_1 &= -\frac{2}{L_1}\left[(y_3 - y_2)\mathbf{e}_x + (x_2 - x_3)\mathbf{e}_y\right] \\ \mathbf{n}_2 &= -\frac{2}{L_2}\left[(y_1 - y_3)\mathbf{e}_x + (x_3 - x_1)\mathbf{e}_y\right] \\ \mathbf{n}_3 &= -\frac{2}{L_3}\left[(y_2 - y_1)\mathbf{e}_x + (x_1 - x_2)\mathbf{e}_y\right]\end{aligned} \tag{7.33}$$

7.3.2 The Boundary Integrals

Looking now at the boundary integral, one has

$$\int_{1-2} dC_1 = L_1 \int_0^1 d\xi, \quad \int_{2-3} dC_2 = L_2 \int_0^1 d\eta = L_2 \int_0^1 d\xi, \quad \int_{3-1} dC_3 = L_3 \int_0^1 d\eta \qquad (7.34)$$

where L_1 is the length of the line joining nodes 1 and 2, etc.

The shape functions along the three edges are given in the following table:

	N_1	N_2	N_3
C_{1-2}	$-1+2\xi$	$1-2\xi$	1
C_{2-3}	1	$1-2\xi$ (or $-1+2\eta$)	$-1+2\xi$ (or $1-2\eta$)
C_{3-1}	$-1+2\eta$	1	$1-2\eta$

Table 7.3: Shape Functions for the Nonconforming Element along Element Edges

For natural boundary conditions of the type

$$\frac{\partial p}{\partial n} = A,$$

were A is a constant, one obtains the boundary vectors

$$\int_{C_{1-2}} N_j dC_1 = L_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \int_{C_{2-3}} N_j dC_2 = L_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \int_{C_{3-1}} N_j dC_3 = L_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (7.35)$$

7.3.3 Some Integrals

The integrals of the shape functions over an element are

$$\int_E N_j dS = \frac{\Delta}{3}, \quad i = 1, 2, 3 \tag{7.36}$$

The shape functions have a useful orthogonality property; when integrated over an element,

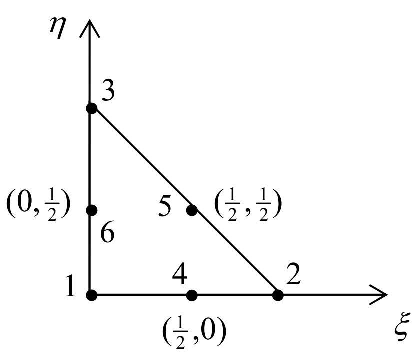
$$\int_E N_i N_j dS = \begin{cases} 0 & \text{if } i \neq j \\ \dfrac{\Delta}{3} & \text{if } i = j \end{cases} \tag{7.37}$$

7.4 The Quadratic Triangular Element

The quadratic triangular element has midside nodes in addition to those at the vertices, and a function is interpolated as

$$p = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \tag{7.38}$$

The shape functions are



$$\begin{aligned} N_1 &= (1-\xi-\eta)(1-2\xi-2\eta) \\ N_2 &= \xi(2\xi-1) \\ N_3 &= \eta(2\eta-1) \\ N_4 &= 4\xi(1-\xi-\eta) \\ N_5 &= 4\xi\eta \\ N_6 &= 4\eta(1-\xi-\eta) \end{aligned} \tag{7.39}$$

For example, taking $\eta = 0$, the non-zero shape functions are

$$N_1 = 1 - 3\xi + 2\xi^2, \quad N_2 = -\xi + 2\xi^2, \quad N_4 = 4\xi - 4\xi^2 \tag{7.40}$$

The one-dimensional quadratic 3-noded element is recovered by letting $\xi \to (\xi + 1)/2$ (so that the interval is now $[-1,+1]$).

7.5 Problems

1. Derive the integral relations (7.17).