3.9 The Principles of Virtual Work and Power

The principle of virtual work was introduced and discussed in Part I, §5.5. As mentioned there, it is yet another re-statement of the work – energy principle, only it is couched in terms of virtual displacements, and the principle of virtual power to be introduced below is an equivalent statement based on **virtual velocities**.

On the one hand, the principle of virtual work/power can be regarded as the fundamental law of dynamics for a continuum, and from it can be derived the equations of motion. On the other hand, one can regard the principle of linear momentum as the fundamental law, derive the equations of motion, and hence derive the principle of virtual work.

3.9.1 Overview of The Principle of Virtual Work

Consider a material under the action of external forces: body forces **b** and tractions **t**. The body undergoes a displacement $\mathbf{u}(\mathbf{x})$ due to these forces and now occupies its current configuration, Fig. 3.9.1. The problem is to find this displacement function $\mathbf{u}$.

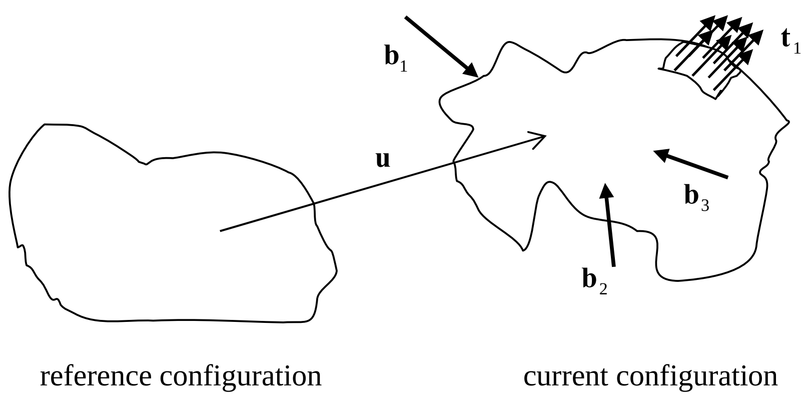

Figure 3.9.1: a material displacing to its current configuration under the action of body forces and surface forces

Imagine the material to undergo a small displacement $\delta\mathbf{u}$ from the *current configuration*, Fig. 3.9.2, $\delta\mathbf{u}$ not necessarily constant throughout the body; $\delta\mathbf{u}$ is a virtual displacement, meaning that it is an *imaginary* displacement, and in no way is it related to the applied external forces – *it does not actually occur physically*.

As each material particle moves through these virtual displacements, the external forces do virtual work δW. If the force $\mathbf{b}$ acts at position $\mathbf{x}$ and this point undergoes a virtual displacement $\delta u(\mathbf{x})$, the virtual work is $\delta W = \mathbf{b}\cdot\delta\mathbf{u}\,\Delta v$. Similarly for the surface tractions, and the total external virtual work is

$$\boxed{\delta W_{\text{ext}} = \int_v \mathbf{b}\cdot\delta\mathbf{u}\,dv + \int_s \mathbf{t}\cdot\delta\mathbf{u}\,ds = 0} \qquad \textbf{External Virtual Work} \qquad (3.9.1)$$

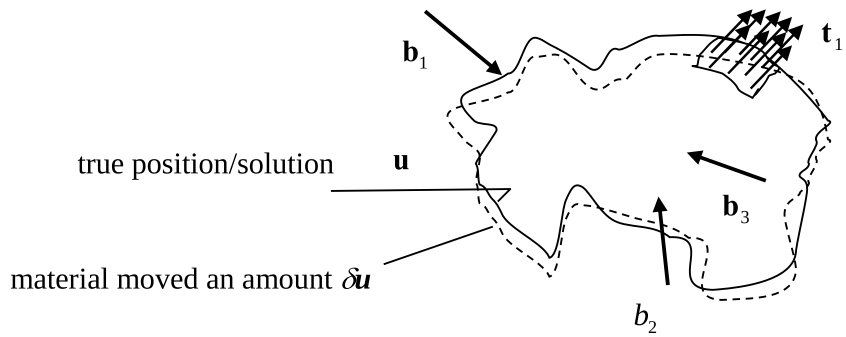

Figure 3.9.2: a virtual displacement field applied to a material in the current configuration

There is also an internal virtual work W_{int} due to the internal forces as they move through virtual displacements and a virtual kinetic energy δK. The principle of virtual work then says that

$$\delta W_{\text{ext}} + \delta W_{\text{int}} = \delta K \tag{3.9.2}$$

And this equation is then solved for the actual displacement **u**. Expressions for the internal virtual work and virtual kinetic energy will be derived below.

3.9.2 Derivation of The Principle of Virtual Work

As mentioned above, one can simply write down the principle of virtual work, regarding it as the fundamental principle of mechanics, and then from it derive the equations of motion. This will be done further below. To begin, though, the starting point will be the equations of motion, and from it will be derived the principle of virtual work.

Kinematically and Statically Admissible Fields

A **kinematically admissible displacement field** is defined to be one which satisfies the displacement boundary condition 3.7.7c, $\mathbf{u} = \overline{\mathbf{u}}$ on s_u (see Part I, §5.5.1). Such a displacement field would induce some stress field within the body, but this resulting stress field might not satisfy the equations of motion 3.7.7a. In other words, it might not be the actual displacement field, but it does not violate the boundary conditions.

A **statically admissible stress field** is one which satisfies the equations of motion 3.7.7a and the traction boundary conditions 3.7.7b, $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \overline{\mathbf{t}}$, on s_σ. Again, it might not be the actual stress field, since it is not specified how this stress field should be related to the actual displacement field.

Derivation from the Equations of Motion (Spatial Form)

Let $\boldsymbol{\sigma}$ be a statically admissible stress field corresponding to a kinematically admissible displacement field $\mathbf{u}$, so $\boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}$ on s_σ, $\mathbf{u} = \bar{\mathbf{u}}$ on s_u and $\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}}$. Multiplying the equations of motion by $\mathbf{u}$ and integrating leads to

$$\int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u} dv = \int_v (\mathrm{div}\boldsymbol{\sigma} + \mathbf{b}) \cdot \mathbf{u} dv \tag{3.9.3}$$

Using the identity 1.14.16b, $\mathrm{div}(\mathbf{A}\mathbf{v}) = \mathbf{v} \cdot \mathrm{div}\mathbf{A}^{\mathrm{T}} + \mathrm{tr}(\mathbf{A}\mathrm{grad}\mathbf{v})$, 1.10.10e, $\mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \mathbf{A} : \mathbf{B}$, and the symmetry of stress,

$$\int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u} dv = \int_v \{\mathrm{div}(\boldsymbol{\sigma}\mathbf{u}) - \boldsymbol{\sigma} : \mathrm{grad}(\mathbf{u}) + \mathbf{b} \cdot \mathbf{u}\} dv \tag{3.9.4}$$

and the divergence theorem 1.14.22c and Cauchy's law lead to

$$\int_s \mathbf{t} \cdot \mathbf{u} ds + \int_v \mathbf{b} \cdot \mathbf{u} dv = \int_v \boldsymbol{\sigma} : \mathrm{grad}\mathbf{u} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u} dv \tag{3.9.5}$$

Splitting the surface integral into one over $s_{\mathbf{u}}$ and one over $s_{\boldsymbol{\sigma}}$ gives

$$\int_{s_{\mathbf{u}}} \mathbf{t} \cdot \bar{\mathbf{u}} ds + \int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \mathbf{u} ds + \int_v \mathbf{b} \cdot \mathbf{u} dv = \int_v \boldsymbol{\sigma} : \mathrm{grad}\mathbf{u} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u} dv \tag{3.9.6}$$

Next, consider a second kinematically admissible displacement field $\mathbf{u}^*$, so $\mathbf{u}^* = \bar{\mathbf{u}}$ on s_u, which is completely arbitrary, in the sense that it is unrelated to either $\boldsymbol{\sigma}$ or $\mathbf{u}$. This time multiplying $\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}}$ across by $\mathbf{u}^*$, and following the same procedure, one arrives at

$$\int_{s_{\mathbf{u}}} \mathbf{t} \cdot \bar{\mathbf{u}} ds + \int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \mathbf{u}^* ds + \int_v \mathbf{b} \cdot \mathbf{u}^* dv = \int_v \boldsymbol{\sigma} : \mathrm{grad}\mathbf{u}^* dv + \int_v \rho\ddot{\mathbf{u}} \cdot \mathbf{u}^* dv \tag{3.9.7}$$

Let $\delta\mathbf{u} = \mathbf{u}^* - \mathbf{u}$, so the difference between $\mathbf{u}^*$ and $\mathbf{u}$ is infinitesimal, then subtracting 3.9.6 from 3.9.7 gives the principle of virtual work,

$$\boxed{\int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \delta\mathbf{u} ds + \int_v \mathbf{b} \cdot \delta\mathbf{u} dv = \int_v \boldsymbol{\sigma} : \mathrm{grad}(\delta\mathbf{u}) dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv}$$

Principle of Virtual Work (spatial form) (3.9.8)

Note that since $\mathbf{u}, \mathbf{u}^*$, are kinematically admissible, $\delta\mathbf{u} = \mathbf{u}^* - \mathbf{u} = \mathbf{o}$ on s_u.

If one considers the $\mathbf{u}$ in 3.9.8 to be the actual displacement of the body, then $\delta\mathbf{u}$ can be considered to be a virtual displacement from the current configuration, Fig. 3.9.2. Again, it is emphasized that this virtual displacement leaves the stress, body force and applied traction unchanged.

One also has the transformed initial conditions: from 3.7.7d-e,

$$\int_v \mathbf{u}(\mathbf{x},\mathbf{t})_{t=0} \cdot \delta\mathbf{u}dv = \int_v \mathbf{u}_0(\mathbf{x}) \cdot \delta\mathbf{u}dv$$
$$\int_v \dot{\mathbf{u}}(\mathbf{x},\mathbf{t})_{t=0} \cdot \delta\mathbf{u}dv = \int_v \dot{\mathbf{u}}_0(\mathbf{x}) \cdot \delta\mathbf{u}dv \qquad (3.9.9)$$

Eqns. 3.9.8 and 3.9.9 together constitute the **weak form** of the initial BVP 3.7.7.

The principle of virtual work can be grouped into three separate terms: the external virtual work:

$$\boxed{\delta W_{\text{ext}} = \int_{s_\sigma} \bar{\mathbf{t}} \cdot \delta\mathbf{u}ds + \int_v \mathbf{b} \cdot \delta\mathbf{u}dv} \quad \textbf{External Virtual Work} \qquad (3.9.10)$$

the internal virtual work,

$$\boxed{\delta W_{\text{int}} = -\int_v \boldsymbol{\sigma} : \text{grad}(\delta\mathbf{u})dv} \quad \textbf{Internal Virtual Work} \qquad (3.9.11)$$

and the virtual kinetic energy,

$$\boxed{\delta K = \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}dv} \quad \textbf{Virtual Kinetic Energy} \qquad (3.9.12)$$

corresponding to the statement 3.9.2.

Derivation from the Equations of Motion (Material Form)

The derivation in the spatial form follows exactly the same lines as for the spatial form.

This time, let $\mathbf{P}$ be a statically admissible stress field corresponding to a kinematically admissible displacement field $\mathbf{U}$, so $\mathbf{PN} = \overline{\mathbf{T}}$ on $S_\mathbf{P}$, $\mathbf{U} = \overline{\mathbf{U}}$ on $S_\mathbf{u}$ and $\text{div}\mathbf{P} + \mathbf{B} = \rho_0\ddot{\mathbf{U}}$. This time one arrives at

$$\int_{S_\mathbf{P}} \overline{\mathbf{T}} \cdot \delta\mathbf{U}dS + \int_V \mathbf{B} \cdot \delta\mathbf{U}dV = \int_V \mathbf{P} : \text{Grad}(\delta\mathbf{U})dV + \int_V \rho_0\ddot{\mathbf{U}} \cdot \delta\mathbf{U}dV \qquad (3.9.13)$$

Again, one can consider $\mathbf{U}$ to be the actual displacement of the body, so that $\delta\mathbf{U}$ represents a virtual displacement from the current configuration. With

$$\mathbf{U} = \mathbf{x} - \mathbf{X}$$
$$\delta\mathbf{U} = \delta\mathbf{x} - \delta\mathbf{X} = \delta\mathbf{x} \qquad (3.9.14)$$

the virtual work equation can be expressed in terms of the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$,

$$\boxed{\int_{S_\mathbf{P}} \overline{\mathbf{T}} \cdot \delta\boldsymbol{\chi} dS + \int_V \mathbf{B} \cdot \delta\boldsymbol{\chi} dV = \int_V \mathbf{P} : \mathrm{Grad}(\delta\boldsymbol{\chi}) dV + \int_V \rho_0 \ddot{\mathbf{U}} \cdot \delta\boldsymbol{\chi} dV}$$

Principle of Virtual Work (material form) (3.9.15)

3.9.3 Principle of Virtual Work in terms of Strain Tensors

The principle of virtual work, in particular the internal virtual work term, can be expressed in terms of strain tensors.

Spatial Form

Using the commutative property of the variation 2.13.2, the term $\mathrm{grad}(\delta\mathbf{u})$ in the internal virtual work expression 3.9.8 can be written as

$$\begin{aligned} \mathrm{grad}(\delta\mathbf{u}) &= \frac{1}{2}\left(\mathrm{grad}(\delta\mathbf{u}) + (\mathrm{grad}(\delta\mathbf{u}))^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathrm{grad}(\delta\mathbf{u}) - (\mathrm{grad}(\delta\mathbf{u}))^{\mathrm{T}}\right) \\ &= \delta\frac{1}{2}\left(\mathrm{grad}\mathbf{u} + (\mathrm{grad}\mathbf{u})^{\mathrm{T}}\right) + \delta\frac{1}{2}\left(\mathrm{grad}\mathbf{u} - (\mathrm{grad}\mathbf{u})^{\mathrm{T}}\right) \\ &= \delta\boldsymbol{\varepsilon} + \delta\boldsymbol{\Omega} \end{aligned} \qquad (3.9.16)$$

where $\boldsymbol{\varepsilon}$ is the (symmetric) small strain tensor and $\boldsymbol{\Omega}$ is the (skew-symmetric) small rotation tensor, Eqn 2.7.2. Using the fact that the double contraction of a symmetric tensor ($\boldsymbol{\sigma}$) and a skew-symmetric one ($\boldsymbol{\Omega}$) is zero, 1.10.31c, one has

$$\delta W_{\mathrm{int}} = -\int_v \boldsymbol{\sigma} : \mathrm{grad}(\delta\mathbf{u}) dv = -\int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dv \qquad (3.9.17)$$

Thus the stresses do internal virtual work along the virtual strains $\delta\boldsymbol{\varepsilon}$. One has

$$\int_{s_\sigma} \bar{\mathbf{t}} \cdot \delta\mathbf{u} ds + \int_v \mathbf{b} \cdot \delta\mathbf{u} dv = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv \qquad (3.9.18)$$

Note that, although the small strain has been introduced here, this formulation is not restricted to small-strain theory. It is only the virtual strains that must be infinitesimal – there is no restriction on the magnitude of the *actual* strains.

From 2.13.15, the Lie-variation of the Euler-Almansi strain **e** is $\delta_{\mathrm{L}}\mathbf{e} = \delta\boldsymbol{\varepsilon}$, so the internal; virtual work can be expressed as

$$\delta W_{\mathrm{int}} = -\int_v \boldsymbol{\sigma} : \delta_{\mathrm{L}}\mathbf{e}\, dv \qquad (3.9.19)$$

Material Form

From Eqn. 3.9.15 and Eqn. 2.13.9,

$$\delta W_{\text{int}} = -\int_V \mathbf{P} : \delta\mathbf{F} dV \tag{3.9.20}$$

so

$$\int_{S_\mathbf{P}} \overline{\mathbf{T}} \cdot \delta\boldsymbol{\chi} dS + \int_V \mathbf{B} \cdot \delta\boldsymbol{\chi} dV = \int_V \mathbf{P} : \delta\mathbf{F} dV + \int_V \rho_0 \ddot{\mathbf{U}} \cdot \delta\boldsymbol{\chi} dV \tag{3.9.21}$$

Derivation of the Material Form directly from the Spatial Form

To transform the spatial form of the virtual work equation into the material form, first note that, with 3.9.14b,

$$\boldsymbol{\sigma} : \text{grad}(\delta\mathbf{u}) = \boldsymbol{\sigma} : \text{grad}(\delta\mathbf{x}) \tag{3.9.22}$$

Then, using 2.2.8b, $\text{grad}\mathbf{v} = \text{Grad}\mathbf{V}\,\mathbf{F}^{-1}$, 2.13.9, $\delta\mathbf{F} = \text{Grad}(\delta\mathbf{u})$, 1.10.3h, $\mathbf{A} : (\mathbf{BC}) = (\mathbf{AC}^{\text{T}}) : \mathbf{B}$, and 3.5.10, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\text{T}}$,

$$\begin{aligned} \boldsymbol{\sigma} : \text{grad}(\delta\mathbf{x}) &= \boldsymbol{\sigma} : \left(\text{Grad}(\delta\mathbf{x})\mathbf{F}^{-1}\right) \\ &= \boldsymbol{\sigma} : \left(\delta\mathbf{F}\mathbf{F}^{-1}\right) \\ &= \left(\boldsymbol{\sigma}\mathbf{F}^{-\text{T}}\right) : \delta\mathbf{F} \\ &= \left(J^{-1}\mathbf{P}\right) : \delta\mathbf{F} \end{aligned} \tag{3.9.23}$$

which converts 3.9.17 into 3.9.120.

Also, again comparing 3.9.17 and 3.9.20, using the trace properties 1.10.10, and Eqns. 3.5.9 and 2.13.11b,

$$\mathbf{P} : \delta\mathbf{F} = J\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = J\text{tr}(\boldsymbol{\sigma}\delta\boldsymbol{\varepsilon}) = J\text{tr}\left(\mathbf{F}^{-1}\boldsymbol{\sigma}\delta\boldsymbol{\varepsilon}\mathbf{F}\right) = J\left(\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\text{T}}\right) : \left(\mathbf{F}^{\text{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}\right) = \mathbf{S} : \delta\mathbf{E} \tag{3.9.24}$$

and so the internal work can also be expressed as an integral of $\mathbf{S} : \delta\mathbf{E}$ over the reference volume.

The Internal Virtual Work and Work Conjugate Tensors

The expressions for stress power 3.8.15, 3.8.29, and internal virtual work are very similar. For the material description, the time derivatives in the former are simply replaced with the variation to get the latter:

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} \quad \to \quad \mathbf{P} : \delta\mathbf{F} = \mathbf{S} : \delta\mathbf{E} \tag{3.9.25}$$

For spatial tensors, the rate of strain tensor, e.g. $\mathbf{d}$, is replaced with a Lie variation 2.13.14. For example, $J\boldsymbol{\sigma} : \mathbf{d} = J\boldsymbol{\sigma} : \text{L}_\text{v}^b\mathbf{e}$ (see 2.12.41-42) becomes:

$$J\boldsymbol{\sigma} : \mathbf{d} = J\boldsymbol{\sigma} : \text{L}_\text{v}^b\mathbf{e} \quad \to \quad J\boldsymbol{\sigma} : \delta_\text{L}\mathbf{e} \tag{3.9.26}$$

3.9.4 Derivation of the Strong Form from the Weak Form

Just as the strong form (equations of motion and boundary conditions) was converted into the weak form (principle of virtual work), the weak form can be converted back into the strong form. For example,

$$\begin{aligned}\int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv &= \int_v \boldsymbol{\sigma} : \delta \mathrm{grad}\mathbf{u} dv + \int_v \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u} dv \\ &= \int_v \left\{ \mathrm{div}\left(\boldsymbol{\sigma} \cdot \delta\mathbf{u}\right) - \left(\mathrm{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}}\right) \cdot \delta\mathbf{u} \right\} dv \\ &= \int_s \mathbf{t} \cdot \delta\mathbf{u} ds - \int_v \left\{ \left(\mathrm{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}}\right) \cdot \delta\mathbf{u} \right\} dv \\ &= \int_{s_{\boldsymbol{\sigma}}} \mathbf{t} \cdot \delta\mathbf{u} ds - \int_v \left\{ \left(\mathrm{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}}\right) \cdot \delta\mathbf{u} \right\} dv\end{aligned} \tag{3.9.27}$$

and the last line follows from the fact that $\delta\mathbf{u} = \mathbf{o}$ on $s_{\mathbf{u}}$. Thus the weak form now reads

$$\int_{s_{\boldsymbol{\sigma}}} \left(\mathbf{t} - \bar{\mathbf{t}}\right) \cdot \delta\mathbf{u} ds - \int_v \left(\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} - \rho\ddot{\mathbf{u}}\right) \cdot \delta\mathbf{u} dv = 0 \tag{3.9.28}$$

and, since $\delta\mathbf{u}$ is arbitrary, one finds that the expressions in the parentheses are zero, and so 3.7.7 is recovered.

3.9.5 Conservative Systems

Thus far, no assumption has been made about the nature of the internal forces acting in the material. Indeed, the principle of virtual work applies to all types of materials.

Now, however, attention is restricted to the special case where the system is conservative, in the sense that the work done by the external loads and the internal forces can be written in terms of potential energy functions[1]. Further, for brevity, assume also that the material is in static equilibrium, i.e. the kinetic energy term is zero.

In other words, it is assumed that the internal virtual work term can be expressed in the form of a virtual potential energy function:

$$\int_v \boldsymbol{\sigma} : \mathrm{grad}(\delta\mathbf{u}) dv = \int_v \delta U dv \tag{3.9.29}$$

Here, U is considered to be a function of $\mathbf{u}$, and the variation is to be understood as in Eqn. 2.13.5, $\delta U\left(\mathbf{u}, \delta\mathbf{u}\right) \equiv \partial_{\mathbf{u}} U[\delta\mathbf{u}]$.

If the loads can be regarded as functions of $\mathbf{u}$ only then, since they are conservative, they may be written as the gradient of a scalar potential:

[1] The external loads being conservative would exclude, for example, cases of frictional loading

$$\mathbf{b} = -\frac{\partial U_b}{\partial \mathbf{u}}, \quad \bar{\mathbf{t}} = -\frac{\partial U_t}{\partial \mathbf{u}} \tag{3.9.30}$$

Then, with

$$\delta U_b = \frac{\partial U_b}{\partial \mathbf{u}} \cdot \delta \mathbf{u}, \quad \delta U_t = \frac{\partial U_t}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \tag{3.9.31}$$

and using the commutative property 2.13.3 of the variational operator, one arrives at

$$\delta \left\{ \int_v U dv + \int_{s_\sigma} U_t ds + \int_v U_b dv \right\} = \delta \overline{U}(\mathbf{u}) = 0 \tag{3.9.32}$$

The quantity inside the brackets is the total potential energy of the system. This statement is the **principle of stationary potential energy**: the value of the quantity inside the parentheses, i.e. $\overline{U}(\mathbf{u})$, is stationary at the true solution $\mathbf{u}$.

Eqn. 3.9.32 is an example of a **Variational Principle**, that is, a principle expressed in the form of a variation of a functional. Note that the principle of virtual work in the form 3.9.8 is not a variational principle, since it is not expressed as the variation of one functional.

Body Forces

Body forces can usually be expressed in the form 3.9.30. For example, with gravity loading, $\mathbf{b} = \rho \mathbf{g}$, where $\mathbf{g}$ is the constant acceleration due to gravity. Then $U_b = -\rho \mathbf{g} \cdot \mathbf{u}$ ($\delta \mathbf{b} = \delta \mathbf{g} = \mathbf{o}$ and $\mathbf{b} \cdot \delta \mathbf{u} = \delta(\mathbf{b} \cdot \mathbf{u})$, so $\int \mathbf{b} \cdot \delta \mathbf{u} dv = \delta \int \mathbf{b} \cdot \mathbf{u} dv$).

Material Form

In the material form, one again has a stationary principle if one can write $\mathbf{B} = -\partial U_B(\mathbf{U}) / \partial \mathbf{U}$, $\overline{\mathbf{T}} = -\partial U_T(\mathbf{U}) / \partial \mathbf{U}$ (or, equivalently, replacing $\mathbf{U}$ with the motion $\boldsymbol{\chi}$). In the case of dead loading, §3.7.1, $\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X})$ is independent of the motion so (similar to the case of gravity loading above) $U_T = -\overline{\mathbf{T}} \cdot \mathbf{u}$ with $\delta \overline{\mathbf{T}} = \mathbf{o}$ and $\int \overline{\mathbf{T}} \cdot \delta \boldsymbol{\chi} dV = \delta \int \overline{\mathbf{T}} \cdot \boldsymbol{\chi} dV$.

Deformation Dependent Traction

In many practical cases, the traction will depend on not only the motion, but also the strain. In that case, one can write

$$\int_{s_\sigma} \bar{\mathbf{t}} \cdot \delta \mathbf{u} ds = \int_s \bar{\boldsymbol{\sigma}} \mathbf{n} \cdot \delta \mathbf{u} ds = \int_s \mathbf{n} \bar{\boldsymbol{\sigma}} \delta \mathbf{u} ds = \int_v \operatorname{div}(\bar{\boldsymbol{\sigma}} \delta \mathbf{u}) dv = \int_v (\operatorname{div} \bar{\boldsymbol{\sigma}} \cdot \delta \mathbf{u} + \bar{\boldsymbol{\sigma}} : \operatorname{grad} \delta \mathbf{u}) dv$$

One might be able to then introduce a scalar function ϕ such that

$$\delta\phi(\mathbf{u},\boldsymbol{\varepsilon}) = \frac{\partial \phi}{\partial \mathbf{u}} \cdot \delta\mathbf{u} + \frac{\partial \phi}{\partial \boldsymbol{\varepsilon}} : \delta\boldsymbol{\varepsilon} \quad \text{with} \quad \frac{\partial \phi}{\partial \mathbf{u}} = \operatorname{div}\overline{\boldsymbol{\sigma}}, \ \frac{\partial \phi}{\partial \boldsymbol{\varepsilon}} = \overline{\boldsymbol{\sigma}} \tag{3.9.33}$$

In the material form, one would have $\overline{\mathbf{T}} = \overline{\mathbf{P}}\mathbf{N}$ with

$$\int_{S_\mathbf{P}} \overline{\mathbf{T}} \cdot \delta\boldsymbol{\chi} dS = \int_V \left(\operatorname{Div} \overline{\mathbf{P}} \cdot \delta\boldsymbol{\chi} + \overline{\mathbf{P}} : \mathbf{F}\right) dV$$

and then one might be able to introduce a scalar function $\phi(\boldsymbol{\chi}, \mathbf{F})$ such that

$$\delta\phi = \frac{\partial \phi}{\partial \boldsymbol{\chi}} \cdot \delta\boldsymbol{\chi} + \frac{\partial \phi}{\partial \mathbf{F}} : \delta\mathbf{F} \quad \text{with} \quad \frac{\partial \phi}{\partial \boldsymbol{\chi}} = \operatorname{Div}\overline{\mathbf{P}}, \ \frac{\partial \phi}{\partial \mathbf{F}} = \overline{\mathbf{P}} \tag{3.9.34}$$

For example, considering again the fluid pressure example of §3.7.1, one can let $\phi = -pJ$ so that, using 1.15.7, $\partial\phi / \partial\mathbf{F} = -pJ\mathbf{F}^{-\mathrm{T}}$. Then $\overline{\mathbf{P}} = -p\mathbf{F}^{-\mathrm{T}} = \partial\phi / \partial\mathbf{F}\big|_{J=1}$, $\operatorname{Div}\overline{\mathbf{P}} = \rho g \mathbf{E}_2$ and $\int \overline{\mathbf{T}} \cdot \delta\boldsymbol{\chi} dS = -\delta \int pJdV$.

3.9.6 The Principle of Virtual Power

The principle of virtual power is similar to the principle of virtual work, the only difference between them being that a virtual velocity $\delta\mathbf{v}$ is used in the former rather than a virtual displacement. To derive the virtual power equation, multiply the equations of motion by the virtual velocity function, and integrate over the current configuration, giving

$$\begin{aligned}
\int_v \rho \frac{d\mathbf{v}}{dt} \cdot \delta\mathbf{v} dv = \int_v \left(\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}\right) \cdot \delta\mathbf{v} dv &= \int_v \left\{ \operatorname{div}(\boldsymbol{\sigma}\delta\mathbf{v}) - \boldsymbol{\sigma} : \frac{\partial(\delta\mathbf{v})}{\partial\mathbf{x}} + \mathbf{b} \cdot \delta\mathbf{v} \right\} dv \\
&= \int_v \left\{ \operatorname{div}(\boldsymbol{\sigma}\delta\mathbf{v}) - \boldsymbol{\sigma} : \delta\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + \mathbf{b} \cdot \delta\mathbf{v} \right\} dv \\
&= \int_s \mathbf{t} \cdot \delta\mathbf{v} ds - \int_v \boldsymbol{\sigma} : \delta\mathbf{d} dv + \int_v \mathbf{b} \cdot \delta\mathbf{v} dv
\end{aligned} \tag{3.9.35}$$

These equations are identical to the mechanical balance equations 3.8.16, except that the actual velocity is replaced with a virtual velocity. The term $-\int_v \boldsymbol{\sigma} : \delta\mathbf{d} dv$ is called the **internal virtual power**.

Note that here, unlike the virtual displacement function in the work equation, the virtual velocity does not have to be infinitesimal. This can be seen more clearly if one derives this equation directly from the virtual work equation. If the infinitesimal virtual displacement $\delta\mathbf{u}$ occurs over an infinitesimal time interval δt, the virtual velocity is the finite quantity $\delta\mathbf{u} / \delta t$, which here is labelled $\delta\mathbf{v}$. The virtual power equation can thus be obtained by dividing the virtual work equation through by δt.

Again, supposing that the velocities are specified over that part of the surface s_v and tractions over $s_{\boldsymbol{\sigma}}$, the principle of virtual power can be written for the case of a kinematically admissible virtual velocity field:

$$\boxed{\int_{s_{\boldsymbol{\sigma}}} \bar{\mathbf{t}} \cdot \delta \mathbf{v} ds + \int_v \mathbf{b} \cdot \delta \mathbf{v} dv = \int_v \boldsymbol{\sigma} : \delta \mathbf{d} dv + \int_v \rho \frac{d\mathbf{v}}{dt} \cdot \delta \mathbf{v} dv} \quad \textbf{Principle of Virtual Power} \quad (3.9.36)$$

In words, the principle of virtual power states that *at any time t, the total virtual power of the external, internal and inertia forces is zero in any admissible virtual state of motion*.

3.9.7 Linearisation of the Internal Virtual Work

In order to solve the virtual work equations in anything but the most simple cases, one must apply some approximate numerical method. This will usually involve linearising the non-linear virtual work equations. To this end, the internal virtual work term will be linearised in what follows.

Material Description

In the material description, one has

$$\delta W_{\text{int}} = \int_V \mathbf{S}(\mathbf{E}(\mathbf{u})) : \delta \mathbf{E}(\mathbf{u}) dV \qquad (3.9.37)$$

in which the Green-Lagrange strain is considered to be a function of the displacement, Eqn. 2.2.46, and the PK2 stress is a function of the Green-Lagrange strain; the precise functional dependence of **S** on **E** will depend on the material under study (see Part IV).

The linearisation of the variation of a function is given by (see §2.13.2)

$$\mathrm{L}\ \delta W_{\text{int}}(\mathbf{u}, \Delta\mathbf{u}) = \delta W_{\text{int}}(\mathbf{u}) + \Delta\delta W_{\text{int}}(\mathbf{u}, \Delta\mathbf{u}) \qquad (3.9.38)$$

where

$$\begin{aligned}
\Delta\delta W_{\text{int}}(\mathbf{u}, \Delta\mathbf{u}) &= \partial_{\mathbf{u}} \delta W_{\text{int}}[\Delta\mathbf{u}] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta W_{\text{int}}(\mathbf{u} + \varepsilon\Delta\mathbf{u}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_V \mathbf{S}(\mathbf{E}(\mathbf{u} + \varepsilon\Delta\mathbf{u})) : \delta\mathbf{E}(\mathbf{u} + \varepsilon\Delta\mathbf{u}) dV \\
&= \int_V \left\{ \mathbf{S}(\mathbf{E}(\mathbf{u})) : \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta\mathbf{E}(\mathbf{u} + \varepsilon\Delta\mathbf{u}) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{S}(\mathbf{E}(\mathbf{u} + \varepsilon\Delta\mathbf{u})) : \delta\mathbf{E}(\mathbf{u}) \right\} dV \\
&= \int_V \{ \mathbf{S}(\mathbf{E}(\mathbf{u})) : \Delta\delta\mathbf{E}(\mathbf{u}, \Delta\mathbf{u}) + \Delta\mathbf{S}(\mathbf{E}(\mathbf{u}, \Delta\mathbf{u})) : \delta\mathbf{E}(\mathbf{u}) \} dV
\end{aligned} \qquad (3.9.39)$$

The linearization of the variation of the Green-Lagrange strain is given by 2.13.24, $\Delta\delta\mathbf{E} = \mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\,\mathrm{Grad}\delta\mathbf{u}\right)$. With the PK2 stress symmetric, one has, with 1.10.3h, 1.10.31c,

$$\begin{aligned}\mathbf{S}(\mathbf{E}(\mathbf{u})) : \Delta\delta\mathbf{E}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{S}(\mathbf{E}(\mathbf{u})) : \mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\,\mathrm{Grad}\delta\mathbf{u}\right) \\ &= \mathbf{S}(\mathbf{E}(\mathbf{u})) : (\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\,\mathrm{Grad}\delta\mathbf{u} \\ &= \mathrm{Grad}\delta\mathbf{u} : (\mathrm{Grad}\Delta\mathbf{u})\mathbf{S}(\mathbf{E}(\mathbf{u}))\end{aligned} \tag{3.9.40}$$

For the second term in 3.9.39, from 2.13.22, the variation of $\mathbf{E}$ is

$$\begin{aligned}\delta\mathbf{E} &= \tfrac{1}{2}\left[(\mathrm{Grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right] \\ &= \tfrac{1}{2}\left[(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u})^{\mathrm{T}} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right] \\ &= \mathrm{sym}\left(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)\end{aligned} \tag{3.9.41}$$

What remains is the calculation of the linearisation of the PK2 stress. One has using the chain rule,

$$\begin{aligned}\Delta\mathbf{S}(\mathbf{E}(\mathbf{u},\Delta\mathbf{u})) &= \partial_{\mathbf{u}}\mathbf{S}[\Delta\mathbf{u}] \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{S}(\mathbf{E}(\mathbf{u}+\varepsilon\Delta\mathbf{u})) \\ &= \frac{\partial\mathbf{S}(\mathbf{E})}{\partial\mathbf{E}} : \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{E}((\mathbf{u}+\varepsilon\Delta\mathbf{u})) \\ &= \frac{\partial\mathbf{S}(\mathbf{E})}{\partial\mathbf{E}} : \Delta\mathbf{E}(\mathbf{u},\Delta\mathbf{u})\end{aligned} \tag{3.9.42}$$

Denote the fourth order tensor $\partial\mathbf{S}(\mathbf{E})/\partial\mathbf{E}$ by $\mathbf{C}$ and assume that it has the minor symmetries1.12.10. Then (see 3.9.41), with

$$\Delta\mathbf{E} = \mathrm{sym}\left(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u}\right) \tag{3.9.43}$$

the linear increments in 3.9.38 become

$$\boxed{\begin{aligned}\Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_V \Big\{\mathrm{Grad}\delta\mathbf{u} : (\mathrm{Grad}\Delta\mathbf{u})\mathbf{S}(\mathbf{E}(\mathbf{u})) \\ &\qquad + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u} : \mathbf{C} : \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u}\Big\}dV \\ \Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_V \left\{\frac{\partial\delta u_i}{\partial X_b}\frac{\partial\Delta u_i}{\partial X_d}S_{bd} + F_{ka}\frac{\partial\delta u_k}{\partial X_b}C_{abcd}F_{jc}\frac{\partial\Delta u_j}{\partial X_d}\right\}dV \\ &= \int_V \frac{\partial\delta u_i}{\partial X_b}\left\{\delta_{ij}S_{bd} + F_{ia}F_{jc}C_{abcd}\right\}\frac{\partial\Delta u_j}{\partial X_d}\,dV\end{aligned}} \tag{3.9.44}$$

The first term is due to the current stress and is called the (**initial**) **stress contribution**. The second term depends on the material properties and is called the **material contribution**. Solution formulations based on 3.9.44 are called **total Lagrangian**.

Spatial Description

The spatial description can be obtained by pushing forward the material description. First note that the linearization of the Kirchhoff stress is, from 3.5.13,

$$\begin{aligned} \mathrm{L}\ \boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) &= \chi_*(\mathrm{L}\ \mathbf{S}(\mathbf{u},\Delta\mathbf{u}))^{\#} \\ &= \chi_*(\mathbf{S}(\mathbf{u},\Delta\mathbf{u}))^{\#} + \chi_*(\Delta\ \mathbf{S}(\mathbf{u},\Delta\mathbf{u}))^{\#} \\ &= \boldsymbol{\tau}(\mathbf{u}) + \mathbf{F}\Delta\mathbf{S}(\mathbf{u},\Delta\mathbf{u})\mathbf{F}^{\mathrm{T}} \end{aligned} \tag{3.9.45}$$

so that, as in the derivation of the material term in 3.9.44, and using 2.4.8,

$$\begin{aligned} \Delta\boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{F}\left(\mathbf{C} : \mathbf{F}^{\mathrm{T}}\mathrm{grad}\Delta\mathbf{u}\mathbf{F}\right)\mathbf{F}^{\mathrm{T}} \\ \Delta\boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) &= F_{ia}F_{jb}F_{kc}F_{ld}C_{abcd}\frac{\partial \Delta u_k}{\partial x_l} \end{aligned} \tag{3.9.46}$$

Define the fourth-order spatial tensor $\mathbf{c}$ through

$$c_{ijkl} = J^{-1}F_{ia}F_{jb}F_{kc}F_{ld}C_{abcd} \tag{3.9.47}$$

so that

$$\Delta\boldsymbol{\tau}(\mathbf{u},\Delta\mathbf{u}) = J\,\mathbf{c} : \mathrm{grad}\Delta\mathbf{u} \tag{3.9.48}$$

Then, from 3.9.39,

$$\boxed{\begin{aligned} \Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_V \left\{\chi_*(\mathbf{S})^{\#} : \chi_*(\Delta\delta\mathbf{E})^{b} + \chi_*(\Delta\mathbf{S})^{\#} : \chi_*(\delta\mathbf{E})^{b}\right\}dV \\ &= \int_V \left\{\boldsymbol{\tau} : \mathrm{sym}\left((\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{grad}\delta\mathbf{u}\right) + J\,\mathbf{c} : \mathrm{grad}\Delta\mathbf{u} : \mathrm{sym}(\mathrm{grad}\delta\mathbf{u})\right\}dV \\ &= \int_v \left\{\boldsymbol{\sigma} : (\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{grad}\delta\mathbf{u} + \mathbf{c} : \mathrm{grad}\Delta\mathbf{u} : \mathrm{grad}\delta\mathbf{u}\right\}dv \\ &= \int_v \left\{\mathrm{grad}\delta\mathbf{u} : \mathrm{grad}\Delta\mathbf{u}\,\boldsymbol{\sigma} + \mathrm{grad}\delta\mathbf{u} : \mathbf{c} : \mathrm{grad}\Delta\mathbf{u}\right\}dv \\[2ex] \Delta\delta W_{\mathrm{int}}(\mathbf{u},\Delta\mathbf{u}) &= \int_v \left\{\frac{\partial \delta u_a}{\partial x_b}\frac{\partial \Delta u_a}{\partial x_d}\sigma_{bd} + \frac{\partial \delta u_a}{\partial x_b}c_{abcd}\frac{\partial \Delta u_c}{\partial x_d}\right\}dv \\ &= \int_v \frac{\partial \delta u_a}{\partial x_b}\left\{\delta_{ac}\sigma_{bd} + c_{abcd}\right\}\frac{\partial \Delta u_c}{\partial x_d}\,dv \end{aligned}}$$

(3.9.49)

Solution formulations based on 3.9.49 are called **updated-Lagrangian**.