# 3.8 Balance of Mechanical Energy

## 3.8.1 The Balance of Mechanical Energy

First, from Part I, Chapter 5, recall work and kinetic energy are related through

$$W_{\text{ext}} + W_{\text{int}} = \Delta K \tag{3.8.1}$$

where $W_{\text{ext}}$ is the work of the external forces and $W_{\text{int}}$ is the work of the internal forces. The *rate* form is

$$P_{\text{ext}} + P_{\text{int}} = \dot{K} \tag{3.8.2}$$

where the external and internal *powers* and rate of change of kinetic energy are

$$P_{\text{ext}} = \frac{d}{dt} W_{\text{ext}}, \quad P_{\text{int}} = \frac{d}{dt} W_{\text{int}}, \quad \dot{K} = \frac{d}{dt} \Delta K \tag{3.8.3}$$

This expresses the *mechanical* energy balance for a material. Eqn. 3.8.2 is equivalent to the equations of motion (see below).

The total external force acting on the material is given by 3.2.6:

$$\mathbf{F}_{\text{ext}} = \int_s \mathbf{t}\, ds + \int_v \mathbf{b}\, dv \tag{3.8.4}$$

The increment in work done $dW$ when an element subjected to a body force (per unit volume) $\mathbf{b}$ undergoes a displacement $d\mathbf{u}$ is $\mathbf{b} \cdot d\mathbf{u}\, dv$. The rate of working is $dP = \mathbf{b} \cdot (d\mathbf{u} / dt) dv$. Thus, and similarly for the traction, the power of the external forces is

$$P_{\text{ext}} = \int_s \mathbf{t} \cdot \mathbf{v}\, ds + \int_v \mathbf{b} \cdot \mathbf{v}\, dv \tag{3.8.5}$$

where $\mathbf{v}$ is the velocity. Also, the total kinetic energy of the matter in the volume is

$$K = \int_v \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv \tag{3.8.6}$$

Using Reynold's transport theorem,

$$\frac{d}{dt} K = \int_v \tfrac{1}{2} \rho \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dv = \int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv \tag{3.8.7}$$

Thus the expression 3.8.2 becomes

$$\int_s \mathbf{t}\cdot\mathbf{v}\,ds + \int_v \mathbf{b}\cdot\mathbf{v}\,dv + P_{\text{int}} = \int_v \rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\,dv \tag{3.8.8}$$

power of surface forces — power of body forces — power of internal forces — rate of change of kinetic energy

It can be seen that some of the power exerted by the external forces alters the kinetic energy of the material and the remainder changes its internal energy state.

### Conservative Force System

In the special case where the internal forces are conservative, that is, no energy is dissipated as heat, but all energy is stored as internal energy, one can express the power of the internal forces in terms of a potential function *u* (see Part I, §5.1), and rewrite this equation as

$$\int_s \mathbf{t}\cdot\mathbf{v}\,ds + \int_v \mathbf{b}\cdot\mathbf{v}\,dv = \int_v \rho\frac{du}{dt}\,dv + \int_v \rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\,dv \tag{3.8.9}$$

rate of change of internal energy

Here, the rate of change of the internal energy has been written in the form

$$\frac{d}{dt}U = \frac{d}{dt}\int_v \rho u\,dv = \int_v \rho\frac{du}{dt}dv \tag{3.8.10}$$

where $u$ is the internal energy per unit mass, or the **specific internal energy**.

## 3.8.2 The Stress Power

To express the power of the internal forces $P_{\text{int}}$ in terms of stresses and strain-rates, first re-write the rate of change of kinetic energy using the equations of motion,

$$\frac{d}{dt}K = \int_v \rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\,dv = \int_v \mathbf{v}\cdot\left(\text{div}\boldsymbol{\sigma} + \mathbf{b}\right)dv \tag{3.8.11}$$

Also, using the product rule of differentiation,

$$\mathbf{v}\cdot\text{div}\boldsymbol{\sigma} = \text{div}(\mathbf{v}\boldsymbol{\sigma}) - \boldsymbol{\sigma}:\mathbf{l}, \qquad v_i\frac{\partial\sigma_{ij}}{\partial x_j} = \frac{\partial(v_i\sigma_{ij})}{\partial x_j} - \sigma_{ij}\frac{\partial v_i}{\partial x_j} \tag{3.8.12}$$

where $\mathbf{l}$ is the spatial velocity gradient, $l_{ij} = \partial v_i / \partial x_j$. Decomposing $\mathbf{l}$ into its symmetric part $\mathbf{d}$, the rate of deformation, and its antisymmetric part $\mathbf{w}$, the spin tensor, gives

$$\boldsymbol{\sigma} : \mathbf{l} = \boldsymbol{\sigma} : \mathbf{d} + \boldsymbol{\sigma} : \mathbf{w} = \boldsymbol{\sigma} : \mathbf{d}, \qquad \sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} \frac{1}{2}\left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{3.8.13}$$

since the double contraction of any symmetric tensor ($\boldsymbol{\sigma}$) with any skew-symmetric tensor ($\mathbf{w}$) is zero, 1.10.31c. Also, using Cauchy's law and the divergence theorem 1.14.21,

$$\begin{gathered} \int_s \mathbf{t} \cdot \mathbf{v}\, ds = \int_s \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{v}\, ds = \int_s (\mathbf{v}\boldsymbol{\sigma}) \cdot \mathbf{n}\, ds = \int_v \operatorname{div}(\mathbf{v}\boldsymbol{\sigma}) dv \\ \int_s t_i v_i\, ds = \int_s \sigma_{ik} n_k v_i\, ds = \int_v \frac{\partial(\sigma_{ik} v_i)}{\partial x_k} dv \end{gathered} \tag{3.8.14}$$

Thus, finally, from Eqn. 3.8.8,

$$P_{\text{int}} = -\int_v \boldsymbol{\sigma} : \mathbf{d}\, dv, \qquad P_{\text{int}} = -\int_v \sigma_{ij} \left\{ \frac{1}{2}\left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} dv \qquad \textbf{Stress Power} \tag{3.8.15}$$

The term $\boldsymbol{\sigma} : \mathbf{d}$ is called the **stress power**; the stress power is the (negative of the) rate of working of the internal forces, per unit volume. The complete equation for the conservation of mechanical energy is then

$$\int_s \mathbf{t} \cdot \mathbf{v}\, ds + \int_v \mathbf{b} \cdot \mathbf{v}\, dv = \int_v \boldsymbol{\sigma} : \mathbf{d}\, dv + \int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv \qquad \textbf{Mechanical Energy Balance} \tag{3.8.16}$$

The stress power is that part of the externally supplied power which is not converted into kinetic energy; it is converted into heat and a change in internal energy.

Note that, as with the law of conservation of mechanical energy for a particle, this equation does not express a separate law of continuum mechanics; it is merely a re-arrangement of the equations of motion (see below), which themselves follows from the principle of linear momentum (Newtons second law).

## Conservative Force System

If the internal forces are conservative, one has

$$\int_v \boldsymbol{\sigma} : \mathbf{d}\, dv = \frac{d}{dt} U = \int_v \rho \frac{du}{dt} dv \tag{3.8.17}$$

or, in local form,

$$\boxed{\boldsymbol{\sigma} : \mathbf{d} = \rho \frac{du}{dt}} \quad \textbf{Mechanical Energy Balance (Conservative System)} \quad (3.8.18)$$

This is the local form of the energy equation for the case of a purely mechanical conservative process.

## 3.8.3 Derivation from the Equations of Motion

As mentioned, the conservation of mechanical energy equation can be derived directly from the equations of motion. The derivation is similar to that used above (where the mechanical energy equations were used to derive an expression for the stress power using the equations of motion). One has, multiplying the equations of motion by **v** and integrating,

$$\begin{aligned}\int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv = \int_v \mathbf{v} \cdot (\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}) dv &= \int_v \{\operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \mathbf{l} + \mathbf{v} \cdot \mathbf{b}\} dv \\ &= \int_v \{\operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \mathbf{d} + \mathbf{v} \cdot \mathbf{b}\} dv \\ &= \int_v -\boldsymbol{\sigma} : \mathbf{d} dv + \int_s \mathbf{t} \cdot \mathbf{v} ds + \int_v \mathbf{v} \cdot \mathbf{b} dv\end{aligned} \quad (3.8.19)$$

## 3.8.4 Stress Power and the Continuum Element

In the above, the stress power was derived using a global (integral) form of the equations. The stress power can also be deduced by considering a differential mass element. For example, consider such an element whose boundary particles are moving with velocity **v** and whose boundary is subjected to stresses $\boldsymbol{\sigma}$, Fig. 3.8.1.

Consider first the components of force and velocity acting in the $x_1$ direction. The external forces act on the six sides. On three of them (the ones that can be seen in the illustration) the stress and velocity act in the same direction, so the power is positive; on the other three they act in opposite directions, so there the power is negative.

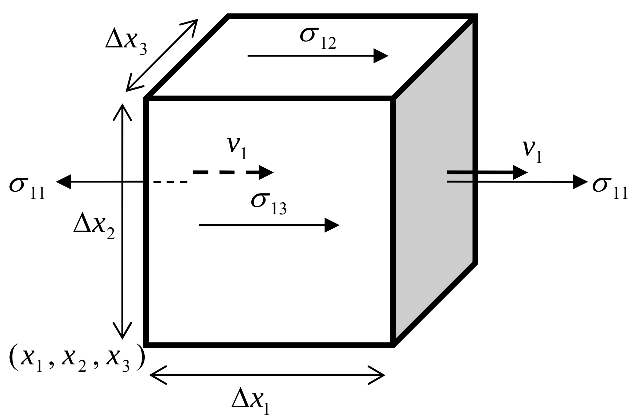

**Figure 3.8.1: A differential mass element subjected to stresses**

As usual (see §1.6.6), the element is assumed to be small enough so that the product of stress and velocity varies linearly over the element, so that the average of this product over an element face can be taken to be representative of the power of the surface forces on that element. The power of the external surface forces acting on the three faces to the front is then

$$\begin{aligned} P_{\text{surf}} = \Delta x_2 \Delta x_3 (\sigma_{11} v_1)_{x_1+\Delta x_1, x_2+\frac{1}{2}\Delta x_2, x_3+\frac{1}{2}\Delta x_3} + \Delta x_1 \Delta x_3 (\sigma_{12} v_1)_{x_1+\frac{1}{2}\Delta x_1, x_2+\Delta x_2, x_3+\frac{1}{2}\Delta x_3} \\ + \Delta x_1 \Delta x_2 (\sigma_{13} v_1)_{x_1+\frac{1}{2}\Delta x_1, x_2+\frac{1}{2}\Delta x_2, x_3+\Delta x_3} \end{aligned} \quad (3.8.20)$$

Using a Taylor's series expansion, and neglecting higher order terms, then leads to

$$\begin{aligned} P_{\text{surf}} \approx \Delta x_2 \Delta x_3 &\left\{ (\sigma_{11} v_1)_{x_1,x_2,x_3} + \Delta x_1 \frac{\partial(\sigma_{11} v_1)}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial(\sigma_{11} v_1)}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial(\sigma_{11} v_1)}{\partial x_3} \right\} \\ + \Delta x_1 \Delta x_3 &\left\{ (\sigma_{12} v_1)_{x_1,x_2,x_3} + \tfrac{1}{2}\Delta x_1 \frac{\partial(\sigma_{12} v_1)}{\partial x_1} + \Delta x_2 \frac{\partial(\sigma_{12} v_1)}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial(\sigma_{12} v_1)}{\partial x_3} \right\} \\ + \Delta x_1 \Delta x_2 &\left\{ (\sigma_{13} v_1)_{x_1,x_2,x_3} + \tfrac{1}{2}\Delta x_1 \frac{\partial(\sigma_{13} v_1)}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial(\sigma_{13} v_1)}{\partial x_2} + \Delta x_3 \frac{\partial(\sigma_{13} v_1)}{\partial x_3} \right\} \end{aligned} \quad (3.8.21)$$

The net power *per unit volume* (subtracting the power of the stresses on the other three surfaces and dividing through by the volume) is then

$$P_{\text{surf}} = \frac{\partial(\sigma_{11} v_1)}{\partial x_1} + \frac{\partial(\sigma_{12} v_1)}{\partial x_2} + \frac{\partial(\sigma_{13} v_1)}{\partial x_3} = \frac{\partial(\sigma_{1j} v_1)}{\partial x_j} \quad (3.8.22)$$

Assume the body force **b** to act at the centre of the element. Neglecting higher order terms which vanish as the element size is allowed to shrink towards zero, the power of the body force in the $x_1$ direction, per unit volume, is simply $b_1 v_1$.

The total power of the external forces is then (including the other two components of stress and velocity), using the equations of motion,

$$\begin{aligned} P_{\text{ext}} &= \frac{\partial(\sigma_{ij} v_i)}{\partial x_j} + b_i v_i \\ &= \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_j} v_i + b_i v_i \\ &= \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \left\{ -b_i + \rho \frac{dv_i}{dt} \right\} v_i + b_i v_i \\ &= \sigma_{ij} \frac{1}{2}\left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \rho \frac{1}{2} \frac{d(v_i v_i)}{dt} \end{aligned} \qquad \begin{aligned} P_{\text{ext}} &= \operatorname{div}(\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{v}) + \mathbf{b} \cdot \mathbf{v} \\ &= \boldsymbol{\sigma} : \mathbf{l} + \operatorname{div}\boldsymbol{\sigma} \cdot \mathbf{v} + \mathbf{b} \cdot \mathbf{v} \\ &= \boldsymbol{\sigma} : \mathbf{l} + \left\{ -\mathbf{b} + \rho \frac{d\mathbf{v}}{dt} \right\} \cdot \mathbf{v} + \mathbf{b} \cdot \mathbf{v} \\ &= \boldsymbol{\sigma} : \mathbf{d} + \rho \frac{1}{2} \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt} \end{aligned} \quad (3.8.23)$$

which again equals the stress power term plus the change in kinetic energy.

The power of the internal forces is $-\boldsymbol{\sigma}:\mathbf{d}$, a result of the forces acting *inside* the differential element, *reacting* to the applied forces $\boldsymbol{\sigma}$ and **b**.

### 3.8.5 The Balance of Mechanical Energy (Material form)

The material form of the power of the external forces is written as a function of the PK1 traction **T** and the reference body force **B**, 3.6.7, and the kinetic energy as a function of the velocity $\mathbf{V}(\mathbf{X})$:

$$\int_S \mathbf{T}\cdot\mathbf{V}\,dS + \int_V \mathbf{B}\cdot\mathbf{V}\,dV + P_{\text{int}} = \int_V \rho_0 \mathbf{V}\cdot\frac{d\mathbf{V}}{dt}dV \tag{3.8.24}$$

Next, using the identities 2.5.4, $\dot{\mathbf{F}} = \mathbf{lF}$ and 1.10.3h, $\mathbf{A}:(\mathbf{BC}) = \left(\mathbf{AC}^{\mathrm{T}}\right):\mathbf{B}$, gives

$$\boldsymbol{\sigma}:\mathbf{d} = \boldsymbol{\sigma}:\mathbf{l} - \boldsymbol{\sigma}:\mathbf{w} = \boldsymbol{\sigma}:\mathbf{l} = \boldsymbol{\sigma}:\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right) = \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right):\dot{\mathbf{F}}, \tag{3.8.25}$$

and so

$$\begin{aligned}\int_v \boldsymbol{\sigma}:\mathbf{d}\,dv &= \int_v \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right):\dot{\mathbf{F}}\,dv = \int_V \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right):\dot{\mathbf{F}}\,J\,dV \\ &= \int_V \mathbf{P}:\dot{\mathbf{F}}\,dV\end{aligned} \tag{3.8.26}$$

and

$$\boxed{\int_S \mathbf{T}\cdot\mathbf{V}\,dS + \int_V \mathbf{B}\cdot\mathbf{V}\,dV = \int_V \mathbf{P}:\dot{\mathbf{F}}\,dV + \int_V \rho_0 \mathbf{V}\cdot\frac{d\mathbf{V}}{dt}dV} \quad \textbf{Mechanical Energy Balance (Material Form)} \tag{3.8.27}$$

For a conservative system, this can be written in terms of the internal energy

$$\int_S \mathbf{T}\cdot\mathbf{V}\,dS + \int_V \mathbf{B}\cdot\mathbf{V}\,dV = \int_V \rho_0 \frac{du}{dt}dV + \int_V \rho_0 \mathbf{V}\cdot\frac{d\mathbf{V}}{dt}dV \tag{3.8.28}$$

### 3.8.6 Work Conjugate Variables

Since the stress power is the double contraction of the Cauchy stress and rate-of-deformation, one says that the Cauchy stress and rate of deformation are **work conjugate** (or **power conjugate** or **energy conjugate**). Similarly, from 3.8.26, the PK1 stress **P** is power conjugate to $\dot{\mathbf{F}}$. It can also be shown that the PK2 stress **S** is power conjugate to the rate of Euler-Lagrange strain, $\dot{\mathbf{E}}$ (and hence also the right Cauchy-Green strain) {▲Problem 1} :

$$J\boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S} : \tfrac{1}{2}\dot{\mathbf{C}} \tag{3.8.29}$$

Note that, for conservative systems, these quantities represent the rate of change of internal energy per unit *reference* volume.

Using the polar decomposition and the relation $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$,

$$\begin{aligned}\dot{\mathbf{F}} &= \dot{\mathbf{R}}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}} \\ &= \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}} \\ &= \boldsymbol{\Omega}_{\mathbf{R}}\mathbf{F} + \mathbf{R}\dot{\mathbf{U}}\end{aligned} \tag{3.8.30}$$

where $\boldsymbol{\Omega}_{\mathbf{R}}$ is the angular velocity tensor 2.6.1. Then, using 1.11.3h, 1.10.31c, and the definitions 3.5.8, 3.5.12 and 3.5.18,

$$\begin{aligned}\mathbf{P} : \dot{\mathbf{F}} &= \mathbf{P} : \boldsymbol{\Omega}_{\mathbf{R}}\mathbf{F} + \mathbf{P} : \mathbf{R}\dot{\mathbf{U}} \\ &= \mathbf{P}\mathbf{F}^{\mathrm{T}} : \boldsymbol{\Omega}_{\mathbf{R}} + \mathbf{R}^{\mathrm{T}}\mathbf{P} : \dot{\mathbf{U}} \\ &= \boldsymbol{\tau} : \boldsymbol{\Omega}_{\mathbf{R}} + \mathbf{R}^{\mathrm{T}}\mathbf{P} : \dot{\mathbf{U}} \\ &= \mathbf{T}_{\mathrm{B}} : \dot{\mathbf{U}}\end{aligned} \tag{3.8.31}$$

so that the Biot stress is power conjugate to the right stretch tensor. Since $\mathbf{U}$ is symmetric, $\mathbf{P} : \dot{\mathbf{F}} = \mathrm{sym}\mathbf{T}_{\mathrm{B}} : \dot{\mathbf{U}}$. Also, the Biot stress is conjugate to the Biot strain tensor $\overline{\mathbf{B}} = \mathbf{U} - \mathbf{I}$ introduced in §2.2.5.

From 3.5.14 and 1.10.3h,

$$\boldsymbol{\sigma} : \mathbf{d} = \mathbf{R}\hat{\boldsymbol{\sigma}}\mathbf{R}^{\mathrm{T}} : \mathbf{d} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{d}} \tag{3.8.32}$$

so that the corotational stress is power conjugate to the **rotated deformation rate**, defined by

$$\hat{\mathbf{d}} = \mathbf{R}^{\mathrm{T}}\mathbf{d}\mathbf{R} \tag{3.8.33}$$

### Pull Back and Push Forward

From 2.12.12-13, the double contraction of two tensors can be expressed as push-forwards and pull-backs of those tensors. For example, the stress power (per unit reference volume) in the material description is $\mathbf{S} : \dot{\mathbf{E}}$. Then, using 3.5.13, 2.12.9a and 2.5.18b, $\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F}$,

$$\mathbf{S} : \dot{\mathbf{E}} = \chi_*(\mathbf{S})^{\#} : \chi_*(\dot{\mathbf{E}})^{b} = \boldsymbol{\tau} : \mathbf{d} = J\boldsymbol{\sigma} : \mathbf{d} \tag{3.8.34}$$

This means that the material and spatial descriptions of the internal power can be transformed into each other using push-forward and pull-back operations.

Similarly, pulling back the corotational stress and rotated deformation rate to the intermediate configuration of Fig. 2.10.8, using 2.12.13, 2.12.27,

$$\boldsymbol{\sigma} : \mathbf{d} = \chi_*^{-1}(\boldsymbol{\sigma})^{\#}{}_{\mathbf{R}(\mathbf{g})} : \chi_*^{-1}(\mathbf{d})^{b}{}_{\mathbf{R}(\mathbf{g})} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{d}} \qquad (3.8.35)$$

The stress power in terms of spatial tensors can also be expressed as a derivative of a tensor, using the Lie derivative. From 2.12.42, the Lie derivative of the Euler-Almansi strain is the rate of deformation and hence (note that there is no universal function whose derivative is $\mathbf{d}$), so

$$J\boldsymbol{\sigma} : \mathbf{d} = J\boldsymbol{\sigma} : \mathrm{L}_{\mathrm{v}}^{b}\mathbf{e} \qquad (3.8.36)$$

## 3.8.7 Problems

1. Show that the rate of internal energy per unit reference volume $J\boldsymbol{\sigma} : \mathbf{d}$ is equivalent to $\mathbf{S} : \dot{\mathbf{E}}$ (without using push-forwards/pull-backs).