5.2 Plasticity I: Hypoelastic-Plastic Models

The two main types of classical plasticity model for large strains are the **hypoelastic-plastic** model and the **hyperelastic-plastic** model. The first of these is discussed in this section.

5.2.1 Hypoelasticity

In the hypoelastic-plastic models of plasticity, the elastic response is assumed to be hypoelastic. As discussed in §4.1.4, hypoelastic materials are characterised by constitutive relations of the form

$$\dot{\boldsymbol{\sigma}} = f(\boldsymbol{\sigma}, \mathbf{d}) \tag{5.2.1}$$

and, in mutiaxial problems, this implies that the response cannot be expressed in terms of an elastic strain energy function. Thus the response is path-dependent and dissipation may occur even though the material is supposed to be elastic. However, the idea with the hypoelastic-plastic models is that the elastic strains are assumed to be relatively small so that any error in the conservation of energy is very small, and so can be neglected.

The stress-rate $\dot{\boldsymbol{\sigma}}$ in the hypoelastic equation 5.2.1 must be objective. Thus the material derivative of the Cauchy stress can not be used, but any of the many objective rates, for example the Jaumann, Green-Naghdi, Truesdell, etc., can be used.

A large class of hypoelastic materials is encompassed in the *linear* relation between objective stress-rate and the rate of deformation:

$$\boldsymbol{\sigma}^{\nabla} = \mathbf{C}^{\nabla} : \mathbf{d} \tag{5.2.2}$$

where $\boldsymbol{\sigma}^{\nabla}$ is an objective stress rate and $\mathbf{C}^{\nabla}$ is the corresponding fourth order tensor of elastic moduli, which may itself depend on the stress, in which case it must be an objective function of the stress. For a given finitely deformed state, the (small) increments in stress and strain are linearly related and are recovered upon unloading. However, for finite deformations, the work done in a closed path may not be zero.

The elastic modulus tensor $\mathbf{C}^{\nabla}$ is also called the **tangent modulus**. It possesses the minor symmetries due to the symmetry of $\mathbf{d}$ and $\boldsymbol{\sigma}^{\nabla}$. It is usually assumed to possess also the major symmetries.

Example

Consider the following hypoelastic constitutive equation:

$$\boldsymbol{\sigma}^{\nabla J} = \mathbf{C}^{\sigma J} : \mathbf{d} \tag{5.2.3}$$

where $\boldsymbol{\sigma}^{\nabla J}$ is the Jaumann stress-rate, Eqn. 3.5.20, $\dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}$. The Truesdell stress-rate $\boldsymbol{\sigma}^{\nabla T}$, defined by Eqn. 3.5.22, $\dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} + \mathrm{tr}(\mathbf{d})\boldsymbol{\sigma}$, is then

$$
\begin{aligned}
\boldsymbol{\sigma}^{\nabla T} &= \boldsymbol{\sigma}^{\nabla J} + \mathbf{w}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{w} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} + \mathrm{tr}(\mathbf{d})\boldsymbol{\sigma} \\
&= \boldsymbol{\sigma}^{\nabla J} - \mathbf{d}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{d}^{\mathrm{T}} + \mathrm{tr}(\mathbf{d})\boldsymbol{\sigma} \\
&= \left(\mathbf{C}^{\sigma J} - \hat{\mathbf{C}} + \boldsymbol{\sigma} \otimes \mathbf{I}\right) : \mathbf{d}
\end{aligned}
\tag{5.2.4}
$$

where $\hat{\mathbf{C}} : \mathbf{d} = \mathbf{d}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{d}$. Thus 5.2.3 can be expressed as $\boldsymbol{\sigma}^{\nabla T} = \mathbf{C}^{\sigma T} : \mathbf{d}$ with the Truesdell tangent modulus given by $\mathbf{C}^{\sigma J} - \hat{\mathbf{C}} + \boldsymbol{\sigma} \otimes \mathbf{I}$.

It is interesting to note that, if $\mathbf{C}^{\sigma J}$ is constant, then $\boldsymbol{\sigma}^{\nabla T}$ is not. Further, if $\mathbf{C}^{\sigma J}$ has the major symmetries, then $\boldsymbol{\sigma}^{\nabla T}$ does not (since $\boldsymbol{\sigma} \otimes \mathbf{I}$ does not). For this reason one often uses the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ rather than the Cauchy stress. For example, the Jaumann rate of the Kirchhoff stress is $\boldsymbol{\tau}^{\nabla J} = \dot{\boldsymbol{\tau}} - \mathbf{w}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{w}$. Then, with $\dot{J} = J\mathrm{tr}(\mathbf{d})$,

$$
\begin{aligned}
\boldsymbol{\sigma}^{\nabla T} &= J^{-1}\left(\boldsymbol{\tau}^{\nabla J} - J\mathbf{d}\boldsymbol{\sigma} - J\boldsymbol{\sigma}\mathbf{d}^{\mathrm{T}}\right) \\
&= \left(J^{-1}\mathbf{C}^{\tau J} - \hat{\mathbf{C}}\right) : \mathbf{d} \\
&\equiv \mathbf{C}^{\sigma T} : \mathbf{d}
\end{aligned}
\tag{5.2.5}
$$

Now if $\mathbf{C}^{\sigma T}$ has the major symmetries, then so does $\mathbf{C}^{\tau J}$.

5.2.2 Hypoelastic – Plastic Model

Additive Decomposition of the Rate of deformation

The rate of deformation is now decomposed additively into elastic and plastic parts according to

$$
\mathbf{d} = \mathbf{d}^{e} + \mathbf{d}^{p} \tag{5.2.6}
$$

The elastic response is then

$$
\boldsymbol{\sigma}^{\nabla} = \mathbf{C}_{e}^{\sigma} : \left(\mathbf{d} - \mathbf{d}^{p}\right) \tag{5.2.7}
$$

The yield condition is

$$
f\left(\boldsymbol{\sigma}, \boldsymbol{\alpha}\right) = 0 \tag{5.2.8}
$$

where $\boldsymbol{\alpha}$ represents any other variable(s) besides $\boldsymbol{\sigma}$ upon which f depends.

The flow rule is

$$
\mathbf{d}^{p} = \dot{\lambda}\mathbf{G}\left(\boldsymbol{\sigma}, \boldsymbol{\alpha}\right) \tag{5.2.9}
$$

The tensor $\mathbf{G}$ is usually expressed in the form $\mathbf{G} = \partial g(\boldsymbol{\sigma}, \boldsymbol{\alpha}) / \partial \boldsymbol{\sigma}$, where g is the plastic potential; λ is the plastic multiplier. For associative flow-rules, $f = g$. The variables $\boldsymbol{\alpha}$ are assumed to follow an evolution law of the form

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) \tag{5.2.10}$$

The loading and unloading conditions may be expressed as

$$\dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{\lambda} f = 0 \tag{5.2.11}$$

These are known as the **Kuhn-Tucker conditions**. The first of these states that the plastic multiplier rate is non-negative; for plastic loading, $\dot{\lambda} > 0$, otherwise $\dot{\lambda} = 0$. The second states that the stress must lie on or inside the yield surface. The last condition assures that the stress state remains on the yield surface during plastic flow. This last condition can also be expressed in rate form, $\dot{f} = 0$. This is known as the **consistency condition**:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} = 0 \tag{5.2.12}$$

Isotropic Materials

From the objectivity requirement, the yield function $f(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ in Eqn. 5.2.8 must be an objective scalar function of $\boldsymbol{\sigma}$. This implies that f must be a function only of the invariants of $\boldsymbol{\sigma}$. Thus the form $f(\boldsymbol{\sigma})$ necessarily represents the yield function of an isotropic material. For anisotropic materials, one must use a different stress measure, for example the yield function could be expressed in term of the PK2 stress, $\bar{f}(\mathbf{S}) = 0$.

Consider then an isotropic material, with $f = f(I_1, I_2, I_3)$, where I_i are the invariants of $\boldsymbol{\sigma}$. Then it can be shown that {▲Problem 1}

$$\boldsymbol{\sigma} \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \boldsymbol{\sigma} \tag{5.2.13}$$

that is, the tensors $\boldsymbol{\sigma}$ and $\partial f / \partial \boldsymbol{\sigma}$ are coaxial.

Consider now a formulation in terms of the Jaumann stress-rate. From 5.2.13, it follows that {▲Problem 2}

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^{\nabla J} \tag{5.2.14}$$

The consistency condition 5.2.12 can now be expressed in terms of the Jaumann stress rate:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^{\nabla J} + \frac{\partial f}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} = 0 \tag{5.2.15}$$

The plastic multiplier can now be solved for, using the hypoelastic relation 5.2.7, the flow rule 5.2.9, the evolution equation 5.2.10 and 5.2.15 {▲Problem 3}:

$$\dot{\lambda} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} : \mathbf{d}}{-\dfrac{\partial f}{\partial \boldsymbol{\alpha}} : \mathbf{A} + \dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} : \mathbf{G}} \tag{5.2.16}$$

Substituting this expression into the flow rule 5.2.9 and using the hypoelastic relation 5.2.7 then leads to {▲Problem 4}

$$\boldsymbol{\sigma}^{\nabla J} = \mathbf{C}^{\sigma J} : \mathbf{d}, \qquad \mathbf{C}^{\sigma J} = \mathbf{C}_e^{\sigma J} - \frac{\left(\mathbf{C}_e^{\sigma J} : \mathbf{G}\right) \otimes \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J}\right)}{-\dfrac{\partial f}{\partial \boldsymbol{\alpha}} : \mathbf{A} + \dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} : \mathbf{G}} \tag{5.2.17}$$

The tensor $\mathbf{C}^{\sigma J}$ is the **elasto-plastic tangent modulus**. It consists of an elastic component $\mathbf{C}_e^{\sigma J}$ and a component which results from plastic flow.

When the flow is associative, so that $\mathbf{G} = \partial f / \partial \boldsymbol{\sigma}$, the elasto-plastic modulus possesses the major symmetries. The plasticity equations can also be based upon other stress-rates; for example, the Truesdell stress-rate. For the same reasons as discussed earlier, although $\mathbf{C}^{\sigma J}$ has the major symmetries for associative flow rules, the Truesdell modulus will not. If, however, the equations 5.2.6-17 are formulated in terms of the Kirchhoff stress, then the Truesdell modulus will possess the major symmetries (in fact, all the relevant relations above are valid for the Kirchhoff stress, with $\boldsymbol{\sigma}$ simply being replaced by $\boldsymbol{\tau}$). Note that, if the elastic strains are small and plastic deformations are isochoric (volume preserving), then $J \approx 1$ and $\boldsymbol{\tau} \approx \boldsymbol{\sigma}$.

Small Strains

In the case of small strains, in the above relations one replaces **d** with $\dot{\boldsymbol{\varepsilon}}$ and decomposes the small strain rate according to $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$. The stress rate is the time derivative of the Cauchy stress, since objective stress-rates are not now a consideration. The hypoelastic relation is $\dot{\boldsymbol{\sigma}} = \mathbf{C} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p\right)$ and the remaining relations follow, for example the elasto-plastic tangent modulus is given by 5.2.17b with $\mathbf{C}_e^{\sigma}$ replaced with the small-strain elastic modulus tensor **C**. The small-strain formulation is valid for anisotropic elastic moduli.

5.2.3 J_2 Flow Theory

In the J_2 Flow Theory, one assumes the material is isotropic, plastic flow is independent of the hydrostatic pressure, plastic flow is incompressible and the yield surface used is the

Von Mises yield surface. An effective stress is used to generalise the uniaxial behaviour to multiaxial stress states.

First to be examined is the isotropic hardening model.

Isotropic Hardening

It is useful to express the equations 5.2.6-17 now in terms of the Kirchhoff stress and the Jaumann stress-rate. In that case, one again has Eqn. 5.2.6, $\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p$, with the hypoelastic Eqn. 5.2.7 now reading

$$\boldsymbol{\tau}^{\nabla J} = \mathbf{C}_e^{\tau J} : \left(\mathbf{d} - \mathbf{d}^p\right) \tag{5.2.18}$$

The Von Mises yield function can be expressed as

$$f(\boldsymbol{\tau}, \boldsymbol{\alpha}) = \sqrt{3J_2} - Y = \sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}} - Y = 0 \tag{5.2.19}$$

where J_2 is the second invariant of the deviatoric Kirchhoff stress and $\mathbf{s} = \operatorname{dev}\boldsymbol{\tau}$. The term $\sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}}$ acts as an effective stress $\hat{\sigma}$.

The (associative) flow rule 5.2.9 can be expressed as (see the Appendix to this section for details of the differentiation involved here)

$$\mathbf{d}^p = \dot{\lambda}\mathbf{G}(\boldsymbol{\tau}), \qquad \mathbf{G}(\boldsymbol{\tau}) = \frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{3\mathbf{s}}{2\sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}}} \tag{5.2.20}$$

The only variable $\boldsymbol{\alpha}$ necessary in the model is the scalar accumulated effective plastic strain:

$$\alpha \equiv \hat{\varepsilon}^p, \qquad \hat{\varepsilon}^p = \int d\hat{\varepsilon}^p = \int \dot{\hat{\varepsilon}}^p dt, \qquad d\hat{\varepsilon}^p = \sqrt{\frac{2}{3} d\boldsymbol{\varepsilon}^p : d\boldsymbol{\varepsilon}^p} \tag{5.2.21}$$

The proposed evolution law for α is simply (Eqn. 5.2.10 with the function $A = 1$)

$$\dot{\alpha} = \dot{\lambda} \quad \left(= \dot{\hat{\varepsilon}}^p\right) \tag{5.2.22}$$

With the definition of the plastic modulus being

$$H\left(\hat{\varepsilon}^p\right) = \frac{dY\left(\hat{\varepsilon}^p\right)}{d\hat{\varepsilon}^p} \tag{5.2.23}$$

the consistency condition 5.2.12 is, from 5.2.19,

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} : \dot{\boldsymbol{\tau}} - H\left(\hat{\varepsilon}^p\right)\dot{\hat{\varepsilon}}^p = 0 \tag{5.2.24}$$

The plastic multiplier rate is now given by Eqn. 5.2.16,

$$\dot{\lambda} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \mathbf{d}}{H\left(\hat{\varepsilon}^p\right) + \dfrac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \dfrac{\partial f}{\partial \boldsymbol{\tau}}} \tag{5.2.25}$$

Finally, the elasto-plastic tangent modulus, Eqn. 5.2.17, is

$$\boldsymbol{\tau}^{\nabla J} = \mathbf{C}^{\tau J} : \mathbf{d}, \qquad \mathbf{C}^{\tau J} = \mathbf{C}_e^{\tau J} - \frac{\left(\mathbf{C}_e^{\tau J} : \dfrac{\partial f}{\partial \boldsymbol{\tau}}\right) \otimes \left(\dfrac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J}\right)}{H\left(\hat{\varepsilon}^p\right) + \dfrac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \dfrac{\partial f}{\partial \boldsymbol{\tau}}} \tag{5.2.26}$$

It is useful to decompose the response into volumetric and deviatoric components. First express the elastic moduli as in Eqns. 4.1.19,

$$\mathbf{C}_e^{\tau J} = \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu \left[\mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right] \tag{5.2.27}$$

where $\mathbb{I} = \delta_{im}\delta_{jn} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n$. Note then that {▲Problem 5}

$$\mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}} = 2\mu \frac{\partial f}{\partial \boldsymbol{\tau}}, \qquad \frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}} = 3\mu \tag{5.2.28}$$

so that the tangent modulus 5.2.26b can be expressed as

$$\mathbf{C}^{\tau J} = \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu \left[\mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right] - \frac{(4/3)\mu}{1 + H/3\mu} \frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}} \tag{5.2.29}$$

Kinematic Hardening

To accommodate kinematic hardening, one introduces as another hardening parameter the back stress $\overline{\boldsymbol{\alpha}}$. The yield condition 5.2.8 is now (in terms of the Kirchhoff stress)

$$f\left(\boldsymbol{\tau}, \overline{\boldsymbol{\alpha}}, \hat{\varepsilon}^p\right) = \sqrt{\tfrac{3}{2}\left(\mathbf{s} - \overline{\boldsymbol{\alpha}}\right) : \left(\mathbf{s} - \overline{\boldsymbol{\alpha}}\right)} - Y\left(\hat{\varepsilon}^p\right) = 0 \tag{5.2.30}$$

The flow rule is

$$\mathbf{d}^p = \dot{\lambda} \mathbf{G}\left(\boldsymbol{\tau}, \overline{\boldsymbol{\alpha}}\right), \qquad \mathbf{G}\left(\boldsymbol{\tau}, \overline{\boldsymbol{\alpha}}\right) = \frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{3\left(\mathbf{s} - \overline{\boldsymbol{\alpha}}\right)}{2\sqrt{\tfrac{3}{2}\left(\mathbf{s} - \overline{\boldsymbol{\alpha}}\right) : \left(\mathbf{s} - \overline{\boldsymbol{\alpha}}\right)}} \tag{5.2.31}$$

The evolution equation for the back-stress is, for example using a linear hardening rule and the Jaumann stress-rate,

$$\overline{\boldsymbol{\alpha}}^{\nabla J} = c\mathbf{d}^{p} \left(= c\dot{\lambda}\mathbf{G}\right) \tag{5.2.32}$$

Note that the use of the Jaumann rate in the back-stress evolution law can lead to physically unreasonable stress oscillations in simple shear for large deformations (Nagtegaal and DeJong, 1981). However, this behaviour is not significant provided the elastic strains are not too large. A formulation based upon the Green-Naghdi strain can eliminate such stress oscillations (Johnson and Bammann, 1984).

5.2.4 Drucker – Prager Model

The yield criterion for a Drucker-Prager material is a modification of the Von Mises function so as to incorporate a plastic response due to a hydrostatic pressure:

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \sqrt{3J_2} + \alpha I_1 - Y = 0 \tag{5.2.33}$$

where $I_1 = \mathrm{tr}\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{I}$. The associated flow rule is then

$$\mathbf{d}^{p} = \dot{\lambda}\mathbf{G}(\boldsymbol{\sigma}), \qquad \mathbf{G}(\boldsymbol{\sigma}) = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}}} + \alpha\mathbf{I} \tag{5.2.34}$$

where $\mathbf{s} = \mathrm{dev}\boldsymbol{\sigma}$. A suitable non-associative flow-rule might be

$$\mathbf{d}^{p} = \dot{\lambda}\mathbf{G}(\boldsymbol{\sigma}), \qquad \mathbf{G}(\boldsymbol{\sigma}) = \frac{\partial g}{\partial \boldsymbol{\sigma}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}}}, \quad g = \sqrt{3J_2} \tag{5.2.35}$$

5.2.5 Elastic Moduli and Objectivity

It was mentioned above that objectivity requires that a yield function of the form $f(\boldsymbol{\sigma})$ necessarily represents an isotropic material. Objectivity also places restrictions on the elastic moduli. First, consider a *constant* modulus tensor $\mathbf{C}_e^{\tau J}$. Objectivity of a constitutive equation requires that

$$\left(\boldsymbol{\tau}^{\nabla J}\right)^{*} = \mathbf{C}_e^{\tau J} : \left(\mathbf{d}^{e}\right)^{*} \tag{5.2.36}$$

or

$$\mathbf{Q}\boldsymbol{\tau}^{\nabla J}\mathbf{Q}^{\mathrm{T}} = \mathbf{C}_e^{\tau J} : \left(\mathbf{Q}\mathbf{d}^{e}\mathbf{Q}^{\mathrm{T}}\right) \tag{5.2.37}$$

so, using the index notation,

$$\left(\boldsymbol{\tau}^{\nabla J}\right)_{ij} = \left[Q_{pi}Q_{qj}Q_{rk}Q_{sl}\left(\mathbf{C}_e^{\tau J}\right)_{pqrs}\right]\left(\left(\mathbf{d}^{e}\right)_{kl}\right) \tag{5.2.38}$$

In order that objectivity be satisfied, one must have

$$\left(\mathbf{C}_e^{\tau J}\right)_{ijkl} = Q_{pi}Q_{qj}Q_{rk}Q_{sl}\left(\mathbf{C}_e^{\tau J}\right)_{pqrs} \tag{5.2.39}$$

which shows that, if the modulus is constant, *it must be isotropic*.

If one requires an anisotropic elastic modulus, one must use a formulation based on a configuration other than the spatial configuration, as discussed in the next sub-section.

5.2.6 Corotational Stress Formulation

The models discussed thus far have two drawbacks:

1. the yield function must be an isotropic function of the stress
2. if the elastic moduli are constant, then the moduli must be isotropic

To overcome these restrictions to isotropic materials, one can formulate a plasticity model in terms of, for example, the corotational stress.

The corotational stress is defined by Eqn. 3.5.12,

$$\boldsymbol{\sigma}_{\mathrm{U}} = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} \tag{5.2.40}$$

Using a formulation based on the Kirchhoff stress, the corotational Kirchhoff stress is

$$\boldsymbol{\tau}_{\mathrm{U}} = \mathbf{R}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{R} \tag{5.2.41}$$

Recall from §3.5.3 that the PK2 stress **S** is the pull-back of the Kirchhoff stress $\boldsymbol{\tau}$, $\mathbf{S} = \chi_*^{-1}(\boldsymbol{\tau})^\# = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}}$; the corotational stress $\boldsymbol{\tau}_{\mathrm{U}}$ is the pull-back of $\boldsymbol{\tau}$ but with respect to **R** and not **F**: $\boldsymbol{\tau}_{\mathrm{U}} = \chi_*^{-1}(\boldsymbol{\tau})^\#_{\mathbf{F}=\mathbf{R}} = \mathbf{R}^{-1}\boldsymbol{\tau}\mathbf{R}^{-\mathrm{T}}$.

Define now the corotational rate of deformation:

$$\mathbf{d}_{\mathrm{U}} = \mathbf{R}^{\mathrm{T}}\mathbf{d}\mathbf{R} \tag{5.2.42}$$

with the decomposition

$$\mathbf{d}_{\mathrm{U}} = \mathbf{d}_{\mathrm{U}}^e + \mathbf{d}_{\mathrm{U}}^p \tag{5.2.43}$$

Note that the corotational stress and rate of deformation are insensitive to rigid body rotations **Q** to the current configuration:

$$\begin{aligned}(\boldsymbol{\tau}_{\mathrm{U}})^* &= \left(\mathbf{R}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{R}\right)^* = (\mathbf{QR})^{\mathrm{T}}\left(\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}}\right)(\mathbf{QR}) = \boldsymbol{\tau}_{\mathrm{U}}\\ (\mathbf{d}_{\mathrm{U}})^* &= \left(\mathbf{R}^{\mathrm{T}}\mathbf{d}\mathbf{R}\right)^* = (\mathbf{QR})^{\mathrm{T}}\left(\mathbf{Q}\mathbf{d}\mathbf{Q}^{\mathrm{T}}\right)(\mathbf{QR}) = \mathbf{d}_{\mathrm{U}}\end{aligned} \tag{5.2.44}$$

The corotational stress-rate is

$$\dot{\boldsymbol{\tau}}_{\mathrm{U}} = \overline{\mathbf{R}^{\mathrm{T}} \boldsymbol{\tau} \mathbf{R}}^{\cdot} = \dot{\mathbf{R}}^{\mathrm{T}} \boldsymbol{\tau} \mathbf{R} + \mathbf{R}^{\mathrm{T}} \dot{\boldsymbol{\tau}} \mathbf{R} + \mathbf{R}^{\mathrm{T}} \boldsymbol{\tau} \dot{\mathbf{R}} \tag{5.2.45}$$

A rigid body rotation **Q** to the current configuration also results in {▲Problem 6}

$$\left(\dot{\boldsymbol{\tau}}_{\mathrm{U}}\right)^{*} = \dot{\boldsymbol{\tau}}_{\mathrm{U}} \tag{5.2.46}$$

With the corotational stress-rate objective, the elastic response given by

$$\dot{\boldsymbol{\tau}}_{\mathrm{U}} = \mathbf{C}_{\mathrm{U}e}^{\tau} : \mathbf{d}_{\mathrm{U}}^{e} \tag{5.2.47}$$

Note that rigid body rotations to the current configuration leave $\dot{\boldsymbol{\tau}}_{\mathrm{U}}$ and $\mathbf{d}_{\mathrm{U}}^{e}$ unchanged and so $\mathbf{C}_{\mathrm{U}e}^{\tau}$ remains unchanged also. Thus, unlike the isotropic restriction 5.2.39, there is no objectivity restriction here on the form of $\mathbf{C}_{\mathrm{U}e}^{\tau}$ and so $\mathbf{C}_{\mathrm{U}e}^{\tau}$ can represent in general an anisotropic response.

Another way of looking at this is as follows: for an anisotropic material and using a *fixed* coordinate system, the components of an elastic modulus tensor **C** will in general change as the material rotates. With the corotational formulation, however, the axes rotate with the material, and so material rotation has no effect on $\mathbf{C}_{\mathrm{U}e}^{\tau}$.

The remainder of the formulation follows as before, for example the yield condition would be $f(\boldsymbol{\tau}_{\mathrm{U}}, \boldsymbol{\alpha}_{\mathrm{U}}) = 0$ and the flow rule $\mathbf{d}_{\mathrm{U}}^{e} = \dot{\lambda}\mathbf{G}(\boldsymbol{\tau}_{\mathrm{U}}, \boldsymbol{\alpha}_{\mathrm{U}})$. The evolution law for the $\boldsymbol{\alpha}_{\mathrm{U}}$ variable(s) is $\dot{\boldsymbol{\alpha}}_{\mathrm{U}} = \dot{\lambda}\mathbf{A}(\tau_{\mathrm{U}}, \boldsymbol{\alpha}_{\mathrm{U}})$. Again, the scalar yield function may now represent in general anisotropic material behaviour. The loading and unloading conditions are again given by 5.2.11.

The plastic multiplier rate is now (see 5.2.16)

$$\dot{\lambda} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\tau}_{\mathrm{U}}} : \mathbf{C}_{\mathrm{U}e}^{\tau} : \mathbf{d}_{\mathrm{U}}}{-\dfrac{\partial f}{\partial \boldsymbol{\alpha}_{\mathrm{U}}} : \mathbf{A} + \dfrac{\partial f}{\partial \boldsymbol{\tau}_{\mathrm{U}}} : \mathbf{C}_{\mathrm{U}e}^{\tau} : \mathbf{G}} \tag{5.2.48}$$

and

$$\boldsymbol{\tau}^{\nabla J} = \mathbf{C}_{\mathrm{U}}^{\tau} : \mathbf{d}_{\mathrm{U}}, \qquad \mathbf{C}_{\mathrm{U}}^{\tau} = \mathbf{C}_{\mathrm{Ue}}^{\tau} - \frac{\left(\mathbf{C}_{\mathrm{Ue}}^{\tau} : \mathbf{G}\right) \otimes \left(\dfrac{\partial f}{\partial \boldsymbol{\tau}_{\mathrm{U}}} : \mathbf{C}_{\mathrm{Ue}}^{\tau}\right)}{-\dfrac{\partial f}{\partial \boldsymbol{\alpha}_{\mathrm{U}}} : \mathbf{A} + \dfrac{\partial f}{\partial \boldsymbol{\tau}_{\mathrm{U}}} : \mathbf{C}_{\mathrm{Ue}}^{\tau} : \mathbf{G}} \tag{5.2.49}$$

5.2.7 Problems

1. The derivatives of the invariants of the stress tensor are (see Eqn. 1.11.33)

$$\frac{\partial \mathrm{I}_1}{\partial \boldsymbol{\sigma}} = \mathbf{I}, \quad \frac{\partial \mathrm{I}_2}{\partial \boldsymbol{\sigma}} = \mathrm{I}_1 \mathbf{I} - \boldsymbol{\sigma}, \quad \frac{\partial \mathrm{I}_3}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma}^2 - \mathrm{I}_1 \boldsymbol{\sigma} + \mathrm{I}_2 \mathbf{I} = \mathrm{I}_3 \boldsymbol{\sigma}^{-1}$$

Use these relations to show that, for the yield function $f = f(I_1, I_2, I_3)$, the tensors $\boldsymbol{\sigma}$ and $\partial f / \partial \boldsymbol{\sigma}$ are coaxial.

2. Use the fact that $\boldsymbol{\sigma}$ and $\partial f / \partial \boldsymbol{\sigma}$ are coaxial, and Eqn. 1.9.3h, $\mathbf{A} : (\mathbf{BC}) = (\mathbf{B}^{\mathrm{T}} \mathbf{A}) : \mathbf{C} = (\mathbf{A} \mathbf{C}^{\mathrm{T}}) : \mathbf{B}$, to show that

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^{\nabla J}$$

where $\boldsymbol{\sigma}^{\nabla J} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}$.

3. Derive Eqn. 5.2.16

4. Use the index notation to verify that

$$\mathbf{C} : (\mathbf{A} : \mathbf{C} : \mathbf{d}) : \mathbf{G} = (\mathbf{C} : \mathbf{G}) \otimes (\mathbf{A} : \mathbf{C}) : \mathbf{d}$$

where $\mathbf{A}, \mathbf{d}, \mathbf{G}$ are second order tensors and $\mathbf{C}$ is a fourth-order tensor. Hence derive the elasto-plastic modulus 5.2.17b.

5. Use the relations 1.9.62 and 1.9.64 to show that, for an arbitrary tensor $\mathbf{A}$,

$$\mathbf{C}_e^{\tau J} : \mathbf{A} = \kappa (\mathrm{tr}\mathbf{A}) \mathbf{I} + 2\mu \,\mathrm{dev}\mathbf{A}$$

$$\mathbf{A} : \mathbf{C}_e^{\tau J} : \mathbf{A} = \kappa (\mathrm{tr}\mathbf{A})^2 + 2\mu \,\mathrm{dev}\mathbf{A} : \mathrm{dev}\mathbf{A}$$

with $\mathbf{C}_e^{\tau J}$ given by 5.2.27. Hence, using 1.9.31, show that, for a deviatoric tensor $\mathbf{A}$,

$$\mathbf{C}_e^{\tau J} : \mathbf{A} = 2\mu \mathbf{A}$$

$$\mathbf{A} : \mathbf{C}_e^{\tau J} : \mathbf{A} = 2\mu \mathbf{A} : \mathbf{A}$$

Finally, use 5.2.20,

$$\frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}}}$$

to derive Eqns. 5.2.28,

$$\frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}} = 3\mu$$

6. The application of a rigid body rotation $\mathbf{Q}$ to the current configuration results in a change to the corotational stress rate 5.2.45,

$$\begin{aligned} (\dot{\boldsymbol{\tau}}_{\mathrm{U}})^* &= \dot{\mathbf{R}}^{*\mathrm{T}} \boldsymbol{\tau}^* \mathbf{R}^* + \mathbf{R}^{*\mathrm{T}} \dot{\boldsymbol{\tau}}^* \mathbf{R}^* + \mathbf{R}^{*\mathrm{T}} \boldsymbol{\tau}^* \dot{\mathbf{R}}^* \\ &= \left(\dot{\overline{\mathbf{QR}}} \right)^{\mathrm{T}} (\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}})(\mathbf{QR}) + (\mathbf{QR})^{\mathrm{T}} \left(\dot{\overline{\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}}}} \right) (\mathbf{QR}) + (\mathbf{QR})^{\mathrm{T}} (\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^{\mathrm{T}}) \left(\dot{\overline{\mathbf{QR}}} \right) \end{aligned}$$

Use the relation $\dot{\overline{\mathbf{Q}^{\mathrm{T}}\mathbf{Q}}} = \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q} + \mathbf{Q}^{\mathrm{T}} \dot{\mathbf{Q}} = \mathbf{0}$ to show that $(\dot{\boldsymbol{\tau}}_{\mathrm{U}})^* = \dot{\overline{\mathbf{R}^{\mathrm{T}} \boldsymbol{\tau} \mathbf{R}}} = \dot{\boldsymbol{\tau}}_{\mathrm{U}}$.

5.2.8 Appendix to §5.2

Differentiation of the Von Mises Yield Function

The Von Mises yield criterion is $f = \sqrt{J_2} - k = 0$ where $J_2 = \frac{1}{2} s_{ij} s_{ij}$. Using the product rule of differentiation, in index notation:

$$
\begin{aligned}
\frac{\partial\sqrt{J_2}}{\partial\sigma_{ij}} = \frac{\partial\sqrt{\frac{1}{2}s_{mn}s_{mn}}}{\partial\sigma_{ij}} &= \frac{1}{2}\frac{1}{\sqrt{\frac{1}{2}s_{pq}s_{pq}}}\frac{1}{2}\frac{\partial\left(s_{mn}s_{mn}\right)}{\partial\sigma_{ij}} \\
&= \frac{1}{2\sqrt{\frac{1}{2}s_{pq}s_{pq}}}s_{mn}\frac{\partial s_{mn}}{\partial\sigma_{ij}} \\
&= \frac{s_{mn}}{\sqrt{2s_{pq}s_{pq}}}\frac{\partial\left(\sigma_{mn}-\frac{1}{3}\sigma_{kk}\delta_{mn}\right)}{\partial\sigma_{ij}} \\
&= \frac{s_{mn}}{\sqrt{2s_{pq}s_{pq}}}\left(\delta_{mi}\delta_{nj}-\frac{1}{3}\delta_{ki}\delta_{kj}\delta_{mn}\right) \\
&= \frac{s_{mn}}{\sqrt{2s_{pq}s_{pq}}}\left(\delta_{mi}\delta_{nj}-\frac{1}{3}\delta_{ij}\delta_{mn}\right) \\
&= \frac{s_{ij}-\frac{1}{3}\delta_{ij}s_{mm}}{\sqrt{2s_{pq}s_{pq}}} \\
&= \frac{s_{ij}}{\sqrt{2s_{pq}s_{pq}}} = \frac{s_{ij}}{2\sqrt{J_2}}
\end{aligned}
\tag{5.2.50}
$$

In tensor notation:

$$
\begin{aligned}
\frac{\partial\sqrt{J_2}}{\partial\boldsymbol{\sigma}} = \frac{\partial\sqrt{\frac{1}{2}\mathbf{s}:\mathbf{s}}}{\partial\boldsymbol{\sigma}} &= \frac{1}{2}\frac{1}{\sqrt{\frac{1}{2}\mathbf{s}:\mathbf{s}}}\frac{1}{2}\frac{\partial\left(\mathbf{s}:\mathbf{s}\right)}{\partial\boldsymbol{\sigma}} \\
&= \frac{1}{2\sqrt{\frac{1}{2}\mathbf{s}:\mathbf{s}}}\mathbf{s}:\frac{\partial\mathbf{s}}{\partial\boldsymbol{\sigma}} \\
&= \frac{\mathbf{s}}{\sqrt{2\mathbf{s}:\mathbf{s}}}:\frac{\partial\left(\boldsymbol{\sigma}-\frac{1}{3}\left(\mathrm{tr}\boldsymbol{\sigma}\right)\mathbf{I}\right)}{\partial\boldsymbol{\sigma}} \\
&= \frac{\mathbf{s}}{\sqrt{2\mathbf{s}:\mathbf{s}}}:\left(\mathbb{I}-\frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right) \\
&= \frac{\mathbf{s}-\frac{1}{3}\left(\mathrm{tr}\mathbf{s}\right)\mathbf{I}}{\sqrt{2\mathbf{s}:\mathbf{s}}} \\
&= \frac{\mathbf{s}}{\sqrt{2\mathbf{s}:\mathbf{s}}} = \frac{\mathbf{s}}{2\sqrt{J_2}}
\end{aligned}
\tag{5.2.51}
$$