2.4 Material Time Derivatives

The motion is now allowed to be a function of time, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, and attention is given to time derivatives, both the **material time derivative** and the **local time derivative**.

2.4.1 Velocity & Acceleration

The velocity of a moving particle is the time rate of change of the position of the particle. From 2.1.3, by definition,

$$\mathbf{V}(\mathbf{X}, t) \equiv \frac{d\boldsymbol{\chi}(\mathbf{X}, t)}{dt} \tag{2.4.1}$$

In the motion expression $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X}$ and t are independent variables and $\mathbf{X}$ is independent of time, denoting the particle for which the velocity is being calculated. The velocity can thus be written as $\partial\boldsymbol{\chi}(\mathbf{X}, t)/\partial t$ or, denoting the motion by $\mathbf{x}(\mathbf{X}, t)$, as $d\mathbf{x}(\mathbf{X}, t)/dt$ or $\partial\mathbf{x}(\mathbf{X}, t)/\partial t$.

The spatial description of the velocity field may be obtained from the material description by simply replacing $\mathbf{X}$ with $\mathbf{x}$, i.e.

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right) \tag{2.4.2}$$

As with displacements in both descriptions, there is only *one* velocity, $\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$ – they are just given in terms of different coordinates.

The velocity is most often expressed in the spatial description, as

$$\boxed{\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}} \qquad \textbf{velocity} \tag{2.4.3}$$

To be precise, the right hand side here involves $\mathbf{x}$ which is a function of the material coordinates, but it is understood that the substitution back to spatial coordinates, as in 2.4.2, is made (see example below).

Similarly, the acceleration is defined to be

$$\mathbf{A}(\mathbf{X}, t) = \frac{d^2\boldsymbol{\chi}(\mathbf{X}, t)}{dt^2} = \frac{d^2\mathbf{x}}{dt^2} = \frac{d\mathbf{V}}{dt} = \frac{\partial^2\boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2} \tag{2.4.4}$$

Example

Consider the motion

$$x_1 = X_1 + t^2 X_2, \quad x_2 = X_2 + t^2 X_1, \quad x_3 = X_3$$

The velocity and acceleration can be evaluated through

$$\mathbf{V}(\mathbf{X},t) = \frac{d\mathbf{x}}{dt} = 2tX_2\mathbf{e}_1 + 2tX_1\mathbf{e}_2, \qquad \mathbf{A}(\mathbf{X},t) = \frac{d^2\mathbf{x}}{dt^2} = 2X_2\mathbf{e}_1 + 2X_1\mathbf{e}_2$$

One can write the motion in the spatial description by inverting the material description:

$$X_1 = \frac{x_1 - t^2 x_2}{1-t^4}, \quad X_2 = \frac{x_2 - t^2 x_1}{1-t^4}, \quad X_3 = x_3$$

Substituting in these equations then gives the spatial description of the velocity and acceleration:

$$\begin{aligned}
\mathbf{v}(\mathbf{x},t) &= \mathbf{V}\left(\chi^{-1}(\mathbf{x},t),t\right) = 2t\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2t\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2 \\
\mathbf{a}(\mathbf{x},t) &= \mathbf{A}\left(\chi^{-1}(\mathbf{x},t),t\right) = 2\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2
\end{aligned}$$

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2.4.2 The Material Derivative

One can analyse deformation by examining the current configuration only, discounting the reference configuration. This is the viewpoint taken in Fluid Mechanics – one focuses on material as it flows at the *current time*, and does not consider "where the fluid was". In order to do this, quantities must be cast in terms of the velocity. Suppose that the velocity in terms of spatial coordinates, $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ is known; for example, one could have a measuring instrument which records the velocity at a specific location, but the motion $\boldsymbol{\chi}$ itself is unknown. In that case, to evaluate the acceleration, the chain rule of differentiation must be applied:

$$\dot{\mathbf{v}} \equiv \frac{d}{dt}\mathbf{v}\left(\mathbf{x}(t),t\right) = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt}$$

or

$$\boxed{\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \left(\operatorname{grad}\mathbf{v}\right)\mathbf{v}} \qquad \textbf{acceleration (spatial description)} \qquad (2.4.5)$$

The acceleration can now be determined, because the derivatives can be determined (measured) without knowing the motion.

In the above, the **material derivative**, or **total derivative**, of the particle's velocity was taken to obtain the acceleration. In general, one can take the time derivative of any physical or kinematic property $(\bullet)$ expressed in the spatial description:

$$\boxed{\frac{d}{dt}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \mathrm{grad}(\bullet)\mathbf{v}} \qquad \textbf{Material Time Derivative} \tag{2.4.6}$$

For example, the rate of change of the density $\rho = \rho(\mathbf{x}, t)$ of a particle instantaneously at $\mathbf{x}$ is

$$\dot{\rho} \equiv \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathrm{grad}\,\rho \cdot \mathbf{v} \tag{2.4.7}$$

The Local Rate of Change

The first term, $\partial\rho/\partial t$, gives the **local rate of change** of density at **x** whereas the second term $\mathbf{v} \cdot \mathrm{grad}\,\rho$ gives the change due to the particle's motion, and is called the **convective rate of change**.

Note the difference between the material derivative and the local derivative. For example, the material derivative of the velocity, 2.4.5 (or, equivalently, $d\mathbf{V}(\mathbf{X}, t)/dt$ in 2.4.4, with **X** fixed) is not the same as the derivative $\partial \mathbf{v}(\mathbf{x}, t)/\partial t$ (with **x** fixed). The former is the acceleration of a material particle **X**. The latter is the time rate of change of the velocity of particles *at a fixed location* in space; in general, *different* material particles will occupy position **x** at different times.

The material derivative d/dt can be applied to any scalar, vector or tensor:

$$\begin{aligned} \dot{\alpha} &\equiv \frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + \mathrm{grad}\,\alpha \cdot \mathbf{v} \\ \dot{\mathbf{a}} &\equiv \frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + (\mathrm{grad}\,\mathbf{a})\mathbf{v} \\ \dot{\mathbf{A}} &\equiv \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathrm{grad}\,\mathbf{A})\mathbf{v} \end{aligned} \tag{2.4.8}$$

Another notation often used for the material derivative is D/Dt:

$$\frac{Df}{Dt} \equiv \frac{df}{dt} \equiv \dot{f} \tag{2.4.9}$$

Steady and Uniform Flows

In a **steady flow**, quantities are independent of time, so the local rate of change is zero and, for example, $\dot{\rho} = \mathrm{grad}\,\rho \cdot \mathbf{v}$. In a **uniform flow**, quantities are independent of position so that, for example, $\dot{\rho} = \partial\rho/\partial t$

Example

Consider again the previous example. This time, with only the velocity $\mathbf{v}(\mathbf{x}, t)$ known, the acceleration can be obtained through the material derivative:

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{v}$$

$$= \frac{\partial}{\partial t}\left(2t\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2t\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2\right) + \begin{bmatrix} -\dfrac{2t^3}{1-t^4} & \dfrac{2t}{1-t^4} & 0 \\ \dfrac{2t}{1-t^4} & -\dfrac{2t^3}{1-t^4} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2t\dfrac{x_2 - t^2 x_1}{1-t^4} \\ 2t\dfrac{x_1 - t^2 x_2}{1-t^4} \\ 0 \end{bmatrix}$$

$$= 2\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2$$

as before.

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The Relationship between the Displacement and Velocity

The velocity can be derived directly from the displacement 2.2.42:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{u}+\mathbf{X})}{dt} = \frac{d\mathbf{u}}{dt}, \tag{2.4.10}$$

or

$$\mathbf{v} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad}\mathbf{u})\mathbf{v} \tag{2.4.11}$$

When the displacement field is given in material form one has

$$\mathbf{V} = \frac{d\mathbf{U}}{dt} \tag{2.4.12}$$

2.4.3 Problems

1. The density of a material is given by

$$\rho = \frac{e^{-2t}}{\mathbf{x}\cdot\mathbf{x}}$$

The velocity field is given by

$$v_1 = x_2 + 2x_3, \quad v_2 = x_3 - 2x_1, \quad v_3 = x_1 + 2x_2$$

Determine the time derivative of the density (a) at a certain position $\mathbf{x}$ in space, and (b) of a material particle instantaneously occupying position $\mathbf{x}$.