

## 3.3 The Cauchy Stress Tensor

### 3.3.1 The Traction Vector

The **traction vector** was introduced in Part I, §3.3. To recall, it is the limiting value of the ratio of force over area; for Force  $\Delta F$  acting on a surface element of area  $\Delta S$ , it is

$$\mathbf{t}^{(n)} = \lim_{\Delta S \rightarrow 0} \frac{\Delta F}{\Delta S} \quad (3.3.1)$$

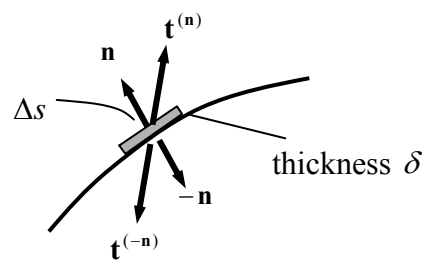
and  $\mathbf{n}$  denotes the normal to the surface element. An infinite number of traction vectors act at a point, each acting on different surfaces through the point, defined by different normals.

### 3.3.2 Cauchy's Lemma

**Cauchy's lemma** states that traction vectors acting on opposite sides of a surface are equal and opposite<sup>1</sup>. This can be expressed in vector form:

$$\boxed{\mathbf{t}^{(n)} = -\mathbf{t}^{(-n)}} \quad \text{Cauchy's Lemma} \quad (3.3.2)$$

This can be proved by applying the principle of linear momentum to a collection of particles of mass  $\Delta m$  instantaneously occupying a small box with parallel surfaces of area  $\Delta s$ , thickness  $\delta$  and volume  $\Delta v = \delta \Delta s$ , Fig. 3.3.1. The resultant *surface* force acting on this matter is  $\mathbf{t}^{(n)} \Delta s + \mathbf{t}^{(-n)} \Delta s$ .



**Figure 3.3.1: traction acting on a small portion of material particles**

The total linear momentum of the matter is  $\int_{\Delta V} \rho \mathbf{v} dv = \int_{\Delta m} \mathbf{v} dm$ . By the mean value theorem (see Appendix A to Chapter 1, §1.B.1), this equals  $\bar{\mathbf{v}} \Delta m$ , where  $\bar{\mathbf{v}}$  is the velocity at some interior point. Similarly, the body force acting on the matter is  $\int_{\Delta V} \mathbf{b} dv = \bar{\mathbf{b}} \Delta v$ , where  $\bar{\mathbf{b}}$  is the body force (per unit volume) acting at some interior point. The total mass

<sup>1</sup> this is equivalent to Newton's (third) law of action and reaction – it seems like a lot of work to prove this seemingly obvious result but, to be consistent, it is supposed that the only fundamental dynamic laws available here are the principles of linear and angular momentum, and not any of Newton's laws

can also be written as  $\Delta m = \int_{\Delta V} \rho dv = \bar{\rho} \Delta v$ . From the principle of linear momentum, Eqn. 3.2.7, and since  $\Delta m$  does not change with time,

$$\mathbf{t}^{(n)} \Delta s + \mathbf{t}^{(-n)} \Delta s + \bar{\mathbf{b}} \Delta v = \frac{d}{dt} [\bar{\mathbf{v}} \Delta m] = \Delta m \frac{d\bar{\mathbf{v}}}{dt} = \bar{\rho} \Delta v \frac{d\bar{\mathbf{v}}}{dt} = \bar{\rho} \delta \Delta s \frac{d\bar{\mathbf{v}}}{dt} \quad (3.3.3)$$

Dividing through by  $\Delta s$  and taking the limit as  $\delta \rightarrow 0$ , one finds that  $\mathbf{t}^{(n)} = -\mathbf{t}^{(-n)}$ . Note that the values of  $\mathbf{t}^{(n)}$ ,  $\mathbf{t}^{(-n)}$  acting on the box with finite thickness are not the same as the final values, but approach the final values *at* the surface as  $\delta \rightarrow 0$ .

### 3.3.3 Stress

In Part I, the components of the traction vector were called stress components, and it was illustrated how there were nine stress components associated with each material particle. Here, the stress is defined more formally,

#### Cauchy's Law

**Cauchy's Law** states that there exists a **Cauchy stress tensor**  $\boldsymbol{\sigma}$  which maps the normal to a surface to the traction vector acting on that surface, according to

$$\boxed{\mathbf{t} = \boldsymbol{\sigma} \mathbf{n}, \quad t_i = \sigma_{ij} n_j} \quad \text{Cauchy's Law} \quad (3.3.4)$$

or, in full,

$$\begin{aligned} t_1 &= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ t_2 &= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ t_3 &= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{aligned} \quad (3.3.5)$$

Note:

- many authors define the stress tensor as  $\mathbf{t} = \mathbf{n} \boldsymbol{\sigma}$ . This amounts to the definition used here since, as mentioned in Part I, and as will be (re-)proved below, the stress tensor is symmetric,  $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ ,  $\sigma_{ij} = \sigma_{ji}$
- the Cauchy stress refers to the *current* configuration, that is, it is a measure of force per unit area acting on a surface in the current configuration.

#### Stress Components

Taking Cauchy's law to be true (it is proved below), the components of the stress tensor with respect to a Cartesian coordinate system are, from 1.9.4 and 3.3.4,

$$\sigma_{ij} = \mathbf{e}_i \boldsymbol{\sigma} \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{t}^{(e_j)} \quad (3.3.6)$$

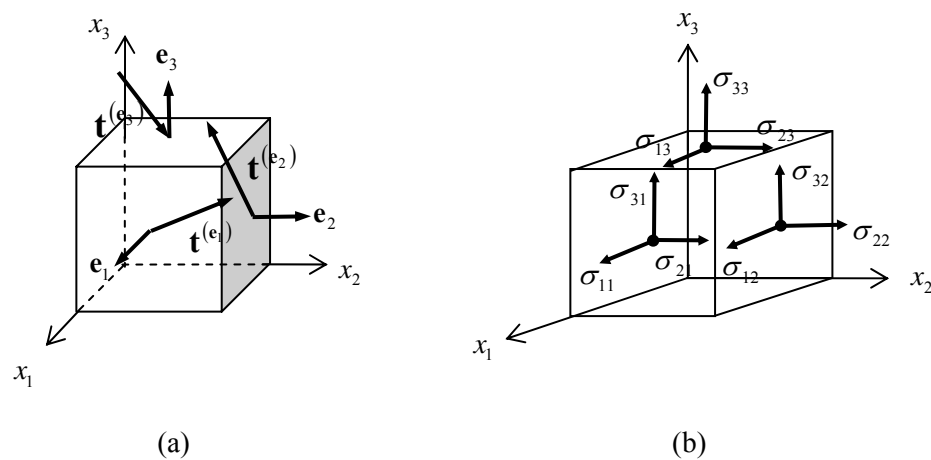
which is the  $i$ th component of the traction vector acting on a surface with normal  $\mathbf{e}_j$ . Note that this definition is inconsistent with that given in Part I, §3.2 – there, the first

subscript denoted the direction of the normal – but, again, the two definitions are equivalent because of the symmetry of the stress tensor.

The three traction vectors acting on the surface elements whose outward normals point in the directions of the three base vectors  $\mathbf{e}_j$  are

$$\begin{aligned} \mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \end{aligned} \quad (3.3.7)$$

Eqns. 3.3.6-7 are illustrated in Fig. 3.3.2.



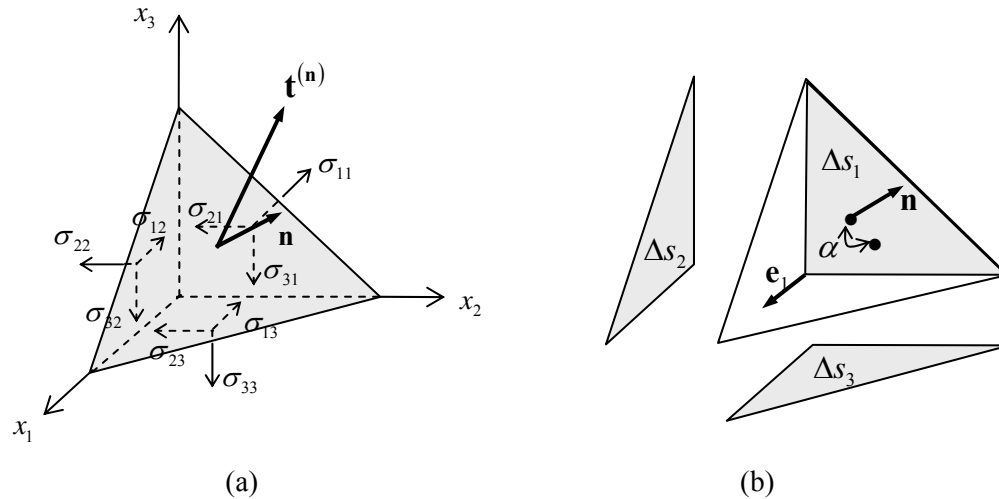
**Figure 3.3.2: traction acting on surfaces with normals in the coordinate directions; (a) traction vectors, (b) stress components**

### Proof of Cauchy's Law

The proof of Cauchy's law essentially follows the same method as used in the proof of Cauchy's lemma.

Consider a small tetrahedral free-body, with vertex at the origin, Fig. 3.3.3. It is required to determine the traction  $\mathbf{t}$  in terms of the nine stress components (which are all shown positive in the diagram).

Let the area of the base of the tetrahedron, with normal  $\mathbf{n}$ , be  $\Delta s$ . The area  $ds_1$  is then  $\Delta s \cos \alpha$ , where  $\alpha$  is the angle between the planes, as shown in Fig. 3.3.3b; this angle is the same as that between the vectors  $\mathbf{n}$  and  $\mathbf{e}_1$ , so  $\Delta s_1 = (\mathbf{n} \cdot \mathbf{e}_1)\Delta s = n_1\Delta s$ , and similarly for the other surfaces:  $\Delta s_2 = n_2\Delta s$  and  $\Delta s_3 = n_3\Delta s$ .



**Figure 3.3.3: free body diagram of a tetrahedral portion of material; (a) traction acting on the material, (b) relationship between surface areas and normal components**

The resultant surface force on the body, acting in the  $x_1$  direction, is

$$t_1 \Delta s - \sigma_{11} n_1 \Delta s - \sigma_{12} n_2 \Delta s - \sigma_{13} n_3 \Delta s$$

Again, the momentum is  $\bar{v} \Delta M$ , the body force is  $\bar{\mathbf{b}} \Delta v$  and the mass is  $\Delta m = \bar{\rho} \Delta v = \bar{\rho} (h/3) \Delta s$ , where  $h$  is the perpendicular distance from the origin (vertex) to the base. The principle of linear momentum then states that

$$t_1 \Delta s - \sigma_{11} n_1 \Delta s - \sigma_{12} n_2 \Delta s - \sigma_{13} n_3 \Delta s + \bar{b}_1 (h/3) \Delta s = \bar{\rho} (h/3) \Delta s \frac{d\bar{v}_1}{dt}$$

Again, the values of the traction and stress components on the faces will in general vary over the faces, so the values used in this equation are average values over the faces.

Dividing through by  $\Delta s$ , and taking the limit as  $h \rightarrow 0$ , one finds that

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

and now these quantities,  $t_1$ ,  $\sigma_{11}$ ,  $\sigma_{12}$ ,  $\sigma_{13}$ , are the values *at* the origin. The equations for the other two traction components can be derived in a similar way.

### Normal and Shear Stress

The stress acting normal to a surface is given by

$$\sigma_N = \mathbf{n} \cdot \mathbf{t}^{(n)} \quad (3.3.8)$$

The shear stress acting on the surface can then be obtained from

$$\sigma_s = \sqrt{|\hat{\mathbf{t}}^{(\hat{\mathbf{n}})}|^2 - \sigma_N^2} \quad (3.3.9)$$

### Example

The state of stress at a point is given in the matrix form

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine

- (a) the traction vector acting on a plane through the point whose unit normal is  $\hat{\mathbf{n}} = (1/3)\hat{\mathbf{e}}_1 + (2/3)\hat{\mathbf{e}}_2 - (2/3)\hat{\mathbf{e}}_3$   
 (b) the component of this traction acting perpendicular to the plane  
 (c) the shear component of traction.

### Solution

- (a) The traction is

$$\begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

$$\text{or } \mathbf{t}^{(\hat{\mathbf{n}})} = (-2/3)\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3.$$

- (b) The component normal to the plane is the projection of  $\mathbf{t}^{(\hat{\mathbf{n}})}$  in the direction of  $\hat{\mathbf{n}}$ , i.e.

$$\sigma_N = \mathbf{t}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

- (c) The shearing component of traction is

$$\begin{aligned} \sigma_s &= \mathbf{t}^{(\hat{\mathbf{n}})} - (22/9)\hat{\mathbf{n}} \\ &= [(-2/3) - (22/27)]\hat{\mathbf{e}}_1 + [3 - (44/27)]\hat{\mathbf{e}}_2 + [-1 + (44/27)]\hat{\mathbf{e}}_3 \\ &= [(-40/27)\hat{\mathbf{e}}_1 + (37/27)\hat{\mathbf{e}}_2 + (17/27)\hat{\mathbf{e}}_3] \end{aligned}$$

i.e. of magnitude  $\sqrt{(-40/27)^2 + (37/27)^2 + (17/27)^2} \approx 2.1$ , which equals

$$\sqrt{|\hat{\mathbf{t}}^{(\hat{\mathbf{n}})}|^2 - \sigma_N^2}.$$

■