4.5 Material Anisotropy

4.5.1 Material Symmetry

The isotropic material was defined as one whose material response was unaffected by rigid body rotations of the reference configuration. Other material symmetries are possible; to generalise the notion, instead of considering orthogonal transformations, consider an arbitrary deformation $\mathbf{F}_0$ of the reference configuration S_0 bringing it to a new configuration $S^\diamond$, Fig. 4.5.1 (compare with the isotropic case, Fig. 2.8.6).

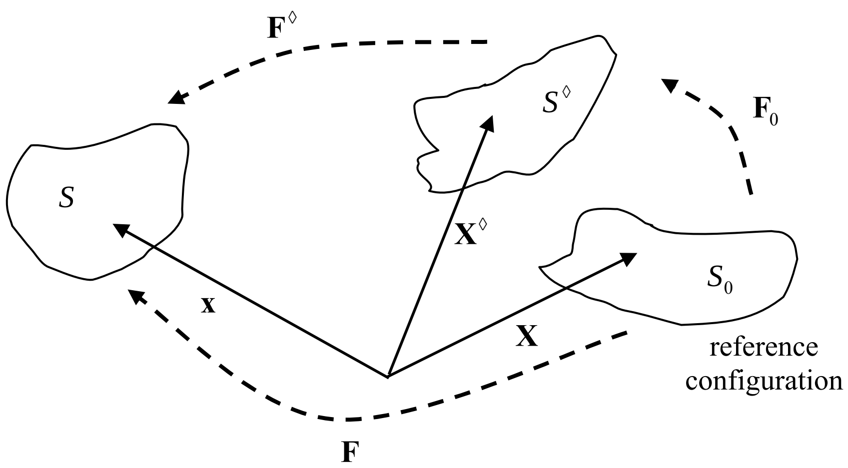

Figure 4.5.1: a deformation of the reference configuration

Considering the Cauchy-elastic material, if the deformation $\mathbf{F}_0$ has no effect on the response of the material, then

$$\boldsymbol{\sigma}(\mathbf{F}^\diamond) = \boldsymbol{\sigma}(\mathbf{F}) \quad \rightarrow \quad \boldsymbol{\sigma}(\mathbf{F}\mathbf{F}_0^{-1}) = \boldsymbol{\sigma}(\mathbf{F}) \tag{4.5.1}$$

When $\mathbf{F}_0 = \mathbf{Q}$, one has the isotropic material. Setting $\mathbf{G} = \mathbf{F}_0^{-1}$, 4.5.1 can be cast in the most usual form:

$$\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{G}) \tag{4.5.2}$$

Note that the restriction $\det \mathbf{G} = \pm 1$ is assumed, since otherwise arbitrary dilatations could occur with no change in material response, which seems physically unreasonable.

Note that the set of all tensors **G** which satisfy 4.5.2 forms a **group** (see the Appendix to this Chapter, §4.A.2) and hence is called the **symmetry group** of the material (with respect to the configuration S_0).

Apart from isotropy, the two most important practical cases of material symmetry are **transverse isotropy** and **orthotropy**.

4.5.2 Transverse Isotropy

Consider first the transversely isotropic material. Such a material has a single preferred direction, defined by a unit vector $\mathbf{a}_0$ in the reference configuration. Such a vector is illustrated in Fig. 4.5.2, showing also the unit vectors $\hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3$ completing an orthonormal set. The symmetry group of the transversely isotropic material is the set of orthogonal tensors **Q** which transform the set $\{\mathbf{a}_0, \mathbf{n}_2, \mathbf{n}_3\}$ into the new orthonormal set $\{\pm\mathbf{a}_0, \mathbf{n}'_2, \mathbf{n}'_3\}$. In particular,

$$\mathbf{Q}\mathbf{a}_0 = \pm\mathbf{a}_0 \qquad (4.5.3)$$

In order to ensure that the sense of $\mathbf{Q}\mathbf{a}_0$ is immaterial, it is best to introduce the **structural tensor** $\mathbf{a}_0 \otimes \mathbf{a}_0$, which transforms as the axes change according to

$$\mathbf{Q}\mathbf{a}_0 \otimes \mathbf{Q}\mathbf{a}_0 = \pm\mathbf{a}_0 \otimes \pm\mathbf{a}_0 \qquad (4.5.4)$$

or

$$\mathbf{Q}(\mathbf{a}_0 \otimes \mathbf{a}_0)\mathbf{Q}^{\mathrm{T}} = \mathbf{a}_0 \otimes \mathbf{a}_0 \qquad (4.5.5)$$

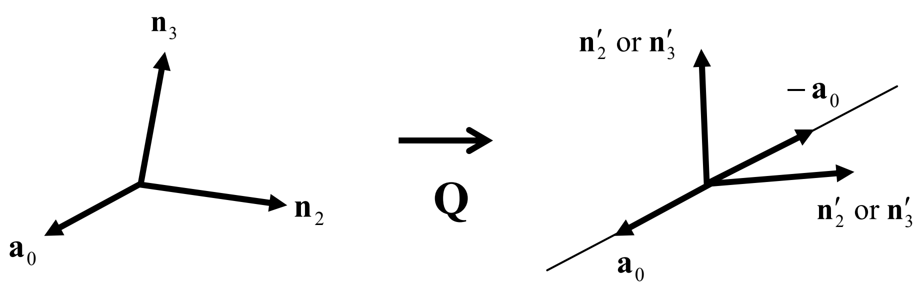

Figure 4.5.2: an orthonormal set of vectors

The strain energy can now be taken to be a function of **C**, as in the isotropic case, and $\mathbf{a}_0 \otimes \mathbf{a}_0$, which characterises the structure of the material:

$$W = W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) \qquad (4.5.6)$$

Allowing for transformations of the undeformed configuration,

$$W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = \mathbf{S}(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{a}_0 \otimes \mathbf{a}_0\mathbf{Q}^{\mathrm{T}}) \qquad (4.5.7)$$

with **Q** here restricted to the symmetry group defined by 4.5.3. Then W is an isotropic scalar function of two symmetric tensors and so, from Table 4.A.1, takes the form

$$W = W\Big(\mathrm{tr}\mathbf{C},\ \mathrm{tr}\mathbf{C}^2,\ \mathrm{tr}\mathbf{C}^3,\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0),\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2,\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^3 \\ \mathrm{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0),\ \mathrm{tr}\mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0),\ \mathrm{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2,\ \mathrm{tr}\mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0)^2\Big) \tag{4.5.8}$$

Since {▲Problem 1}

$$\mathrm{tr}(\mathbf{a} \otimes \mathbf{a}) = 1, \quad \mathrm{tr}(\mathbf{C}(\mathbf{a} \otimes \mathbf{a})) = \mathbf{aCa}, \quad \mathrm{tr}(\mathbf{C}^2(\mathbf{a} \otimes \mathbf{a})) = \mathbf{aC}^2\mathbf{a} \\ \mathbf{a} \otimes \mathbf{a} = (\mathbf{a} \otimes \mathbf{a})^2 = (\mathbf{a} \otimes \mathbf{a})^3 \tag{4.5.9}$$

one arrives at the representation

$$W = W\big(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)\big) \tag{4.5.10}$$

where the fourth and fifth scalar (pseudo-) invariants I_4, I_5 are defined by

$$I_4 = \mathbf{aCa}, \qquad I_5 = \mathbf{aC}^2\mathbf{a} \tag{4.5.11}$$

Note also that, from the definition of the stretch, Eqn. 2.2.17,

$$I_4 = \mathbf{a}_0\mathbf{C}\mathbf{a}_0 = \lambda_{\mathbf{a}}^2 \tag{4.5.12}$$

where $\lambda_{\mathbf{a}}$ is the stretch of the unit line element $\mathbf{a}_0$.

If the preferred direction is $\mathbf{e}_3$, then the fourth and fifth invariants in terms of components are

$$I_4 = \mathbf{a}_0\mathbf{C}\mathbf{a}_0 = C_{33} \\ I_5 = \mathbf{a}_0\mathbf{C}^2\mathbf{a}_0 = C_{13}^2 + C_{23}^2 + C_{33}^2 \tag{4.5.13}$$

in which case the five invariants can be taken as $\{I_1, I_2, I_3, C_{33}, C_{13}^2 + C_{23}^2\}$.

Using the relations {▲Problem 2}

$$\frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{a}_0, \qquad \frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0 \tag{4.5.14}$$

the PK2 stresses for a hyperelastic material are then

$$\begin{aligned} \mathbf{S} &= 2\sum_{i=1}^{5} \frac{\partial W(\mathbf{C}, \mathbf{a}_0)}{\partial I_i} \frac{\partial I_i}{\partial \mathbf{C}} \\ &= 2\Bigg[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\mathbf{I} - \frac{\partial W}{\partial I_2}\mathbf{C} + I_3 \frac{\partial W}{\partial I_3}\mathbf{C}^{-1} \\ &\qquad + \frac{\partial W}{\partial I_4}\mathbf{a}_0 \otimes \mathbf{a}_0 + \frac{\partial W}{\partial I_5}(\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0)\Bigg] \end{aligned} \tag{4.5.15}$$

Let **a** be a unit vector in the current configuration, in the direction of $\mathbf{F}\mathbf{a}_0$, that is,

$$\lambda_{\mathbf{a}}\mathbf{a} = \mathbf{F}\mathbf{a}_0 \tag{4.5.16}$$

Then, using Eqn. 3.5.7, $\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}}$, with $\mathbf{F}\mathbf{I}\mathbf{F}^{\mathrm{T}} = \mathbf{b}$, $\mathbf{F}\mathbf{C}\mathbf{F}^{\mathrm{T}} = \mathbf{b}^2$ (see Eqn.2.2.14), $\mathbf{F}\mathbf{C}^{-1}\mathbf{F}^{\mathrm{T}} = \mathbf{I}$ and noting that **C** and **b** have the same principal invariants, 4.5.13 becomes

$$\begin{aligned}\boldsymbol{\sigma} = 2J^{-1}\Bigg[& I_3\frac{\partial W}{\partial I_3}\mathbf{I} + \left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{b} - \frac{\partial W}{\partial I_2}\mathbf{b}^2 \\ & + I_4\frac{\partial W}{\partial I_4}\mathbf{a}\otimes\mathbf{a} + I_4\frac{\partial W}{\partial I_5}\left(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b}\otimes\mathbf{a}\right)\Bigg]\end{aligned} \tag{4.5.17}$$

Using the Cayley-Hamilton theorem allows one to re-write the Cauchy stress as

$$\begin{aligned}\boldsymbol{\sigma} = 2J^{-1}\Bigg[& \left(I_2\frac{\partial W}{\partial I_2} + I_3\frac{\partial W}{\partial I_3}\right)\mathbf{I} + \frac{\partial W}{\partial I_1}\mathbf{b} - I_3\frac{\partial W}{\partial I_2}\mathbf{b}^{-1} \\ & + I_4\frac{\partial W}{\partial I_4}\mathbf{a}\otimes\mathbf{a} + I_4\frac{\partial W}{\partial I_5}\left(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b}\otimes\mathbf{a}\right)\Bigg]\end{aligned} \tag{4.5.18}$$

Expressing the scalar invariants in terms of **b** rather than **C**, the coefficients of the tensors in 4.5.17-18 are functions of the set

$$\left\{\mathrm{tr}\mathbf{b},\ \mathrm{tr}\mathbf{b}^2,\ \mathrm{tr}\mathbf{b}^3,\ \lambda_{\mathbf{a}}^2\mathbf{a}\otimes\mathbf{a},\ \lambda_{\mathbf{a}}^2\mathbf{a}\mathbf{b}\mathbf{a}\right\} \tag{4.5.19}$$

Transversely Isotropic Materials with Constraints

For an incompressible material, $I_3 = 1$, and, analogous to 4.2.40, the strain energy takes the form

$$W\left(I_1(\mathbf{C}), I_2(\mathbf{C}), I_4(\mathbf{C},\mathbf{a}_0), I_5(\mathbf{C},\mathbf{a}_0)\right) - \frac{1}{2}p(I_3 - 1) \tag{4.5.20}$$

For a material which is inextensible in the direction of $\mathbf{a}_0$, from 4.5.13, $I_4 = 0$, and the strain energy takes the form

$$W\left(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_5(\mathbf{C},\mathbf{a}_0)\right) - \frac{1}{2}q(I_4 - 1) \tag{4.5.21}$$

4.5.3 Orthotropy

Consider now a material which is dependent on two characteristic directions, $\mathbf{a}_0$ and $\mathbf{b}_0$; again the sense of these directions is immaterial. The strain energy is now of the form

$$W = W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0, \mathbf{b}_0 \otimes \mathbf{b}_0) \tag{4.5.22}$$

As isotropic scalar function of three symmetric tensors depends on the following traces (see Table 4.A.1)

$$\begin{gathered}
\mathrm{tr}\mathbf{C},\ \ \mathrm{tr}\mathbf{C}^2,\ \ \mathrm{tr}\mathbf{C}^3 \\
\mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0),\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2,\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^3,\ \ \mathrm{tr}(\mathbf{b}_0 \otimes \mathbf{b}_0),\ \mathrm{tr}(\mathbf{b}_0 \otimes \mathbf{b}_0)^2,\ \mathrm{tr}(\mathbf{b}_0 \otimes \mathbf{b}_0)^3 \\
\mathrm{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0),\ \mathrm{tr}\mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0),\ \mathrm{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2,\ \mathrm{tr}\mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0)^2 \\
\mathrm{tr}\mathbf{C}(\mathbf{b}_0 \otimes \mathbf{b}_0),\ \mathrm{tr}\mathbf{C}^2(\mathbf{b}_0 \otimes \mathbf{b}_0),\ \mathrm{tr}\mathbf{C}(\mathbf{b}_0 \otimes \mathbf{b}_0)^2,\ \mathrm{tr}\mathbf{C}^2(\mathbf{b}_0 \otimes \mathbf{b}_0)^2 \\
\mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)(\mathbf{b}_0 \otimes \mathbf{b}_0),\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2(\mathbf{b}_0 \otimes \mathbf{b}_0), \\
\mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)(\mathbf{b}_0 \otimes \mathbf{b}_0)^2,\ \mathrm{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2(\mathbf{b}_0 \otimes \mathbf{b}_0)^2 \\
\mathrm{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0)(\mathbf{b}_0 \otimes \mathbf{b}_0)
\end{gathered} \tag{4.5.23}$$

Using 4.5.9 (see Eqn. 1.9.10e) this reduces to the set of nine invariants

$$\begin{gathered}
\mathrm{tr}\mathbf{C},\ \ \mathrm{tr}\mathbf{C}^2,\ \ \mathrm{tr}\mathbf{C}^3 \\
\mathbf{a}_0\mathbf{C}\mathbf{a}_0,\ \mathbf{a}_0\mathbf{C}^2\mathbf{a}_0,\ \mathbf{b}_0\mathbf{C}\mathbf{b}_0,\ \mathbf{b}_0\mathbf{C}^2\mathbf{b}_0,\ (\mathbf{a}_0 \cdot \mathbf{b}_0)^2,\ (\mathbf{a}_0 \cdot \mathbf{b}_0)\mathbf{a}_0\mathbf{C}\mathbf{b}_0
\end{gathered} \tag{4.5.24}$$

with $\mathbf{a}_0\mathbf{C}\mathbf{b}_0 = \mathbf{b}_0\mathbf{C}\mathbf{a}_0$. The term $\mathbf{a}_0 \cdot \mathbf{b}_0$ is the cosine of the angle between the two characteristic directions; this does not change during the deformation and so this term can be omitted, leaving

$$\mathrm{tr}\mathbf{C},\ \mathrm{tr}\mathbf{C}^2,\ \mathrm{tr}\mathbf{C}^3,\ \mathbf{a}_0\mathbf{C}\mathbf{a}_0,\ \mathbf{a}_0\mathbf{C}^2\mathbf{a}_0,\ \mathbf{b}_0\mathbf{C}\mathbf{b}_0,\ \mathbf{b}_0\mathbf{C}^2\mathbf{b}_0,\ (\mathbf{a}_0 \cdot \mathbf{b}_0)\mathbf{a}_0\mathbf{C}\mathbf{b}_0 \tag{4.5.25}$$

An orthotropic material is one for which $\mathbf{a}_0$ and $\mathbf{b}_0$ are perpendicular, $\mathbf{a}_0 \cdot \mathbf{b}_0 = 0$, making the last term here zero. This also then defines a third preferred direction, $\mathbf{c}_0$, orthogonal to both $\mathbf{a}_0$ and $\mathbf{b}_0$, which introduces extra terms $\mathbf{c}_0\mathbf{C}\mathbf{c}_0,\ \mathbf{c}_0\mathbf{C}^2\mathbf{c}_{00}$. But

$$\begin{aligned}
\mathrm{tr}\mathbf{C} &= \mathbf{a}_0\mathbf{C}\mathbf{a}_0 + \mathbf{b}_0\mathbf{C}\mathbf{b}_0 + \mathbf{c}_0\mathbf{C}\mathbf{c}_0 \\
\mathrm{tr}\mathbf{C}^2 &= \mathbf{a}_0\mathbf{C}^2\mathbf{a}_0 + \mathbf{b}_0\mathbf{C}^2\mathbf{b}_0 + \mathbf{c}_0\mathbf{C}^2\mathbf{c}_0
\end{aligned} \tag{4.5.26}$$

so that $\mathbf{c}_0\mathbf{C}\mathbf{c}_0,\ \mathbf{c}_0\mathbf{C}^2\mathbf{c}_0$ are redundant. Finally, the strain energy is of the form

$$W = W\left(\mathrm{tr}\mathbf{C},\ \mathrm{tr}\mathbf{C}^2,\ \mathrm{tr}\mathbf{C}^3,\ \mathbf{a}_0\mathbf{C}\mathbf{a}_0,\ \mathbf{a}_0\mathbf{C}^2\mathbf{a}_0,\ \mathbf{b}_0\mathbf{C}\mathbf{b}_0,\ \mathbf{b}_0\mathbf{C}^2\mathbf{b}_0\right) \tag{4.5.27}$$

As before, the stresses in a hyperelastic material can now be obtained by differentiation.

4.5.4 Problems

1. Show that, for unit vector **a**,

(i) $(\mathbf{a}\otimes\mathbf{a})^2 = \mathbf{a}\otimes\mathbf{a}$, (ii) $\mathrm{tr}(\mathbf{a}\otimes\mathbf{a}) = 1$

(iii) $\mathrm{tr}(\mathbf{C}(\mathbf{a}\otimes\mathbf{a})) = \mathbf{aCa}$ (iv) $\mathrm{tr}(\mathbf{C}^2(\mathbf{a}\otimes\mathbf{a})) = \mathbf{aC}^2\mathbf{a}$

2. Show that

$$\frac{\partial I_4}{\partial\mathbf{C}} = \mathbf{a}_0\otimes\mathbf{a}_0, \qquad \frac{\partial I_5}{\partial\mathbf{C}} = \mathbf{a}_0\otimes\mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C}\otimes\mathbf{a}_0$$

3. Show that

(i) $\mathbf{F}(\mathbf{a}_0\otimes\mathbf{a}_0)\mathbf{F}^{\mathrm{T}} = \lambda_{\mathbf{a}}^2\mathbf{a}\otimes\mathbf{a}$

(ii) $\mathbf{F}(\mathbf{a}_0\otimes\mathbf{C}\mathbf{a}_0)\mathbf{F}^{\mathrm{T}} = \lambda_{\mathbf{a}}^2\mathbf{a}\otimes\mathbf{ba}$

For (ii), it might help to note the following relations (for vector **b** and second-order tensors **A**, **B**):

$$(\mathbf{Ab})\mathbf{B} \neq \mathbf{A}(\mathbf{bB})$$
$$(\mathbf{AB})\mathbf{b} = \cdot\mathbf{A}(\mathbf{Bb})$$
$$(\mathbf{A}^{\mathrm{T}}\mathbf{b})\mathbf{A}^{\mathrm{T}} = (\mathbf{AA}^{\mathrm{T}})\mathbf{b}$$