4.2 Hyperelasticity

The Hyperelastic material is examined in this section.

4.2.1 Constitutive Equations

The rate of change of internal energy W per unit reference volume is given by the stress power, which can be expressed in a number of different ways (see §3.7.6):

$$\dot{W} = J\boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S} : \dot{\mathbf{C}} \tag{4.2.1}$$

The internal energy is regarded as a function of a deformation variable. For example,

The change in energy is due to the deformation which takes place, so take W to be a function of, say, the deformation gradient $\mathbf{F}(t)$, $W(\mathbf{F})$. It is assumed that in the reference configuration the strain energy is zero, $W(\mathbf{I}) = 0$, and that it grows with deformation, $W(\mathbf{F}) \geq 0$ [1].

The chain rule gives

$$\dot{W} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} \tag{4.2.2}$$

From 4.2.1,

$$\mathbf{P} : \dot{\mathbf{F}} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} \tag{4.2.3}$$

Since $\mathbf{F}$ and $\dot{\mathbf{F}}$ can take on any value independent of the other, one must have

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \tag{4.2.4}$$

This is a constitutive equation relating the kinematic variables to the force variables. From 4.2.1, alternative forms are:

$$\boxed{\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}}, \quad \mathbf{S} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}} \tag{4.2.5}$$

The procedure used in Eqns. 4.2.2-4 cannot be used for the stress power relation $J\boldsymbol{\sigma} : \mathbf{d}$ since there is no function whose derivative is the rate of deformation. Instead, use the relation 3.5.6, $\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}$, 4.2.4, $\mathbf{P} = \partial W / \partial \mathbf{F}$, to get

[1] and that it tends to infinity as the material is either compressed to a point, $J = \det \mathbf{F} \to 0$ or expanded to an infinite range, $J = \det \mathbf{F} \to \infty$.

$$\boldsymbol{\sigma} = J^{-1}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^{\mathrm{T}}, \qquad \sigma_{ij} = J^{-1}\frac{\partial W}{\partial F_{im}}F_{jm} \tag{4.2.6}$$

An alternative relation can be obtained by first deriving a relationship between the partial derivatives of the strain energy function with respect to the deformation gradient, $\partial W / \partial F_{ij}$, and with respect to the right Cauchy-Green tensor, $\partial W / \partial C_{ij}$. Suppose first that the strain energy is a function of the deformation gradient:

$$\frac{\partial W}{\partial \mathbf{F}} \equiv \frac{\partial W}{\partial F_{ij}}\mathbf{e}_i \otimes \mathbf{e}_j \tag{4.2.7}$$

The chain rule for $W = W(\mathbf{C}(\mathbf{F}))$ gives

$$\frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W}{\partial C_{mn}}\frac{\partial C_{mn}}{\partial F_{ij}}\mathbf{e}_i \otimes \mathbf{e}_j \tag{4.2.8}$$

With $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$,

$$\frac{\partial C_{mn}}{\partial F_{ij}} = \frac{\partial F_{km}}{\partial F_{ij}}F_{kn} + F_{km}\frac{\partial F_{kn}}{\partial F_{ij}} = \delta_{ki}\delta_{mj}F_{kn} + \delta_{ki}\delta_{nj}F_{km} = \delta_{mj}F_{in} + \delta_{nj}F_{im} \tag{4.2.9}$$

so that (and using the fact that $\mathbf{C}$ is symmetric),

$$\begin{aligned}
\frac{\partial W}{\partial F_{ij}}\mathbf{e}_i \otimes \mathbf{e}_j &= \delta_{mj}F_{in}\frac{\partial W}{\partial C_{mn}}\mathbf{e}_i \otimes \mathbf{e}_j + \delta_{nj}F_{im}\frac{\partial W}{\partial C_{mn}}\mathbf{e}_i \otimes \mathbf{e}_j \\
&= F_{in}\frac{\partial W}{\partial C_{mn}}\mathbf{e}_i \otimes \mathbf{e}_m + F_{im}\frac{\partial W}{\partial C_{mn}}\mathbf{e}_i \otimes \mathbf{e}_n \\
&= F_{in}\frac{\partial W}{\partial C_{nm}}\mathbf{e}_i \otimes \mathbf{e}_m + F_{im}\frac{\partial W}{\partial C_{mn}}\mathbf{e}_i \otimes \mathbf{e}_n
\end{aligned} \tag{4.2.10}$$

or

$$\frac{\partial W}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}, \qquad \frac{\partial W}{\partial F_{ij}} = 2F_{ik}\frac{\partial W}{\partial C_{kj}} \tag{4.2.11}$$

Now 4.2.6 can be re-written as

$$\boxed{\boldsymbol{\sigma} = 2J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}}} \tag{4.2.12}$$

Similarly, Eqn. 4.2.4 cab be re-written as

$$\boxed{\mathbf{P} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}} \tag{4.2.13}$$

The Strain Energy and the Right Stretch Tensor

The right stretch tensor can be expressed as $\mathbf{C} = \mathbf{U}\mathbf{U}$, where $\mathbf{U}$ is the right stretch, or, since $\mathbf{U}$ is also symmetric, $\mathbf{C} = \mathbf{U}^{\mathrm{T}}\mathbf{U}$. One can see the similarity between this relation and the relation $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ so, using the same arguments as given above, one has for $W = W(\mathbf{C}(\mathbf{U}))$, (see 1.11.36)

$$\frac{\partial W}{\partial \mathbf{U}} = 2\mathbf{U}\frac{\partial W}{\partial \mathbf{C}} \tag{4.2.14}$$

Since $\mathbf{U}$ is symmetric,

$$\frac{\partial W}{\partial \mathbf{U}} = \left(\frac{\partial W}{\partial \mathbf{U}}\right)^{\mathrm{T}} = 2\left(\mathbf{U}\frac{\partial W}{\partial \mathbf{C}}\right)^{\mathrm{T}} = 2\left(\frac{\partial W}{\partial \mathbf{C}}\right)^{\mathrm{T}}\mathbf{U} = 2\frac{\partial W}{\partial \mathbf{C}}\mathbf{U} \tag{4.2.15}$$

showing that $\mathbf{U}$ and $\partial W / \partial \mathbf{C}$ are coaxial.

4.2.2 Objectivity of the Constitutive Equations

An observer transformation (see §2.8) results in $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$ and $\mathbf{C} = \mathbf{C}^*$, so $\partial W(\mathbf{C}^*)/\partial \mathbf{C}^* = \partial W(\mathbf{C})/\partial \mathbf{C}$, and, so, from 4.2.12,

$$\boldsymbol{\sigma}^* = 2J^{-1}\mathbf{Q}\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} \tag{4.2.16}$$

which is the objectivity requirement for a spatial tensor and so this constitutive law satisfies the requirement of material frame-indifference, when W is a function of $\mathbf{C}$. This evidently holds true also when W is a function of $\mathbf{E}$. It does not, however, hold true in general when W is a function of $\mathbf{F}$, as in the constitutive law $\mathbf{P} = \partial W(\mathbf{F})/\partial \mathbf{F}$. However, with W a function of $\mathbf{C}$, the constitutive equation 4.2.13 can be seen to be objective.

The objective constitutive equations in this section are indicated by a box around them.

4.2.3 Elasticity Tensors

The total differential of the PK2 stress can be written as

$$d\mathbf{S} = \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} : d\mathbf{C} = \left(2\frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}}\right) : \left(\frac{1}{2}d\mathbf{C}\right) \equiv \mathsf{C} : \frac{1}{2}d\mathbf{C} \tag{4.2.17}$$

where C is the fourth-order tensor

$$\mathsf{C} = 2\frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \mathbf{S}(\mathbf{E})}{\partial \mathbf{E}} \tag{4.2.18}$$

and is a measure of the change in stress which occurs due to a change in strain. It is called the **elasticity tensor** (in the material description) or the **tangent modulus**.

Since **S** and **E** are symmetric, C possesses the minor symmetries, $C_{ijkl} = C_{jikl} = C_{ijlk}$, and so has 36 independent components. However, if hyperelastic conditions hold, so that $\mathbf{S} = \partial W(\mathbf{E}) / \partial \mathbf{E} = 2\partial W(\mathbf{C}) / \partial \mathbf{C}$, then

$$\mathsf{C} = 4\frac{\partial^2 W(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}} = \frac{\partial^2 W(\mathbf{E})}{\partial \mathbf{E} \partial \mathbf{E}} \tag{4.2.19}$$

and so C possesses the major symmetries, $C_{ijkl} = C_{klij}$, and only only 21 independent components.

The rate form of the above equations is

$$\dot{\mathbf{S}} = \frac{d}{dt}\left(2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}\right) = 2\frac{\partial^2 W(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}} : \dot{\mathbf{C}} = \frac{\partial^2 W(\mathbf{E})}{\partial \mathbf{E} \partial \mathbf{E}} : \dot{\mathbf{E}} \tag{4.2.20}$$

or

$$\dot{\mathbf{S}} = \mathsf{C} : \tfrac{1}{2}\dot{\mathbf{C}} = \mathsf{C} : \dot{\mathbf{E}} \tag{4.2.21}$$

4.2.4 A Note on the Strain Energy Function

Some confusion can arise in expressions such as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}}, \qquad S_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{4.2.22}$$

As mentioned at the end of §1.11.5, one can either

- use the energy function $W^* = W^*(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{33})$, a function of 6 independent variables, in which case

$$S_{11} = \frac{\partial W^*}{\partial E_{11}}, \quad S_{12} = \frac{1}{2}\frac{\partial W^*}{\partial E_{12}}, \quad S_{13} = \frac{1}{2}\frac{\partial W^*}{\partial E_{13}}, \quad \ldots \tag{4.2.23}$$

- use the energy function $W = W(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33})$, a function of 9 variables, not all of them independent, in which case

$$S_{11} = \frac{\partial W}{\partial E_{11}}, \quad S_{12} = \frac{\partial W}{\partial E_{12}}, \quad \ldots \quad S_{21} = \frac{\partial W}{\partial E_{21}}, \quad \ldots \tag{4.2.24}$$

and the symmetry of W with respect to the strains, which must be assumed here, implies that the stress is also symmetric.

4.2.5 Hyperelasticity with Constraints

Consider a Hyperelastic material which is subject to the scalar constraint

$$\varphi(\mathbf{F}) = 0 \quad \text{or} \quad \dot{\varphi} = \frac{\partial \varphi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = 0 \qquad (4.2.25)$$

Without the constraint, one has

$$\dot{W} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} = \mathbf{P} : \dot{\mathbf{F}} \qquad (4.2.26)$$

The constitutive equation $\mathbf{P} = \partial W / \partial \mathbf{F}$ can be derived from 4.2.26 when $\dot{\mathbf{F}}$ is arbitrary. However, for the material with the constraint, the $\dot{\mathbf{F}}$ cannot be "cancelled" out from each side; one has the conditions

$$\left(\mathbf{P} - \frac{\partial W}{\partial \mathbf{F}} \right) : \dot{\mathbf{F}} = 0, \qquad \frac{\partial \phi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = 0 \qquad (4.2.27)$$

In general then,

$$\mathbf{P} - \frac{\partial W}{\partial \mathbf{F}} = \alpha \frac{\partial \phi}{\partial \mathbf{F}} \qquad (4.2.28)$$

where α is some arbitrary scalar. The stress is therefore given by

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} + \alpha \frac{\partial \phi}{\partial \mathbf{F}} \qquad (4.2.29)$$

and the rate of change of internal energy is

$$\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \alpha \frac{\partial \phi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = \dot{W} + \alpha \dot{\phi} \qquad (4.2.30)$$

The strain energy is

$$W(\mathbf{F}) + \alpha \phi(\mathbf{F}) \qquad (4.2.31)$$

The second term here is evidently zero and so does not contribute to the strain energy. However, the stress is the derivative of the strain energy with respect to a kinematic quantity, and the derivative of this last term, the second term in 4.2.29, is not be zero. The scalar α is, or determines the magnitude of, a **workless** reaction stress. It ensures that the constraint is satisfied; it is not set a value in the constitutive equation, rather, it is determined by considering particular problems, through equilibrium and boundary conditions.

For N constraints $\phi_i(\mathbf{F}) = 0,\ i = 1\ldots N$, 4.2.29 generalises to

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} + \sum_{i=1}^{N} \alpha_i \frac{\partial \phi_i}{\partial \mathbf{F}} \tag{4.2.32}$$

Incompressible Materials

An incompressible material is one whose volume remains constant throughout a motion, and so has the following constraint:

$$J = \det \mathbf{F} = 1 \quad \text{or} \quad \dot{J} = 0 \tag{4.2.33}$$

From 1.11.34, $\partial(\det \mathbf{A}) / \partial \mathbf{A} = (\det \mathbf{A})\mathbf{A}^{-\mathrm{T}}$, one has $dJ / d\mathbf{F} = J\mathbf{F}^{-\mathrm{T}}$, which equals $\mathbf{F}^{-\mathrm{T}}$ at $J = 1$. Thus the PK1 stresses can be written as

$$\boxed{\mathbf{P} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}} - p\mathbf{F}^{-\mathrm{T}}} \tag{4.2.34}$$

and the strain energy function is of the form

$$W(\mathbf{F}) - p(J(\mathbf{F}) - 1) \tag{4.2.35}$$

with $p = -\alpha$.

Using 3.5.9, $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$, one also has

$$\boxed{\mathbf{S} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} - p\mathbf{C}^{-1}} \tag{4.2.36}$$

Using 2.5.20 and 2.5.18b,

$$\dot{J} = J\mathrm{tr}\mathbf{d} = J\mathrm{tr}\left(\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}\mathbf{F}^{-1}\right) = J\mathrm{tr}\left(\mathbf{C}^{-1}\dot{\mathbf{E}}\right) = J\mathbf{C}^{-1} : \dot{\mathbf{E}} = \tfrac{1}{2} J\mathbf{C}^{-1} : \dot{\mathbf{C}} \tag{4.2.37}$$

and the stress power is

$$\left.\begin{cases} \mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} = \dfrac{\partial W(\mathbf{C})}{\partial \mathbf{C}} : \dot{\mathbf{C}} - \frac{1}{2} p\mathbf{C}^{-1} : \dot{\mathbf{C}} \\ \mathbf{P} : \dot{\mathbf{F}} = \dfrac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} - p\mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} \end{cases}\right. = \dot{W} - p\dot{J} \tag{4.2.38}$$

consistent with 4.2.30.

The strain energy of 4.2.35 is expressed in terms of $\mathbf{F}$. This can be re-expressed in terms of $\mathbf{C}$. Since $J = \det\mathbf{F} = \sqrt{\det\mathbf{C}} = \sqrt{\mathrm{III}_\mathbf{C}}$, the incompressibility constraint can be expressed as

$$\varphi(\mathbf{C}) = \mathrm{III}_\mathbf{C} - 1 = 0 \tag{4.2.39}$$

and the strain energy can be written as

$$W(\mathbf{C}) - \frac{1}{2} p(\mathrm{III}_\mathbf{C}(\mathbf{C}) - 1) \tag{4.2.40}$$

The factor of 1/2 has been included so that the p in 4.2.35 is the same as the p in 4.2.40, since, from 1.11.33,

$$\frac{\partial \mathrm{III}_\mathbf{C}}{\partial \mathbf{C}} = \mathbf{C}^{-1} \tag{4.2.41}$$

leading to the same expression for the PK2 stress as befoe, Eqn.4.2.36.

Similarly, in terms of the Cauchy stress, one has{▲Problem 1}

$$\boxed{\boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}\mathbf{F}^\mathrm{T} - p\mathbf{I}} \tag{4.2.42}$$

Because of its role in this equation, the scalar p is called the **hydrostatic pressure**.

Inextensible Constraint

Consider a material which is inextensible in a direction defined by a unit vector $\hat{\mathbf{A}}$ in the reference configuration. The constraint for this material is given by Eqn. 2.2.60,

$$\phi(\mathbf{F}) = \hat{\mathbf{A}}\mathbf{C}(\mathbf{F})\hat{\mathbf{A}} - 1 = 0 \tag{4.2.43}$$

Then, from 4.2.11,

$$\begin{aligned} \frac{\partial \phi}{\partial \mathbf{F}} &= 2\mathbf{F}\frac{\partial \phi}{\partial \mathbf{C}} \\ &= 2\mathbf{F}\hat{\mathbf{A}}\frac{\partial \mathbf{C}}{\partial \mathbf{C}}\hat{\mathbf{A}} \\ &= 2\mathbf{F}\hat{\mathbf{A}} \otimes \hat{\mathbf{A}} \end{aligned} \tag{4.2.44}$$

The stress is then

$$\boxed{\mathbf{P} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}} + 2\alpha\mathbf{F}\hat{\mathbf{A}} \otimes \hat{\mathbf{A}}} \tag{4.2.45}$$

Post-contracting with $\mathbf{F}^{\mathrm{T}}$ (with $J = 1$) then gives

$$\boxed{\boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} + 2\alpha \mathbf{F}\hat{\mathbf{A}} \otimes \mathbf{F}\hat{\mathbf{A}}} \tag{4.2.46}$$

For inextensibility in two directions,

$$\phi_1(\mathbf{F}) = \hat{\mathbf{A}}_1 \mathbf{C}(\mathbf{F})\hat{\mathbf{A}}_1 - 1 = 0, \qquad \phi_2(\mathbf{F}) = \hat{\mathbf{A}}_2 \mathbf{C}(\mathbf{F})\hat{\mathbf{A}}_2 - 1 = 0 \tag{4.2.47}$$

one has the stress

$$\boxed{\boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} + \alpha_1 \mathbf{F}\hat{\mathbf{A}}_1 \otimes \mathbf{F}\hat{\mathbf{A}}_1 + \alpha_2 \mathbf{F}\hat{\mathbf{A}}_2 \otimes \mathbf{F}\hat{\mathbf{A}}_2} \tag{4.2.48}$$

4.2.6 Problems

1. Use the equation

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} - p\mathbf{F}^{-\mathrm{T}}$$

to show that the constitutive equation for an incompressible hyperelastic material can be written in terms of the Cauchy stress as

$$\boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} - p\mathbf{I}$$