2.13 Variation and Linearisation of Kinematic Tensors

2.13.1 The Variation of Kinematic Tensors

The Variation

In this section is reviewed the concept of the variation, introduced in Part I, §8.5.

The **variation** is defined as follows: consider a function $\mathbf{u}(\mathbf{x})$, with $\mathbf{u}^*(\mathbf{x})$ a second function which is at most infinitesimally different from $\mathbf{u}(\mathbf{x})$ at every point $\boldsymbol{x}$, Fig. 2.13.1

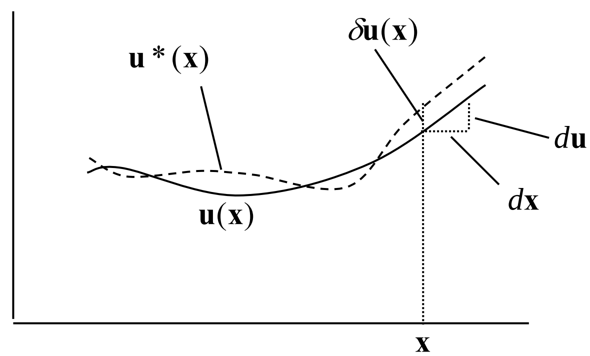

Figure 2.13.1: the variation

Then define

$$\boxed{\delta\mathbf{u} = \mathbf{u}^*(\mathbf{x}) - \mathbf{u}(\mathbf{x})} \qquad \textbf{The Variation} \qquad (2.13.1)$$

The operator δ is called the **variation symbol** and $\delta\mathbf{u}$ is called the variation of $\mathbf{u}(\mathbf{x})$.

The variation of $\mathbf{u}(\mathbf{x})$ is understood to represent an infinitesimal change in the function *at* $\boldsymbol{x}$. Note from the figure that a variation $\delta\mathbf{u}$ of a function $\mathbf{u}$ is different to a differential $d\mathbf{u}$. The ordinary differentiation gives a measure of the change of a function resulting from a specified change in the *independent* variable (in this case $\mathbf{x}$). Also, note that the independent variable does not participate in the variation process; the variation operator imparts an infinitesimal change to the function $\mathbf{u}$ at some *fixed* $\mathbf{x}$ – formally, one can write this as $\delta\mathbf{x} = 0$.

The Commutative Properties of the variation operator

(1) $$\frac{d}{d\mathbf{x}}\delta\mathbf{u} = \delta\frac{d\mathbf{u}}{d\mathbf{x}} \qquad (2.13.2)$$

Proof:

$$\delta \frac{d\mathbf{u}}{d\mathbf{x}} = \left(\frac{d\mathbf{u}}{d\mathbf{x}}\right)^* - \frac{d\mathbf{u}}{d\mathbf{x}} = \frac{d\mathbf{u}^*}{d\mathbf{x}} - \frac{d\mathbf{u}}{d\mathbf{x}} = \frac{d(\mathbf{u}^* - \mathbf{u})}{d\mathbf{x}} = \frac{d}{d\mathbf{x}}\left(\delta \mathbf{u}(\mathbf{x})\right)$$

(2) $$\delta \int_{x_1}^{x_2} \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{x_1}^{x_2} \delta \mathbf{u}(\mathbf{x}) d\mathbf{x} \tag{2.13.3}$$

Proof:

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{u}^*(\mathbf{x}) d\mathbf{x} - \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} [\mathbf{u}^*(\mathbf{x}) - \mathbf{u}(\mathbf{x})] d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \delta \mathbf{u}(\mathbf{x}) d\mathbf{x}$$

Variation of a Function

Consider **A**, a scalar-, vector-, or tensor-valued function of $\mathbf{u}$, $\mathbf{A}(\mathbf{u})$. When we apply a variation to **u**, $\delta\mathbf{u}$, **A** changes to $\mathbf{A}(\mathbf{u} + \delta\mathbf{u})$. The variation of **A** is then defined as

$$\delta \mathbf{A}(\mathbf{u}, \delta\mathbf{u}) = \mathbf{A}(\mathbf{u} + \delta\mathbf{u}) - \mathbf{A}(\mathbf{u}) \tag{2.13.4}$$

(in the limit as $\delta\mathbf{u} \to 0$). This can be expressed using the concept of the directional derivative in the usual way (see §1.6.11): consider the function $\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})$, so that $\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})_{\varepsilon=0} = \mathbf{A}(\mathbf{u})$ and $\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})_{\varepsilon=1} = \mathbf{A}(\mathbf{u} + \delta\mathbf{u})$. A Taylor expansion gives $\mathbf{A}(\varepsilon) = \mathbf{A}(0) + \varepsilon\left(d\mathbf{A} / d\varepsilon\right)_{\varepsilon=0} + \cdots$, or

$$\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u}) = \mathbf{A}(\mathbf{u}) + \varepsilon\left(\frac{d}{d\varepsilon}\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u})\right)_{\varepsilon=0} + \cdots \tag{2.13.5}$$

Setting $\varepsilon = 1$ then gives Eqn. 2.13.4; thus

$$\mathbf{A}(\mathbf{u} + \delta\mathbf{u}) \approx \mathbf{A}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{A}[\delta\mathbf{u}] \tag{2.13.6}$$

where $\partial_{\mathbf{u}}\mathbf{A}[\delta\mathbf{u}]$ is the directional derivative of **A** in the direction $\delta\mathbf{u}$; the directional derivative in this context is the variation of **A**:

$$\delta\mathbf{A}(\mathbf{u}, \delta\mathbf{u}) \equiv \partial_{\mathbf{u}}\mathbf{A}[\delta\mathbf{u}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u}) \tag{2.13.7}$$

For example, consider the scalar function $\phi = \mathbf{P} : \mathbf{E}$, where **P** and **E** are second order tensors. Then

$$\delta\phi(\mathbf{E}, \delta\mathbf{E}) \equiv \partial_{\mathbf{E}}\phi[\delta\mathbf{E}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{P} : (\mathbf{E} + \varepsilon\delta\mathbf{E}) = \mathbf{P} : \delta\mathbf{E} \qquad (2.13.8)$$

The **second variation** is defined as

$$\delta^2 \mathbf{A} = \delta(\delta\mathbf{A}) = \partial_{\mathbf{u}}\delta\mathbf{A}[\delta\mathbf{u}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta\mathbf{A}(\mathbf{u} + \varepsilon\delta\mathbf{u}) \qquad (2.13.9)$$

For example, for a scalar function $\phi(\mathbf{u})$ of a vector **u**, the chain rule and Eqn. 2.13.2 give

$$\begin{aligned} \delta\varphi(\mathbf{u}, \delta\mathbf{u}) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \varphi(\mathbf{u} + \varepsilon\delta\mathbf{u}) = \left.\frac{d\varphi(\mathbf{u} + \varepsilon\delta\mathbf{u})}{d(\mathbf{u} + \varepsilon\delta\mathbf{u})}\right|_{\varepsilon=0} \frac{d(\mathbf{u} + \varepsilon\delta\mathbf{u})}{d\varepsilon} = \frac{\partial\varphi}{\partial\mathbf{u}} \cdot \delta\mathbf{u} \\ \delta^2\varphi &= \frac{\partial\delta\varphi}{\partial\mathbf{u}} \cdot \delta\mathbf{u} = \left(\delta\frac{\partial\varphi}{\partial\mathbf{u}}\right) \cdot \delta\mathbf{u} = \left(\frac{\partial^2\varphi}{\partial\mathbf{u}\partial\mathbf{u}}\delta\mathbf{u}\right) \cdot \delta\mathbf{u} = \delta\mathbf{u}\frac{\partial^2\varphi}{\partial\mathbf{u}\partial\mathbf{u}}\delta\mathbf{u} \end{aligned} \qquad (2.13.10)$$

Variation of Functions of the Displacement

In what follows is discussed the change (variation) in functions $\mathbf{A}(\mathbf{u})$ when the displacement (or velocity) fields undergo a variation. These ideas are useful in formulating variational principles of mechanics (see, for example, §3.9).

Shown in Fig. 2.13.2 is the current configuration frozen at some instant in time. The displacement field is then allowed to undergo a variation $\delta\mathbf{u}$. This change to the displacement field evidently changes kinematic tensors, and these changes are now investigated. Note that this variation to the displacement induces a variation to **x**, $\delta\mathbf{x}$, but **X** remains unchanged, $\delta\mathbf{X} = 0$.

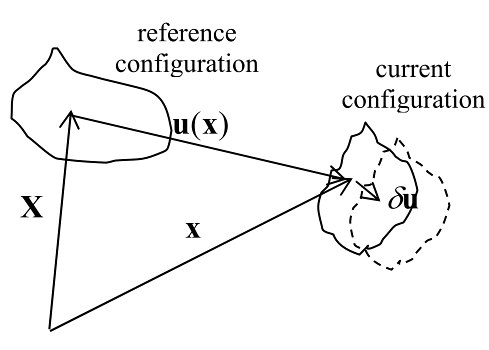

Figure 2.13.2: a variation of the displacement

To evaluate the variation of the deformation gradient $\mathbf{F}$, $\delta\mathbf{F}(\mathbf{u},\delta\mathbf{u})$, where $\mathbf{u}$ is the displacement field, note that $\mathbf{u}=\mathbf{x}-\mathbf{X}$ and Eqn. 2.2.43, $\mathbf{F}(\mathbf{u})=\mathrm{Grad}\mathbf{u}+\mathbf{I}$. One has, from the definition 2.13.7,

$$\begin{aligned}\delta\mathbf{F}(\mathbf{u},\delta\mathbf{u})=\partial_{\mathbf{u}}\mathbf{F}[\delta\mathbf{u}]&=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbf{F}(\mathbf{u}+\varepsilon\delta\mathbf{u})\\&=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left[\mathbf{F}(\mathbf{u})+\varepsilon\mathrm{Grad}(\delta\mathbf{u})+\mathbf{I}\right]\\&=\mathrm{Grad}(\delta\mathbf{u})\end{aligned}\qquad(2.13.11)$$

Noting the first commutative property of the variation, 2.13.2, this can also be expressed as

$$\delta\mathbf{F}(\mathbf{u},\delta\mathbf{u})=\delta(\mathrm{Grad}\mathbf{u})\qquad(2.13.12)$$

Note that $\delta\mathbf{u}$ is completely independent of the function $\mathbf{u}$.

Here are some other examples, involving the inverse deformation gradient, the Green-Lagrange strain, the inverse right Cauchy-Green strain and the spatial line element: {▲Problem 1-3}

$$\begin{aligned}\delta\mathbf{F}^{-1}&=-\mathbf{F}^{-1}\mathrm{grad}\delta\mathbf{u}\\\delta\mathbf{E}&=\mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}\\\delta\mathbf{C}^{-1}&=-2\mathbf{F}^{-1}\boldsymbol{\varepsilon}\mathbf{F}^{-\mathrm{T}}\end{aligned}\qquad(2.13.13)$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor, Eqn. 2.2.48.

One also has, using the chain rule for the directional derivative, Eqn. 1.15.28, the directional derivative for the determinant, Eqn. 1.15.32, the trace relation 1.10.10e, Eqn. 2.2.8b,

$$\begin{aligned}\delta J(\mathbf{u},\delta\mathbf{u})&=\delta\det\mathbf{F}(\mathbf{u},\delta\mathbf{u})\\&=\partial_{\mathbf{u}}\det\mathbf{F}[\delta\mathbf{u}]\\&=\partial_{\mathbf{F}}\det\mathbf{F}[\partial_{\mathbf{u}}\mathbf{F}[\delta\mathbf{u}]]\\&=\partial_{\mathbf{F}}\det\mathbf{F}[\mathrm{Grad}(\delta\mathbf{u})]\\&=\det\mathbf{F}[\mathbf{F}^{-\mathrm{T}}:\mathrm{Grad}(\delta\mathbf{u})]\\&=J\mathrm{tr}(\mathrm{Grad}(\delta\mathbf{u})\mathbf{F}^{-1})\\&=J\mathrm{tr}(\mathrm{grad}(\delta\mathbf{u}))\\&=J\,\mathrm{div}(\delta\mathbf{u})\end{aligned}\qquad(2.13.14)$$

Example

To put some of the above concepts into a simple and less abstract setting, consider the following scenario: a bar over $0 \le \mathbf{X} \le 1$ is extended, as illustrated in Fig. 2.13.3, according to:

$$\begin{aligned} \mathbf{x} &= 2\mathbf{X}^2 + 3 \\ \mathbf{X} &= \sqrt{\tfrac{1}{2}(\mathbf{x}-3)} \end{aligned} \tag{2.13.15}$$

The deformation gradient is

$$\mathbf{F} = \mathrm{Grad}\mathbf{x} = 4\mathbf{X} \tag{2.13.16}$$

So, for example, in the initial configuration (A), an infinitesimal line element at $\mathbf{X}=0$ does not stretch ($\mathbf{F}=0$) whereas a line element at $\mathbf{X}=1$ stretches by 4.

The inverse deformation gradient is

$$\mathbf{F}^{-1} = \mathrm{grad}\mathbf{X} = \frac{1}{\sqrt{8(\mathbf{x}-3)}} \tag{2.13.17}$$

This implies that, in the current configuration (B), an infinitesimal line element at $\mathbf{x}=3$ is the same size as its counterpart in the initial configuration ($\mathbf{F}^{-1}=0$) whereas a line element at $\mathbf{x}=5$ shrinks by a factor of 4 when returning to the initial configuration

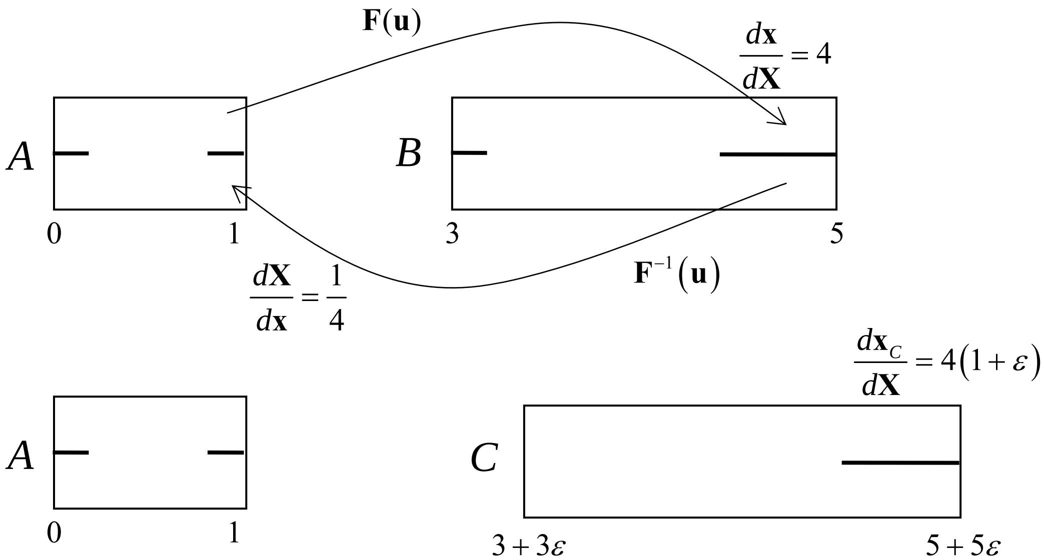

Figure 2.13.3: a motion and a variation

Now introduce a variation, which moves the bar from configuration *B* to configuration *C*:

$$\delta\mathbf{u} = \varepsilon\mathbf{x} = \varepsilon\left(2\mathbf{X}^2 + 3\right) \tag{2.13.18}$$

The point at 3 moves to $3 + 3\varepsilon$ and the point at 5 moves to $5 + 5\varepsilon$. (This variation happens to be a simple linear function of **x**, but it can be anything for our purposes here.)

The variation is plotted below as a function of **X** and **x**.

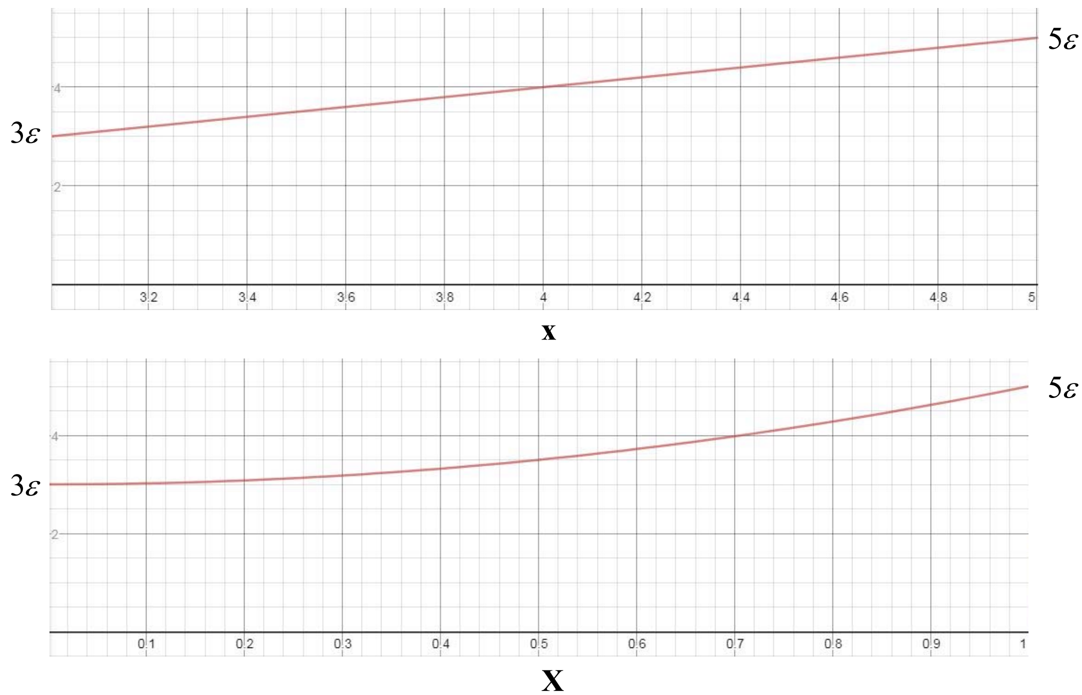

Figure 2.13.4: the variation as a function of x and X

Differentiating Eqns. 2.13.19, the gradients of the variations are

$$\begin{aligned} \operatorname{Grad}(\delta\mathbf{u}) &= \varepsilon(4\mathbf{X}) \\ \operatorname{grad}(\delta\mathbf{u}) &= \varepsilon \end{aligned} \tag{2.13.20}$$

which are the slopes in Figure 2.13.4.

To calculate the **F** associated with the new variation configuration, i.e. $\mathbf{F}(\mathbf{u} + \delta\mathbf{u})$, note that points **X** have now moved to:

$$2\mathbf{X}^2 + 3 + \varepsilon\left(2\mathbf{X}^2 + 3\right) \tag{2.13.21}$$

and so

$$\mathbf{F}(\mathbf{u}+\delta\mathbf{u}) = \mathrm{Grad}\left((1+\varepsilon)\left(2\mathbf{X}^2+3\right)\right) = 4\mathbf{X}+\varepsilon 4\mathbf{X} \tag{2.13.22}$$

This says that an infinitesimal line element at $\mathbf{X}=0$ does not stretch when moving to configuration *C* ($\mathbf{F}=0$) whereas a line element at $\mathbf{X}=1$ stretches by $4+4\varepsilon$.

Subtracting Eqn. 2.13.17 form Eqn. 2.13.22:

$$\delta\mathbf{F} = \mathbf{F}(\mathbf{u}+\delta\mathbf{u}) - \mathbf{F}(\mathbf{u}) = \varepsilon(4\mathbf{X}) \tag{2.13.23}$$

From Eqn. 2.13.20, this verifies Eqn. 2.13.11, that

$$\delta\mathbf{F} = \mathrm{Grad}(\delta\mathbf{u}) \tag{2.13.24}$$

We could also calculate the variation of **F** by moving directly from configuration *B* to configuration *C*. The movement of the particles from *B* to *C* is given by Eqn. 2.13.19: $\varepsilon\left(2\mathbf{X}^2+3\right)$ and so, based on this motion, $\delta\mathbf{F} = \mathrm{Grad}\left(\varepsilon\left(2\mathbf{X}^2+3\right)\right) = \varepsilon(4\mathbf{X})$.

To calculate the $\mathbf{F}^{-1}$ associated with the new variation configuration, i.e. $\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u})$, note that the "new" current position **x** is (Eqn. 2.13.21):

$$\begin{aligned} \mathbf{x}_C &= 2\mathbf{X}^2+3+\varepsilon\left(2\mathbf{X}^2+3\right) \\ \to \mathbf{X} &= \sqrt{\frac{1}{2}\left(\frac{\mathbf{x}_C}{1+\varepsilon}-3\right)} \end{aligned} \tag{2.13.25}$$

This means that the point $3+3\varepsilon$ in configuration *C* corresponds to $\mathbf{X}=0$ and the point $5+5\varepsilon$ corresponds to $\mathbf{X}=1$. Then,

$$\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) = \frac{d}{d\mathbf{x}_C}\sqrt{\frac{1}{2}\left(\frac{\mathbf{x}_C}{1+\varepsilon}-3\right)} = \frac{1}{1+\varepsilon}\frac{1}{\sqrt{8\left(\frac{\mathbf{x}_C}{1+\varepsilon}-3\right)}} \tag{2.13.26}$$

So an element at the point $3+3\varepsilon$ in configuration *C* does not change in size as it is mapped back to the initial configuration, whereas an element at the point $5+5\varepsilon$ shrinks back to the initial configuration by a factor of $1/(4+4\varepsilon)$, as indicated in Fig. 2.13.3.

Alternatively, since $\mathbf{x}_C = \mathbf{x}+\varepsilon\mathbf{x}$, this can be written as

$$\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) = \frac{1}{1+\varepsilon}\frac{1}{\sqrt{8(\mathbf{x}-3)}} \tag{2.13.27}$$

Subtracting Eqn. 2.13.18 from Eqn. 2.13.27, the variation of the inverse deformation gradient is then

$$\begin{aligned}\delta \mathbf{F}^{-1}(\mathbf{u}) = \mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) - \mathbf{F}^{-1}(\mathbf{u}) &= \frac{1}{1+\varepsilon}\frac{1}{\sqrt{8(\mathbf{x}-3)}} - \frac{1}{\sqrt{8(\mathbf{x}-3)}} \\ &= -\frac{\varepsilon}{1+\varepsilon}\frac{1}{\sqrt{8(\mathbf{x}-3)}}\end{aligned} \tag{2.13.28}$$

Using a series expansion, $(1+\varepsilon)^{-1} = 1-\varepsilon+\varepsilon^2-\dots$, for small ε (neglecting terms of order ε^2),

$$\delta \mathbf{F}^{-1}(\mathbf{u}) = -\varepsilon\frac{1}{\sqrt{8(\mathbf{x}-3)}} \tag{2.13.29}$$

From Eqns. 2.13.18 and 2.13.20, this verifies the relation 2.13.13:

$$\delta \mathbf{F}^{-1}(\mathbf{u}) = -\mathbf{F}^{-1}\mathrm{grad}(\delta u) \tag{2.13.30}$$

A formula for the inverse deformation gradient is $\mathbf{F}^{-1} = \mathbf{I} - \mathrm{grad}\mathbf{u}$. However, note that $\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) \neq \mathbf{I} - \partial\mathbf{u}/\partial\mathbf{x}$, but that $\mathbf{F}^{-1}(\mathbf{u}+\delta\mathbf{u}) = \mathbf{I} - \partial\mathbf{u}/\partial\mathbf{x}_C$.

The Lie Variation

The **Lie-variation** is defined for *spatial* vectors and tensors as a variation holding the deformed basis constant. For example,

$$\delta_{\mathrm{L}}^{\mathrm{b}}\mathbf{a} = \delta a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \tag{2.13.31}$$

The object is first pulled-back, the variation is then taken and finally a push-forward is carried out. For example, analogous to 2.12.66,

$$\delta_{\mathrm{L}}\mathbf{a}(\mathbf{u},\delta\mathbf{u}) \equiv \chi_*\left(\partial_{\mathbf{u}}\left(\chi_*^{-1}(\mathbf{a})\right)[\delta\mathbf{u}]\right) \tag{2.13.32}$$

For example, consider the Lie-variation of the Euler-Almansi strain **e**. First, from 2.12.56b, $\chi_{-1}^*(\mathbf{e})^b = \mathbf{E}$. Then 2.13.13b gives $\partial_{\mathbf{u}}\left(\chi_{-1}^*(\mathbf{e})^b\right)[\delta\mathbf{u}] = \delta\mathbf{E} = \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}$. From 2.12.40a,

$$\delta_{\mathrm{L}}\mathbf{e}(\mathbf{u},\delta\mathbf{u}) = \chi_*\left(\partial_{\mathbf{u}}\left(\chi_{-1}^*(\mathbf{e})^b\right)[\delta\mathbf{u}]\right)^b = \chi_*\left(\mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}\right)^b = \delta\boldsymbol{\varepsilon} \tag{2.13.33}$$

2.13.2 Linearisation of Kinematic Functions

Linearisation of a Function

As for the variation, consider **A**, a scalar-, vector-, or tensor-valued function of $\mathbf{u}$. If **u** undergoes an increment $\Delta\mathbf{u}$, then, analogous to 2.13.4,

$$\mathbf{A}(\mathbf{u}+\Delta\mathbf{u}) \approx \mathbf{A}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{A}[\Delta\mathbf{u}] \tag{1.13.34}$$

The directional derivative $\partial_{\mathbf{u}}\mathbf{A}[\Delta\mathbf{u}]$ in this context is also denoted by $\Delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u})$. The **linearization** of **A** with respect to **u** is defined to be

$$\mathrm{L}\ \mathbf{A}(\mathbf{u},\Delta\mathbf{u}) = \mathbf{A}(\mathbf{u}) + \Delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u}) \tag{1.13.35}$$

Using exactly the same method of calculation as was used for the variations above, the linearization of **F** and **E**, for example, are

$$\begin{aligned} \mathrm{L}\,\mathbf{F}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{F}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{F}[\Delta\mathbf{u}] = \mathbf{F} + \mathrm{Grad}\Delta\mathbf{u} \\ \mathrm{L}\,\mathbf{E}(\mathbf{u},\Delta\mathbf{u}) &= \mathbf{E}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{E}[\Delta\mathbf{u}] = \mathbf{E} + \mathbf{F}^{\mathrm{T}}\Delta\boldsymbol{\varepsilon}\mathbf{F} \end{aligned} \tag{2.13.36}$$

where $\Delta\boldsymbol{\varepsilon} = \frac{1}{2}\left((\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}} + (\mathrm{grad}\Delta\mathbf{u})\right)$ is the linearised small strain tensor $\boldsymbol{\varepsilon}$.

Linearisation of Variations of a Function

One can also linearise the variation of a function. For example,

$$\mathrm{L}\ \delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u}) = \delta\mathbf{A}(\mathbf{u},\delta\mathbf{u}) + \Delta\delta\mathbf{A}(\mathbf{u},\Delta\mathbf{u}) \tag{2.13.37}$$

The second term here is the directional derivative

$$\begin{aligned} \Delta\delta\mathbf{A}[\mathbf{u},\Delta\mathbf{u}] &= \partial_{\mathbf{u}}\delta\mathbf{A}[\Delta\mathbf{u}] \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \delta\mathbf{A}(\mathbf{u}+\varepsilon\Delta\mathbf{u}) \end{aligned} \tag{2.13.38}$$

This leads to an expression similar to $\delta^2\mathbf{A}$. For example, for a scalar function $\phi(\mathbf{u})$ of a vector **u**,

$$\Delta\delta\phi = \frac{\partial\delta\phi}{\partial\mathbf{u}}\cdot\Delta\mathbf{u} = \Delta\mathbf{u}\,\frac{\partial^2\phi}{\partial\mathbf{u}\partial\mathbf{u}}\,\delta\mathbf{u} \tag{2.13.39}$$

Consider now the virtual Green-Lagrange strain, 2.13.11b, $\delta\mathbf{E} = \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}$. To carry out the linearization of $\delta\mathbf{E}$, it is convenient to first write it in the form

$$\begin{aligned}\delta\mathbf{E} &= \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F} \\ &= \tfrac{1}{2}\mathbf{F}^{\mathrm{T}}\left[(\operatorname{grad}\delta\mathbf{u})^{\mathrm{T}} + \operatorname{grad}\delta\mathbf{u}\right]\mathbf{F} \\ &= \tfrac{1}{2}\left[(\operatorname{Grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\operatorname{Grad}\delta\mathbf{u}\right]\end{aligned} \tag{2.13.40}$$

Then

$$\Delta\delta\mathbf{E} = \partial_{\mathbf{u}}\delta\mathbf{E}[\Delta\mathbf{u}] = \partial_{\mathbf{u}}\left\{\tfrac{1}{2}\left[(\operatorname{Grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\operatorname{Grad}\delta\mathbf{u}\right]\right\}[\Delta\mathbf{u}] \tag{2.13.41}$$

Recall that the variation $\delta\mathbf{u}$ is *independent* of $\mathbf{u}$; this equation is being linearised with respect to $\mathbf{u}$, and $\delta\mathbf{u}$ is unaffected by the linearization (see Fig. 2.13.3 below). However, the motion, and in particular $\mathbf{F}$, *are* affected by the increment in $\mathbf{u}$. Thus {▲Problem 4}

$$\Delta\delta\mathbf{E} = \operatorname{sym}\left((\operatorname{Grad}\Delta\mathbf{u})^{\mathrm{T}}\operatorname{Grad}\delta\mathbf{u}\right) \tag{2.13.42}$$

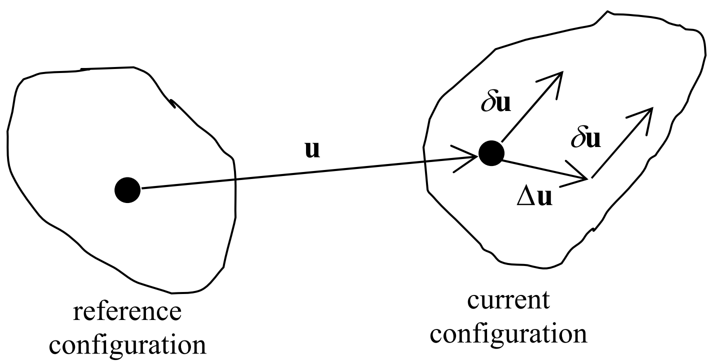

Figure 2.13.3: linearisation

As with the variational operator, one can define the linearization of a spatial tensor as involving a pull back, followed by the directional derivative, and finally the push forward operation. Thus

$$\Delta\mathbf{a}(\mathbf{u},\Delta\mathbf{u}) \equiv \chi_*\left(\partial_{\mathbf{u}}\left(\chi_*^{-1}(\mathbf{a})\right)[\Delta\mathbf{u}]\right) \tag{2.13.43}$$

2.13.3 Problems

1. Use Eqn. 2.2.22, $\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right)$, Eqn. 2.13.11, $\delta\mathbf{F}(\mathbf{u}, \delta\mathbf{u}) = \mathrm{Grad}(\delta\mathbf{u})$, and Eqn. 2.2.8b, $\mathrm{grad}\mathbf{v} = (\mathrm{Grad}\mathbf{v})\mathbf{F}^{-1}$, to show that $\delta\mathbf{E} = \mathbf{F}^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathbf{F}$, where $\boldsymbol{\varepsilon}$ is the small strain tensor, Eqn. 2.2.48.
2. Use 2.13.11 to show that the variation of the inverse deformation gradient $\mathbf{F}^{-1}$ is $\delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathrm{grad}\delta\mathbf{u}$. [Hint: differente the relation $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}$ by the product rule and then use the relation $\mathrm{grad}\mathbf{v} = (\mathrm{Grad}\mathbf{v})\mathbf{F}^{-1}$ for vector $\mathbf{v}$.]
3. Use the definition $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ to show that $\delta\mathbf{C}^{-1} = -2\mathbf{F}^{-1}\boldsymbol{\varepsilon}\mathbf{F}^{-\mathrm{T}}$.
4. Use the relation $\mathrm{sym}\mathbf{A} = \frac{1}{2}\left(\mathbf{A}^{\mathrm{T}} + \mathbf{A}\right)$ to show that
$$\Delta\delta\mathbf{E} = \partial_{\mathbf{u}}\left\{\frac{1}{2}\left[(\mathrm{Grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right]\right\}[\Delta\mathbf{u}] = \mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)$$
5. Use $\delta\mathbf{e} = \delta\boldsymbol{\varepsilon} = \frac{1}{2}\left[(\mathrm{grad}\delta\mathbf{u})^{\mathrm{T}} + \mathrm{grad}\delta\mathbf{u}\right]$ to show that the
$$\begin{aligned}\Delta\delta\mathbf{e} = \chi_*\left(\partial_{\mathbf{u}}\left(\chi_*^{-1}(\delta\mathbf{e})\right)[\Delta\mathbf{u}]\right) &= \chi_*\mathrm{sym}\left((\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}\right)\\ &= \mathrm{sym}\left[(\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\cdot\mathrm{grad}\delta\mathbf{u}\right]\end{aligned}$$