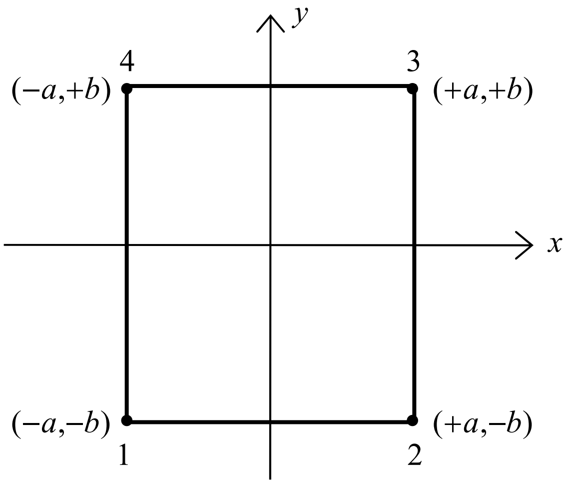
6 Quadrilateral Elements

Two dimensional domains are meshed using quadrilateral (4-sided) and/or triangular (3-sided) elements. Quadrilateral elements are discussed here.

6.1 The (Q4) Bilinear Element

The most basic quadrilateral element is the bilinear element, so-called because it is interpolated using the product of two linear polynomials (see below). It has four nodes, at the corners, and is called the "Q4" element for brevity. Considering here the simple geometry of a rectangular element with sides of lengths $2a$ and $2b$, with nodes at $(x, y) = (\pm a, \pm b)$, the shape functions are



$$
\begin{aligned}
N_1(x, y) &= \frac{1}{4ab}(a - x)(b - y) \\
N_2(x, y) &= \frac{1}{4ab}(a + x)(b - y) \\
N_3(x, y) &= \frac{1}{4ab}(a + x)(b + y) \\
N_4(x, y) &= \frac{1}{4ab}(a - x)(b + y)
\end{aligned}
\tag{6.1}
$$

It can be seen by inspection that each of these shape functions takes the value 1 at one of the four nodes and 0 at the other three nodes. With the nodes labeled as shown - it is conventional to label the nodes $1 \ldots 4$ counterclockwise - shape function N_i is 1 at node *i*.

Another important property follows from this: each shape function is zero along two of the element edges:

along edge 1-2: $N_3 = N_4 = 0$

along edge 2-3: $N_1 = N_4 = 0$ (6.2)
along edge 3-4: $N_1 = N_2 = 0$
along edge 4-1: $N_2 = N_3 = 0$

In these cases, along the edges, the bilinear shape functions reduce to the one-dimensional linear shape functions, for example,

$$N_1\big|_{y=-b} = \frac{1}{2}\left(1 - \frac{x}{a}\right) \tag{6.3}$$

The shape functions can be derived formally by writing them as the product of two linear functions

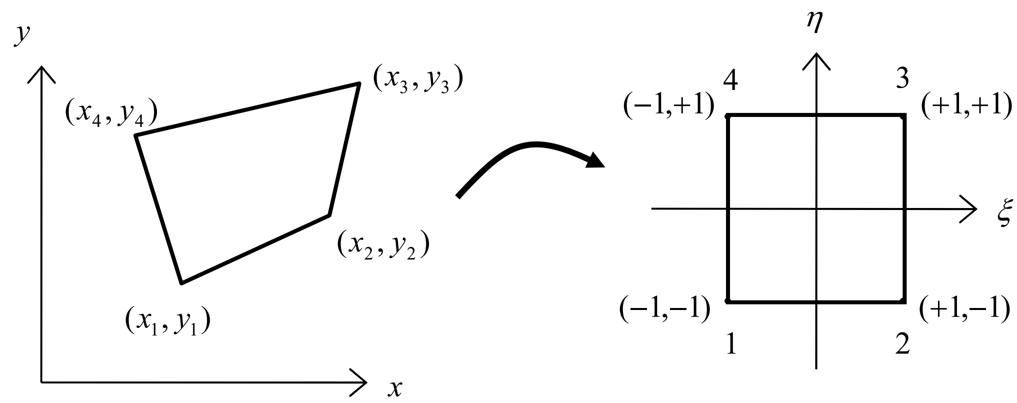
$$N(x,y) = (\alpha_1 x + \beta_1)(\alpha_2 y + \beta_2) \tag{6.4}$$

and solving for the constants α_i, β_i using the conditions $N = 0$ or 1 at the nodes {▲Problem 1}.

6.1.1 Local Coordinates

In simple problems involving rectangular regions, it is possible to use a mesh of rectangular/square Q4 elements and use the above shape functions. In more general problems, where the elements are irregular, it is convenient (necessary) to use the local/natural coordinates (ξ, η), such that the four nodes of each element are located at $(\xi, \eta) = (\pm 1, \pm 1)$. In terms of these local coordinates, the shape functions are

$$\begin{aligned}
N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) \\
N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta) \\
N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta) \\
N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned} \tag{6.5}$$



and again, shape function N_i is 1 at node *i*.

Any Q4 element of arbitrary geometry with nodes at (x_i, y_i), $i = 1 \ldots 4$, can be mapped onto this square element in local coordinates through the transformation (see illustration below)

$$
\begin{aligned}
x(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta) x_i \\
y(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta) y_i
\end{aligned}
\tag{6.6}
$$

6.1.2 Transformation into Local Coordinates

To enable transformation of integrals from the global coordinate system into integrals involving the local coordinates, introduce the Jacobian matrices

$$
\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}, \qquad
\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}, \qquad
\mathbf{J}\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial \xi} & \dfrac{\partial \eta}{\partial \xi} \\ \dfrac{\partial \xi}{\partial \eta} & \dfrac{\partial \eta}{\partial \eta} \end{bmatrix} = \mathbf{I}
\tag{6.7}
$$

and $\mathbf{I}$ is the identity matrix. These matrices relate physical derivatives (with respect to *x* and *y*) and local derivatives (with respect to ξ and η):

$$
\begin{bmatrix} \partial f / \partial \xi \\ \partial f / \partial \eta \end{bmatrix} = \mathbf{J} \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix}, \qquad
\begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \partial f / \partial \xi \\ \partial f / \partial \eta \end{bmatrix}
\tag{6.8}
$$

The Jacobian matrix can be written as the product of two other matrices:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix} = \mathbf{N}_d \mathbf{x} \tag{6.9}$$

where $\mathbf{N}_d$ is the 2×4 matrix of the derivatives of the shape functions, and $\mathbf{x}$ is the 4×2 matrix of nodal coordinates {▲Problem 2}:

$$\mathbf{N}_d = \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & +(1-\eta) & +(1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & +(1+\xi) & +(1-\xi) \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{6.10}$$

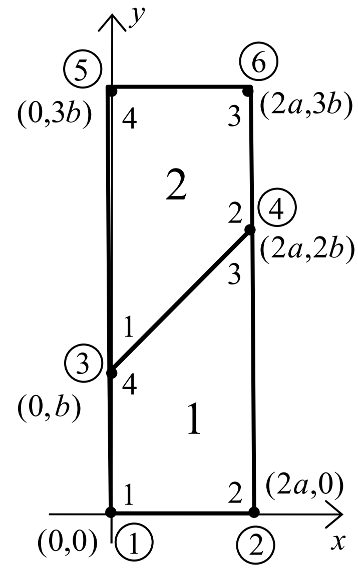
From (6.8), the derivatives with respect to the global coordinates are then

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \mathbf{J}^{-1} \mathbf{N}_d \quad \text{or} \quad \begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x} \\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y} \end{aligned} \tag{6.11}$$

The Jacobian determinant $J = |\mathbf{J}|$ is then {▲Problem 4}

$$\begin{aligned} J = \frac{1}{8} \{ & \xi [(x_4 - x_3)(y_2 - y_1) - (x_2 - x_1)(y_4 - y_3)] \\ & + \eta [(x_3 - x_2)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_2)] \\ & + [(x_3 - x_1)(y_4 - y_2) - (x_4 - x_2)(y_3 - y_1)] \} \end{aligned} \tag{6.12}$$

Example: An Irregular Two-element Mesh



Consider the irregular two-element mesh illustrated here.

The necessary calculations for an implementation in terms of local coordinates are given below for element 1.

The Jacobian matrix is

$$\mathbf{J} = \mathbf{N}_d \mathbf{x} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & +(1-\eta) & +(1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & +(1+\xi) & +(1-\xi) \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2a & 0 \\ 2a & 2b \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & \frac{b}{4}(1+\eta) \\ 0 & \frac{b}{4}(3+\xi) \end{bmatrix} \tag{6.13}$$

with

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{1}{a}\frac{1+\eta}{3+\xi} \\ 0 & \frac{4}{b}\frac{1}{3+\xi} \end{bmatrix}, \quad J = \frac{ab}{4}(3+\xi) \tag{6.14}$$

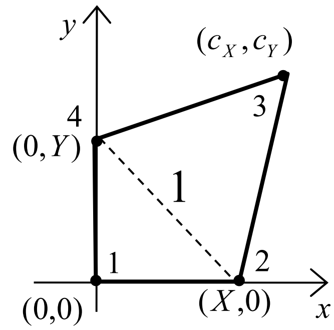
and

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \mathbf{J}^{-1}\mathbf{N}_d = \begin{bmatrix} \frac{1}{a} & -\frac{1}{a}\frac{1+\eta}{3+\xi} \\ 0 & \frac{4}{b}\frac{1}{3+\xi} \end{bmatrix} \frac{1}{4}\begin{bmatrix} -(1-\eta) & +(1-\eta) & +(1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & +(1+\xi) & +(1-\xi) \end{bmatrix}$$

$$= \frac{1}{3+\xi}\begin{bmatrix} \frac{1}{2a}(-1-\xi+2\eta) & \frac{1}{2a}(2+\xi-\eta) & \frac{1}{2a}(1+\eta) & \frac{1}{2a}(-2-2\eta) \\ \frac{1}{b}(-1+\xi) & \frac{1}{b}(-1-\xi) & \frac{1}{b}(1+\xi) & \frac{1}{b}(1-\xi) \end{bmatrix} \tag{6.15}$$

6.1.3 Distorted Elements

Consider the element shown to the right, with $X, Y, c_X, c_d > 0$. From (6.12), the Jacobian determinant is



$$J = \frac{1}{8}\left[X\left(c_Y - Y\right)\xi + Y\left(c_X - X\right)\eta + \left(c_X Y + X c_Y\right)\right]$$

The Jacobian can be zero or negative for certain values of ξ and η, for certain values of c_X, c_Y. For example, although it is positive at nodes 1, 2 and 4, at node 3 it is negative for

$$\frac{c_Y}{Y} + \frac{c_X}{X} < 1,$$

which happens when node 3 crosses "inside" the dotted line. Elements with such distortions should be avoided, since one cannot work with singular Jacobian matrices.

6.1.4 Element Edges aligned with the Global Axes

The above equations simplify for the useful case where all elements have edges parallel to the $x - y$ axes. Taking the lengths of the element sides to be $2a$ and $2b$, and the lower left hand corner at (x_1, y_1), one has

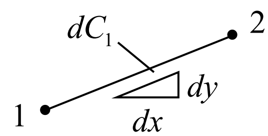
$$\begin{aligned} x &= x_1 + a(1+\xi) \\ y &= y_1 + b(1+\eta) \end{aligned}, \tag{6.16}$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}, \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} = ab \tag{6.17}$$

and

$$\frac{\partial N_i}{\partial x} = \frac{1}{a}\frac{\partial N_i}{\partial \xi}, \quad \frac{\partial N_i}{\partial y} = \frac{1}{b}\frac{\partial N_i}{\partial \eta} \tag{6.18}$$

6.1.5 Boundary Integrals



In 1-D, the boundary term for natural boundary conditions involved simply the value of a derivative at a boundary node. In 2-D, the boundary terms will involve an integration over a line (an element edge). For example, consider a portion dC_1 along the edge 1-2 of a Q4 element. Transforming into local coordinates,

$$dC_1 = \sqrt{(dx)^2 + (dy)^2}\Big|_{d\eta=0} = d\xi\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}\Bigg|_{\eta=-1} = \frac{L_1}{2}d\xi \tag{6.19}$$

where L_1 is the length of the edge,

$$L_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ , \tag{6.20}$$

and similarly for the other edges, summarised as follows:

$$\int_{1-2} dC_1 = \frac{L_1}{2}\int_{-1}^{+!} d\xi, \quad \int_{1-2} dC_2 = \frac{L_2}{2}\int_{-1}^{+!} d\eta, \quad \int_{1-2} dC_3 = \frac{L_3}{2}\int_{-1}^{+!} d\xi, \quad \int_{1-2} dC_4 = \frac{L_4}{2}\int_{-1}^{+!} d\eta \tag{6.21}$$

The shape functions along the four edges are given in the following table:

	N_1	N_2	N_3	N_4
C_1	$\frac{1}{2}(1-\xi)$	$\frac{1}{2}(1+\xi)$	0	0
C_2	0	$\frac{1}{2}(1-\eta)$	$\frac{1}{2}(1+\eta)$	0
C_3	0	0	$\frac{1}{2}(1+\xi)$	$\frac{1}{2}(1-\xi)$
C_4	$\frac{1}{2}(1-\eta)$	0	0	$\frac{1}{2}(1+\eta)$

Table 6.1: Q4 Shape Functions on Element Edges

The most commonly encountered natural boundary condition is one of the form

$$\frac{\partial p}{\partial n} = A \tag{6.22}$$

were A is a constant. In this case, the boundary integrals are

$$\int_{C_1} N_j dC_1 = \frac{L_1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad \int_{C_2} N_j dC_2 = \frac{L_2}{2}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad \int_{C_3} N_j dC_3 = \frac{L_3}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \quad \int_{C_4} N_j dC_4 = \frac{L_4}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \tag{6.23}$$

Another commonly encountered natural boundary condition is one of the form

$$\frac{\partial p}{\partial n} = Ap \tag{6.24}$$

were A is a constant. In this case, the boundary integrals lead to matrices {▲Problem 7}

$$\int_{C_1} N_i N_j dC_1 = \frac{L_1}{6}\begin{bmatrix}2 & 1 & 0 & 0\\1 & 2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}, \quad \text{etc.} \tag{6.25}$$

6.1.6 Internal Element Edges

The boundary integrals over internal edges of elements are taken to cancel each other out for the standard Galerkin FEM in 2-D – one only needs to be concerned with the line integrals along edges at the boundary of the mesh.

6.2 Numerical Integration

It is not possible to evaluate (surface) integrals over the local coordinates exactly (except in a few simple cases) and one must use a numerical approximation. First, consider the *Gauss-Legendre* one-dimensional numerical integration rule; here, an integral is approximated by a series of N terms:

$$\int_{-1}^{+1} f(\xi)d\xi = \sum_{i=1}^{N} W_i f(\xi_i) \tag{6.26}$$

where the W_i are called weights and the function is evaluated at a set of N integration points ξ_i. It is straight forward to calculate these weights and integration points. For example, consider the 1-point rule, $N = 1$. This involves two unknowns and so the rule should integrate exactly a linear function:

$$\int_{-1}^{+1} f(\xi)d\xi = \int_{-1}^{+1} [a_0 + a_1 \xi] d\xi = 2a_0 \tag{6.27}$$

Setting this to equal $W_1 f(\xi_1)$ and equating the coefficients of a_0, a_1, leads to $W_1 = 1, \quad \xi_1 = 0$. This 1-point rule will only give an approximate solution to higher order polynomials.

The 1 to 4 – point rules are given in the table below. The rule integrates polynomials of the order $r = 2N - 1$ exactly using an N – point rule.

N	s_i	W_i	Error	r
1	0	2	$\frac{1}{6}\left(\frac{d^2 f}{ds^2}\right)$	1
2	$\boxed{\mp \frac{1}{\sqrt{3}}}$ -0.577350269189626 $+0.577350269189626$	$\boxed{1}$ 1 1	$\approx 0.7\times10^{-2}\left(\frac{d^4 f}{ds^4}\right)$	3
3	$\boxed{0, \pm\sqrt{\frac{3}{5}}}$ -0.774596669241483 0 $+0.774596669241483$	$\boxed{\frac{8}{9}, \frac{5}{9}}$ 0.555555555555556 0.888888888888889 0.555555555555556	$\approx 0.6\times10^{-4}\left(\frac{d^6 f}{ds^6}\right)$	5
4	$\boxed{\pm\sqrt{\left(3-2\sqrt{\frac{6}{5}}\right)/7}, \pm\sqrt{\left(3+2\sqrt{\frac{6}{5}}\right)/7}}$	$\boxed{\frac{1}{2}+\frac{1}{6\sqrt{6/5}}, \frac{1}{2}-\frac{1}{6\sqrt{6/5}}}$	$\approx 0.3\times10^{-6}\left(\frac{d^8 f}{ds^8}\right)$	7

	-0.861136311594050	0.347854845137450		
	-0.339981043584860	0.652145154862550		
	$+0.339981043584860$	0.652145154862550		
	$+0.861136311594050$	0.347854845137450		

Table 6.2: One dimensional gauss quadrature

$$\int_{-1}^{+1} f(\xi)d\xi = \sum_{i=1}^{N} W_i f(\xi_i)$$

$r =$ maximum polynomial degree for exact results $= 2N - 1$

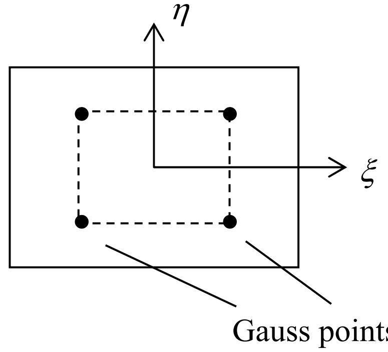
Extending this to two dimensional integrals, one has the rule

$$\int_{-1}^{+1}\int_{-1}^{+1} \Phi(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{N^{(\xi)}}\sum_{j=1}^{N^{(\eta)}} W_i^{(\xi)} W_j^{(\eta)} \Phi(\xi_i,\eta_j) \tag{6.28}$$

with the integration points and weights W as given in the 1-d table. Again, the rule integrates polynomials of the order $2N-1$ exactly using an $N-$ point rule. For example, to evaluate the integral of $\xi^2\eta^4$, one would use $2N^{(\xi)}-1=2,\ 2N^{(\eta)}-1=4$, giving $N^{(\xi)}=1.5,\ N^{(\eta)}=2.5$. Taking the next higher integer, one would choose $N^{(\xi)}=2,\ N^{(\eta)}=3$.

As examples, here follow two numerical quadrature rules obtained directly from (6.28), {▲Problem 9} the 2×2 rule and the 3×3 rule:

Four-Point Rule:



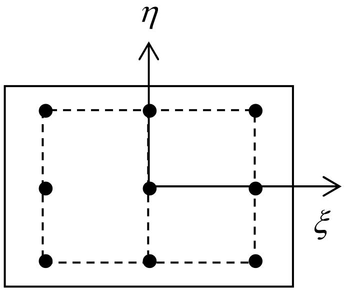
$$\int_{-1}^{+1}\int_{-1}^{+1} \Phi(\xi,\eta)d\xi d\eta \approx \Phi(\xi_1,\eta_1)+\Phi(\xi_1,\eta_2) + \Phi(\xi_2,\eta_1)+\Phi(\xi_2,\eta_2) \tag{6.29}$$

$$\text{where}\quad \begin{aligned} \xi_1 &= -1/\sqrt{3}, & \xi_2 &= +1/\sqrt{3} \\ \eta_1 &= -1/\sqrt{3}, & \eta_2 &= +1/\sqrt{3} \end{aligned} \tag{6.30}$$

This rule integrates exactly functions of the third order in ξ, η, for example terms of the form $\xi^3\eta^2$, $\xi\eta^3$, etc., or of a lower order.

Nine-Point Rule:

$$\int_{-1}^{+1}\int_{-1}^{+1}\Phi(\xi,\eta)d\xi d\eta = \frac{25}{81}\left[\Phi(\xi_1,\eta_1)+\Phi(\xi_1,\eta_3)+\Phi(\xi_3,\eta_1)+\Phi(\xi_3,\eta_3)\right]$$
$$+\frac{40}{81}\left[\Phi(\xi_1,\eta_2)+\Phi(\xi_2,\eta_1)+\Phi(\xi_2,\eta_3)+\Phi(\xi_3,\eta_2)\right]+\frac{64}{81}\Phi(\xi_2,\eta_2) \tag{6.31}$$



where
$$\begin{aligned} \xi_1 &= -\sqrt{0.6}, \quad \xi_2 = 0, \quad \xi_3 = +\sqrt{0.6} \\ \eta_1 &= -\sqrt{0.6}, \quad \eta_2 = 0, \quad \eta_3 = +\sqrt{0.6} \end{aligned} \tag{6.32}$$

This rule integrates exactly functions of the fifth order in ξ, η.

6.2.1 Computational Aspects

The values of the shape functions and their derivatives are required *only at the set of integration points*. These can be evaluated in the pre-processing stage, on a once and for all basis, or extracted from a database, before input of the actual mesh geometry. The Jacobian, on the other hand, depends on the global coordinates of the element, and must be evaluated *for each element*.

6.3 The Q6 Bilinear Element

The Q4 element is a stable low-order element which performs well in many applications. However, when used to model plate bending, it can tend to respond in an overly stiff manner. The Q6 bilinear element is often used in these cases. Its first four shape functions are as for the Q4 element; the other two are given by

$$N_5(\xi,\eta) = \left(1-\xi^2\right), \qquad N_6(\xi,\eta) = \left(1-\eta^2\right) \tag{6.33}$$

A function interpolated with (6.33) then has six unknowns, the four nodal values and two *internal* degrees of freedom. This element is used in Chapter 9 to model plane elasticity.

6.4 Higher Order Quadrilateral Elements

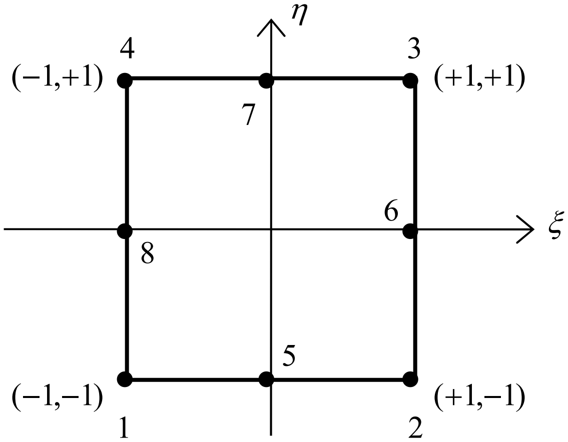
Other popular quadrilateral elements are the 8-noded and 9-noded elements.

6.4.1 The 8-noded Quadrilateral

The Q8 elements consists of the following polynomial representation of an unknown function:

$$p = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2 \tag{6.34}$$

In terms of local coordinates, the shape functions are then found to be



$$\begin{aligned}
N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta) \\
N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta) \\
N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta) \\
N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta) \\
N_5(\xi,\eta) &= \frac{1}{2}\left(1-\xi^2\right)(1-\eta) \\
N_6(\xi,\eta) &= \frac{1}{2}(1+\xi)\left(1-\eta^2\right) \\
N_7(\xi,\eta) &= \frac{1}{2}\left(1-\xi^2\right)(1+\eta) \\
N_8(\xi,\eta) &= \frac{1}{2}(1-\xi)\left(1-\eta^2\right)
\end{aligned} \tag{6.35}$$

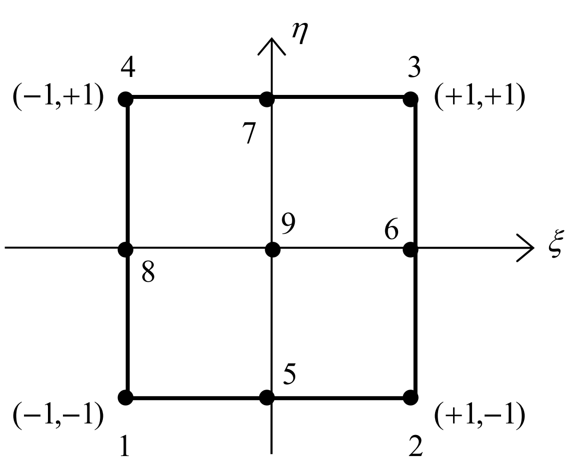
Along any one edge, the variation is quadratic. For example, along $\eta = -1$, the non-zero shape functions are

$$N_1(\xi) = -\frac{1}{2}\xi(1-\xi), \quad N_2(\xi) = +\frac{1}{2}\xi(1+\xi), \quad N_5(\xi) = 1-\xi^2 \tag{6.36}$$

which are the shape functions for the one dimensional 3-noded quadratic element.

6.4.2 The 9-noded Quadrilateral

The shape functions for this element are



$$\begin{aligned}
N_1(\xi,\eta) &= +\frac{1}{4}(1-\xi)(1-\eta)\xi\eta \\
N_2(\xi,\eta) &= -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta \\
N_3(\xi,\eta) &= +\frac{1}{4}(1+\xi)(1+\eta)\xi\eta \\
N_4(\xi,\eta) &= -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta \\
N_5(\xi,\eta) &= -\frac{1}{2}(1-\xi^2)(1-\eta)\eta \\
N_6(\xi,\eta) &= +\frac{1}{2}(1+\xi)(1-\eta^2)\xi \\
N_7(\xi,\eta) &= +\frac{1}{2}(1-\xi^2)(1+\eta)\eta \\
N_8(\xi,\eta) &= -\frac{1}{2}(1-\xi)(1-\eta^2)\xi \\
N_9(\xi,\eta) &= (1-\xi^2)(1-\eta^2)
\end{aligned} \tag{6.37}$$

with shape function N_i is 1 at node *i*.

6.5 Problems

1. Use the general relation for a bilinear polynomial, Eqn. 6.4, to derive the shape functions (6.1).

2. Show that a differentiation of (6.5) leads to (6.10).
3. Consider the derivatives of the shape functions with respect to the global coordinates. Show that their values at the four nodes are given by

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}_1 = \frac{1}{2\Delta_{124}} \begin{bmatrix} y_2 - y_4 & y_4 - y_1 & 0 & y_1 - y_2 \\ x_4 - x_2 & x_1 - x_4 & 0 & x_2 - x_1 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}_2 = \frac{1}{2\Delta_{231}} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}_3 = \frac{1}{2\Delta_{342}} \begin{bmatrix} 0 & y_3 - y_4 & y_4 - y_2 & y_2 - y_3 \\ 0 & x_4 - x_3 & x_2 - x_4 & x_3 - x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}_4 = \frac{1}{2\Delta_{413}} \begin{bmatrix} y_3 - y_4 & 0 & y_4 - y_1 & y_1 - y_3 \\ x_4 - x_3 & 0 & x_1 - x_4 & x_3 - x_1 \end{bmatrix}$$

where Δ_{ijk} is the area of the triangle with nodes i, j, k.
4. Use (6.7-6.10) to derive the closed-form expression (6.12) for the Jacobian determinant.
5. Show that the values of the Jacobian determinant (6.12) at the four nodes of the Q4 element are

$$J(-1,-1) = \frac{1}{2}\Delta_{124}, \quad J(+1,-1) = \frac{1}{2}\Delta_{231}, \quad J(+1,+1) = \frac{1}{2}\Delta_{342}, \quad J(-1,+1) = \frac{1}{2}\Delta_{413}$$

6. Consider the irregular two-element mesh discussed above. Derive the expressions analogous to (6.13–6.15) for element 2.
7. Derive the matrix (6.25) for the natural boundary condition (6.24).
8. Derive the 2-point one-dimensional integration rule for Table 6.2.
9. Use Table 6.2 and the 2-D Gaussian rule (6.28) to derive the 4-point integration rule (6.29–6.30).