2.7 Small Strain Theory

When the deformation is small, from 2.2.43-4,

$$\begin{aligned}\mathbf{F} &= \mathbf{I} + \mathrm{Grad}\mathbf{U} \\ &= \mathbf{I} + (\mathrm{grad}\mathbf{u})\mathbf{F} \\ &\approx \mathbf{I} + \mathrm{grad}\mathbf{u}\end{aligned} \tag{2.7.1}$$

neglecting the product of $\mathrm{grad}\mathbf{u}$ with $\mathrm{Grad}\mathbf{U}$, since these are small quantities. Thus one can take $\mathrm{Grad}\mathbf{U} = \mathrm{grad}\mathbf{u}$ and there is no distinction to be made between the undeformed and deformed configurations. The deformation gradient is of the form $\mathbf{F} = \mathbf{I} + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is small.

2.7.1 Decomposition of Strain

Any second order tensor can be decomposed into its symmetric and antisymmetric part according to 1.10.28, so that

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right) + \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right) = \boldsymbol{\varepsilon} + \boldsymbol{\Omega} \\ \frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij} + \Omega_{ij}\end{aligned} \tag{2.7.2}$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor 2.2.48 and $\boldsymbol{\Omega}$, the anti-symmetric part of the displacement gradient, is the **small rotation tensor**, so that **F** can be written as

$$\boxed{\mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}} \quad \textbf{Small Strain Decomposition of the Deformation Gradient} \tag{2.7.3}$$

It follows that (for the calculation of **e**, one can use the relation $(\mathbf{I} + \boldsymbol{\delta})^{-1} \approx \mathbf{I} - \boldsymbol{\delta}$ for small $\boldsymbol{\delta}$)

$$\begin{aligned}\mathbf{C} &= \mathbf{b} = \mathbf{I} + 2\boldsymbol{\varepsilon} \\ \mathbf{E} &= \mathbf{e} = \boldsymbol{\varepsilon}\end{aligned} \tag{2.7.4}$$

Rotation

Since $\boldsymbol{\Omega}$ is antisymmetric, it can be written in terms of an axial vector $\boldsymbol{\omega}$, *cf.* §1.10.11, so that for any vector **a**,

$$\boldsymbol{\Omega}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}, \qquad \boldsymbol{\omega} = -\Omega_{23}\mathbf{e}_1 + \Omega_{13}\mathbf{e}_1 - \Omega_{12}\mathbf{e}_3 \tag{2.7.5}$$

The relative displacement can now be written as

$$\begin{aligned} d\mathbf{u} &= (\mathrm{grad}\,\mathbf{u})d\mathbf{X} \\ &= \boldsymbol{\varepsilon}d\mathbf{X} + \boldsymbol{\omega} \times d\mathbf{X} \end{aligned} \tag{2.7.6}$$

The component of relative displacement given by $\boldsymbol{\omega} \times d\mathbf{X}$ is perpendicular to $d\mathbf{X}$, and so represents a pure rotation of the material line element, Fig. 2.7.1.

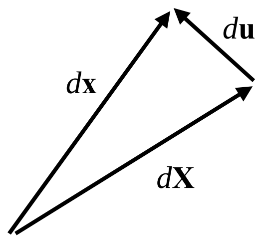

Figure 2.7.1: a pure rotation

Principal Strains

Since $\boldsymbol{\varepsilon}$ is symmetric, it must have three mutually orthogonal eigenvectors, the **principal axes of strain**, and three corresponding real eigenvalues, the **principal strains,** e_1, e_2, e_3), which can be positive or negative, *cf.* §1.11. The effect of $\boldsymbol{\varepsilon}$ is therefore to deform an elemental unit sphere into an elemental ellipsoid, whose axes are the principal axes, and whose lengths are $1+e_1, 1+e_2, 1+e_3$. Material fibres in these principal directions are stretched only, in which case the deformation is called a **pure deformation**; fibres in other directions will be stretched and rotated.

The term $\boldsymbol{\varepsilon}d\mathbf{X}$ in 2.7.6 therefore corresponds to a pure stretch along the principal axes. The total deformation is the sum of a pure deformation, represented by $\boldsymbol{\varepsilon}$, and a rigid body rotation, represented by $\boldsymbol{\Omega}$. This result is similar to that obtained for the exact finite strain theory, but here the decomposition is *additive* rather than *multiplicative*. Indeed, here the corresponding small strain stretch and rotation tensors are $\mathbf{U} = \mathbf{I} + \boldsymbol{\varepsilon}$ and $\mathbf{R} = \mathbf{I} + \boldsymbol{\Omega}$, so that

$$\mathbf{F} = \mathbf{RU} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega} \tag{2.7.7}$$

Example

Consider the simple shear (*c.f.* Eqn. 2.2.40)

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3$$

where k is small. The displacement vector is $\mathbf{u} = kx_2\mathbf{e}_1$ so that

$$\mathrm{grad}\,\mathbf{u} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The deformation can be written as the additive decomposition

$$d\mathbf{u} = \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\Omega} d\mathbf{X} \qquad \text{or} \qquad d\mathbf{u} = \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\omega} \times d\mathbf{X}$$

with

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\boldsymbol{\omega} = -(k/2)\mathbf{e}_3$. For the rotation component, one can write

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\Omega} = \begin{bmatrix} 1 & k/2 & 0 \\ -k/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, since for small θ, $\cos\theta \approx 1, \sin\theta \approx \theta$, can be seen to be a rotation through an angle $\theta = -k/2$ (a clockwise rotation).

The principal values of $\boldsymbol{\varepsilon}$ are $\pm k/2, 0$ with corresponding principal directions $\mathbf{n}_1 = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2$, $\mathbf{n}_2 = -(1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2$ and $\mathbf{n}_3 = \mathbf{e}_3$.

Thus the simple shear with small displacements consists of a rotation through an angle $k/2$ superimposed upon a pure shear with angle $k/2$, Fig. 2.6.2.

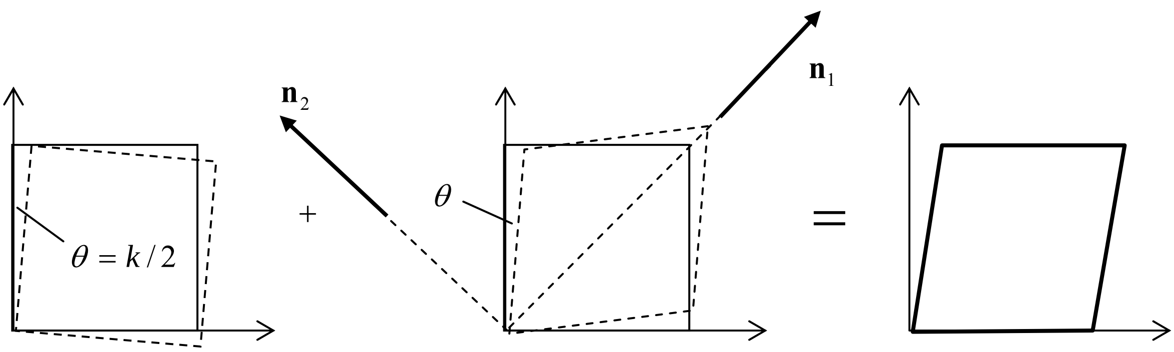

Figure 2.6.2: simple shear

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2.7.2 Rotations and Small Strain

Consider now a pure rotation about the X_3 axis (within the exact finite strain theory), $d\mathbf{x} = \mathbf{R} d\mathbf{X}$, with

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.7.8}$$

This rotation does not change the length of line elements $d\mathbf{X}$. According to the small strain theory, however,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which does predict line element length changes, but which can be neglected if θ is small. For example, if the rotation is of the order 10^{-2} rad, then $\varepsilon_{11} = \varepsilon_{22} = 10^{-4}$. However, if the rotation is large, the errors will be appreciable; in that case, rigid body rotation introduces geometrical non-linearities which must be dealt with using the finite deformation theory.

Thus the small strain theory is restricted to not only the case of small displacement gradients, but also small rigid body rotations.

2.7.3 Volume Change

An elemental cube with edges of unit length in the directions of the principal axes deforms into a cube with edges of lengths $1+e_1, 1+e_2, 1+e_3$, so the unit change in volume of the cube is

$$\frac{dv - dV}{dV} = (1+e_1)(1+e_2)(1+e_3) - 1 = e_1 + e_2 + e_3 + O(2) \tag{2.7.9}$$

Since second order quantities have already been neglected in introducing the small strain tensor, they must be neglected here. Hence the increase in volume per unit volume, called the **dilatation** (or **dilation**) is

$$\boxed{\frac{\delta V}{V} = e_1 + e_2 + e_3 = e_{ii} = \mathrm{tr}\boldsymbol{\varepsilon} = \mathrm{div}\mathbf{u}} \qquad \textbf{Dilatation} \tag{2.7.10}$$

Since any elemental volume can be constructed out of an infinite number of such elemental cubes, this result holds for any elemental volume irrespective of shape.

2.7.4 Rate of Deformation, Strain Rate and Spin Tensors

Take now the expressions 2.4.7 for the rate of deformation and spin tensors. Replacing **v** in these expressions by $\dot{\mathbf{u}}$, one has

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \qquad d_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right)$$
$$\mathbf{w} = \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right), \qquad w_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \frac{\partial \dot{u}_j}{\partial x_i}\right) \tag{2.7.11}$$

For small strains, one can take the time derivative outside (by considering the x_i to be material coordinates independent of time):

$$d_{ij} = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}$$
$$w_{ij} = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right\} \tag{2.7.12}$$

The rate of deformation in this context is seen to be the **rate of strain**, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$, and the spin is seen to be the **rate of rotation**, $\mathbf{w} = \dot{\boldsymbol{\Omega}}$.

The instantaneous motion of a material particle can hence be regarded as the sum of three effects:

(i) a translation given by $\dot{\mathbf{u}}$ (so in the time interval Δt the particle has been displaced by $\dot{\mathbf{u}}\Delta t$)
(ii) a pure deformation given by $\dot{\boldsymbol{\varepsilon}}$
(iii) a rigid body rotation given by $\dot{\boldsymbol{\Omega}}$

2.7.5 Compatibility Conditions

Suppose that the strains ε_{ij} in a body are known. If the displacements are to be determined, then the strain-displacement partial differential equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.7.13}$$

need to be integrated. However, there are six independent strain components but only three displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**, and it can be shown that they are given by

$$\varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0 \tag{2.7.14}$$

These are 81 equations, but only six of them are distinct, and these six equations are necessary and sufficient to evaluate the displacement field.