3.7 Boundary Conditions and The Boundary Value Problem

In order to solve a mechanics problem, one must specify certain conditions around the boundary of the material under consideration. Such **boundary conditions** will be discussed here, together with the resulting **boundary value problem** (**BVP**). (see Part I, 3.5.1, for a discussion of stress boundary conditions.)

3.7.1 Boundary Conditions

There are two types of boundary condition, those on displacement and those on traction. Denote the body in the reference condition by B_0 and in the current configuration by B. Denote the boundary of the body in the reference configuration by S and in the current configuration by s, Fig. 3.7.1.

Displacement Boundary Conditions

The position of particles may be specified over some portion of the boundary in the current configuration. That is, $\mathbf{x} = \chi(\mathbf{X})$ is specified to be $\bar{\mathbf{x}}$ say, over some portion $s_{\mathbf{u}}$ of s, Fig. 3.7.1, which corresponds to the portion $S_{\mathbf{u}}$ of S. With $\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{X}(\mathbf{x})$, or $\mathbf{U}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}$, this can be expressed as

$$
\begin{aligned}
\mathbf{u}(\mathbf{x}) &= \bar{\mathbf{u}}(\mathbf{x}), & \mathbf{x} \in s_{\mathbf{u}} \\
\mathbf{U}(\mathbf{X}) &= \overline{\mathbf{U}}(\mathbf{X}), & \mathbf{X} \in S_{\mathbf{u}}
\end{aligned}
\tag{3.7.1}
$$

These are called **displacement boundary conditions**. The most commonly encountered displacement boundary condition is where some portion of the boundary is fixed, in which case $\bar{\mathbf{u}}(\mathbf{x}) = \mathbf{o}$.
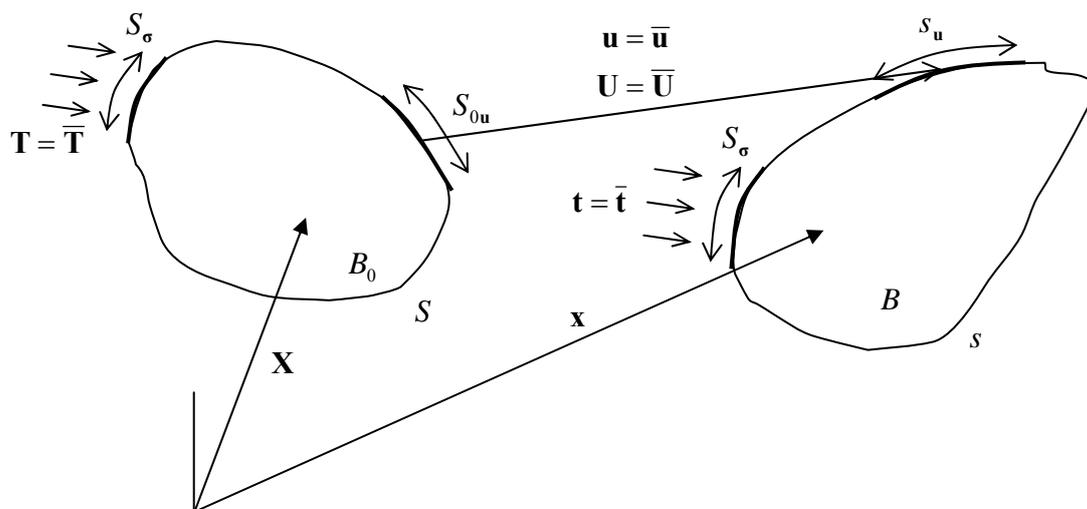

Figure 3.7.1: Boundary conditions

Traction Boundary Conditions

Traction $\mathbf{t} = \bar{\mathbf{t}}$ can be specified over a portion s_σ of the boundary, Fig. 3.7.1. These traction boundary conditions are related to the PK1 traction $\mathbf{T} = \overline{\mathbf{T}}$ over the corresponding surface S_σ in the reference configuration, through Eqns. 3.5.1-4,

$$\mathbf{T}dS = \mathbf{P}\mathbf{N}dS = \mathbf{t}ds = \boldsymbol{\sigma}\mathbf{n}ds \qquad (3.7.2)$$

One usually knows the position of the boundary S and the normal $\mathbf{N}(\mathbf{X})$ in the reference configuration. As deformation proceeds, the PK1 traction develops according to $\overline{\mathbf{T}} = \mathbf{P}\mathbf{N}$ with, from 3.5.8, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$. The PK1 stress will in general depend on the motion $\mathbf{x}$ and the deformation gradient $\mathbf{F}$, so the traction boundary condition can be expressed in the general form

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X}, \mathbf{x}, \mathbf{F}) \qquad (3.7.3)$$

Example: Fluid Pressure

Consider the case of fluid pressure p around the boundary, $\bar{\mathbf{t}} = -p\mathbf{n}$, Fig. 3.7.2. The Cauchy traction $\bar{\mathbf{t}}$ depends through the normal $\mathbf{n}$ on the new position and geometry of the surface s_σ. Also, $\overline{\mathbf{T}} = -pJ\mathbf{F}^{-T}\mathbf{N}$, which is of the general form 3.7.3.
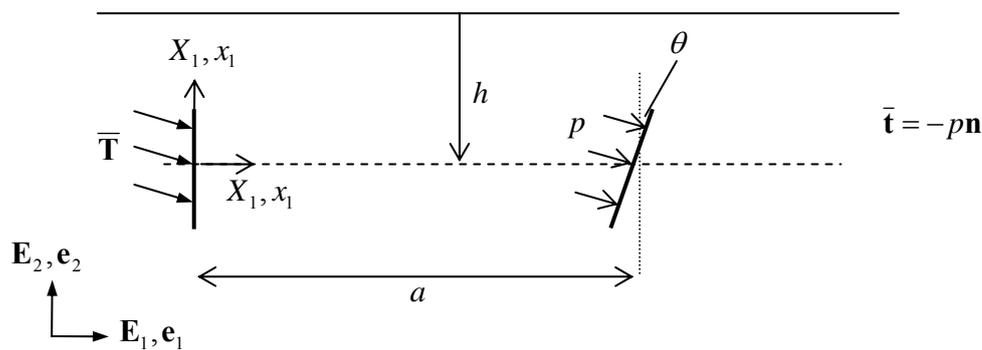

Figure 3.7.2: Fluid pressure on deforming material

Consider a material under water with part of its surface deforming as shown in Fig. 3.7.2. Referring to the figure, $\mathbf{N} = -\mathbf{E}_1$, $\mathbf{n} = -\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, $\boldsymbol{\sigma} = -p\mathbf{I}$, $p = \rho g(h - x_2)$ and

$$\begin{aligned} x_1 &= X_1 + a + X_2 \tan\theta \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned}, \quad \mathbf{F} = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \det\mathbf{F} = 1$$

The traction vectors and PK1 stress are

$$\bar{\mathbf{t}} = -\rho g \left(h - x_2 \right) \begin{bmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad \overline{\mathbf{T}} = -\rho g \left(h - X_2 \right) \begin{bmatrix} -1 \\ \tan\theta \\ 0 \end{bmatrix}, \quad \mathbf{P} = -\rho g \left(h - x_2 \right) \begin{bmatrix} 1 & 0 & 0 \\ -\tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with (note that $dS / ds = \cos\theta$) $\left| \bar{\mathbf{t}} \right| = p$ and $\left| \overline{\mathbf{T}} \right| = p / \cos\theta$. The traction vectors clearly depend on both position, and the deformation through θ. In this example, $\mathrm{grad}\,\mathbf{u} = \mathbf{F} - \mathbf{I} = \mathrm{Grad}\,\mathbf{U} = \mathbf{I} - \mathbf{F}^{-1} = \tan\theta\,\mathbf{e}_1 \otimes \mathbf{e}_2$ and

$$\theta \left(\mathrm{grad}\,\mathbf{u} \right) = \arctan \| \mathrm{grad}\,\mathbf{u} \| = \arctan \sqrt{\mathrm{grad}\,\mathbf{u} : \mathrm{grad}\,\mathbf{u}}$$

∎

Dead Loading

A special case of loading is that of **dead loading**, where

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X}) \tag{3.7.4}$$

Here, the PK1 stress on the boundary does not change with the deformation and an initially normal traction will not remain so as deformation proceeds.

For example, if one considers again the geometry of Fig. 3.7.2, this time take

$$\overline{\mathbf{T}}(\mathbf{X}) = \mathbf{P}\mathbf{N} = -p\mathbf{N} = \rho g \left(h - X_2 \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}(\mathbf{X}) = -\rho g \left(h - X_2 \right) \mathbf{I}$$

Then

$$\bar{\mathbf{t}}(\mathbf{x}, \theta) = \cos\theta\,\rho g \left(h - x_2 \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\sigma}(\mathbf{x}, \theta) = -\rho g \left(h - x_2 \right) \begin{bmatrix} 1 & 0 & 0 \\ \tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.7.2　The Boundary Value Problem

The equations of motion 3.6.2, 3.6.9, are a set of three differential equations. In the solution of any problem, one would have to supplement these equations with others, for example a constitutive equation expressing a relationship between the stress and the kinematic variables (see Part IV). This constitutive relation will typically relate the stress to the strains, or rates of strain, for example $\boldsymbol{\sigma} = f(\mathbf{e}, \mathbf{d})$. Suppose then that the stresses are known in terms of the strains and hence the displacements $\mathbf{u}$. The equations of motion are then a set of three second order differential equations in the three unknowns u_i (assuming that the body force $\mathbf{b}$ is a prescribed function of the problem). They need to be subjected to certain boundary and initial conditions.

Assume that the boundary conditions are such that the displacements are specified over that part of the surface $s_\mathbf{u}$ and tractions are specified over that part s_σ, with the total surface $s = s_\mathbf{u} + s_\sigma$, with $s_\mathbf{u} \cap s_\sigma = 0$ [1]. Thus

$$\begin{aligned} \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, && \text{on } s_\sigma \\ \mathbf{u} = \bar{\mathbf{u}}, && \text{on } s_u \end{aligned}$$

Boundary Conditions (3.7.5)

where the overbar signifies quantities which are prescribed. Initial conditions are also required for the displacement and velocity, so that

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}), && \text{at } t = 0 \\ \dot{\mathbf{u}}(\mathbf{x},t) = \dot{\mathbf{u}}_0(\mathbf{x}), && \text{at } t = 0 \end{aligned}$$

Initial Conditions (3.7.6)

and it is usually taken that $\mathbf{x} = \mathbf{X}$ at $t = 0$. Comparing 3.7.5 and 3.7.6, one also requires that $\mathbf{u}_0 = \bar{\mathbf{u}}$, $\dot{\mathbf{u}}_0 = \dot{\bar{\mathbf{u}}}$ over $s_\mathbf{u}$, so that the boundary and initial conditions are compatible.

These equations together, the differential equations of motion and the boundary and initial conditions, are called the **strong form** of the initial boundary value problem (BVP):

$$\boxed{\begin{aligned} \operatorname{div}\boldsymbol{\sigma} + \mathbf{b} &= \rho\dot{\mathbf{v}} = \rho\ddot{\mathbf{u}} \\ \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, && \text{on } s_\sigma \\ \mathbf{u} = \bar{\mathbf{u}}, && \text{on } s_u \\ \mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}), && \text{at } t = 0 \\ \dot{\mathbf{u}}(\mathbf{x},t) = \dot{\mathbf{u}}_0(\mathbf{x}), && \text{at } t = 0 \end{aligned}}$$

Strong form of the Initial BVP (3.7.7)

When the problem is quasi-static, so the accelerations can be neglected, the equations of motion reduce to the equations of equilibrium 3.6.3. In that case one does not need initial conditions and one has a boundary value problem involving 3.7.5 only.

It is only in certain special cases and in certain simple problems that an exact solution can be obtained to these equations. An alternative solution strategy is to convert these equations into what is known as the **weak form**. The weak form, which is in the form of integrals rather than differential equations, can then be solved approximately using a numerical technique, for example the Finite Element Method[2]. The weak form is discussed in §3.9.

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[1] It is possible to specify both traction and displacement over the same portion of the boundary, but not the same components. For example, if one specified $\mathbf{t} = t_1\mathbf{e}_1$ on a boundary, one could also specify $\mathbf{u} = u_2\mathbf{e}_2$, but not $\mathbf{u} = u_1\mathbf{e}_1$. In that case, one could imagine the boundary to consist of two separate boundaries, one with conditions with respect to $\mathbf{e}_1$ and one with respect to $\mathbf{e}_2$, and still write $s_\mathbf{u} \cap s_\sigma = 0$.

[2] Further, it is often easier to prove results regarding the uniqueness and stability of solutions to the problem when it is cast in the weak form

In the material form, the boundary conditions are

$$\mathbf{T} = \mathbf{PN} = \overline{\mathbf{T}}, \quad \text{on } S_\sigma$$
$$\mathbf{U} = \overline{\mathbf{U}}, \quad \text{on } S_u$$

Boundary Conditions $\qquad$ (3.7.8)

and the initial conditions are

$$\mathbf{U}(\mathbf{X},\mathbf{t}) = \mathbf{U}_0(\mathbf{X}), \quad \text{at } t = 0$$
$$\dot{\mathbf{U}}(\mathbf{X},\mathbf{t}) = \dot{\mathbf{U}}_0(\mathbf{X}), \quad \text{at } t = 0$$

Initial Conditions $\qquad$ (3.7.9)

and the initial vale problem is

$$\boxed{\begin{aligned}
\text{Div}\mathbf{P} + \mathbf{B} &= \rho_0 \dot{\mathbf{V}} = \rho\ddot{\mathbf{U}} \\
\mathbf{T} = \mathbf{PN} &= \overline{\mathbf{T}}, \quad \text{on } S_\sigma \\
\mathbf{U} &= \overline{\mathbf{U}}, \quad \text{on } S_u \\
\mathbf{U}(\mathbf{X},\mathbf{t}) &= \mathbf{U}_0(\mathbf{X}), \quad \text{at } t = 0 \\
\dot{\mathbf{U}}(\mathbf{X},\mathbf{t}) &= \dot{\mathbf{U}}_0(\mathbf{X}), \quad \text{at } t = 0
\end{aligned}}$$

Strong form of the Initial BVP $\qquad$ (3.7.10)