1.12 Higher Order Tensors

In this section are discussed some important higher (third and fourth) order tensors.

1.12.1 Fourth Order Tensors

After second-order tensors, the most commonly encountered tensors are the fourth order tensors A, which have 81 components. Some properties and relations involving these tensors are listed here.

Transpose

The transpose of a fourth-order tensor A, denoted by A^{T}, by analogy with the definition for the transpose of a second order tensor 1.10.4, is defined by

$$\mathbf{B} : \mathsf{A}^{\mathrm{T}} : \mathbf{C} = \mathbf{C} : \mathsf{A} : \mathbf{B} \tag{1.12.1}$$

for all second-order tensors **B** and **C**. It has the property $\left(\mathsf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathsf{A}$ and its components are $(\mathsf{A}^{\mathrm{T}})_{ijkl} = (\mathsf{A})_{klij}$. It also follows that

$$\left(\mathbf{A} \otimes \mathbf{B}\right)^{\mathrm{T}} = \mathbf{B} \otimes \mathbf{A} \tag{1.12.2}$$

Identity Tensors

There are two **fourth-order identity tensors**. They are defined as follows:

$$\begin{aligned} \mathsf{I} : \mathbf{A} &= \mathbf{A} \\ \bar{\mathsf{I}} : \mathbf{A} &= \mathbf{A}^{\mathrm{T}} \end{aligned} \tag{1.12.3}$$

And have components

$$\begin{aligned} \mathsf{I} &\equiv \delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j \\ \bar{\mathsf{I}} &\equiv \delta_{il}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i \end{aligned} \tag{1.12.4}$$

For a *symmetric* second order tensor **S**, $\bar{\mathsf{I}} : \mathbf{S} = \mathsf{I} : \mathbf{S} = \mathbf{S}$.

Another important fourth-order tensor is $\mathbf{I} \otimes \mathbf{I}$,

$$\mathbf{I} \otimes \mathbf{I} = \delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \tag{1.12.5}$$

Functions of the trace can be written in terms of these tensors {▲Problem 1}:

$$\begin{aligned} \mathbf{I} \otimes \mathbf{I} : \mathbf{A} &= (\mathrm{tr}\mathbf{A})\mathbf{I} \\ \mathbf{I} \otimes \mathbf{I} : \mathbf{A} : \mathbf{A} &= (\mathrm{tr}\mathbf{A})^2 \\ \mathsf{I} : \mathbf{A} : \mathbf{A} &= \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) \\ \bar{\mathsf{I}} : \mathbf{A} : \mathbf{A} &= \mathrm{tr}\mathbf{A}^2 \end{aligned} \tag{1.12.6}$$

Projection Tensors

The symmetric and skew-symmetric parts of a second order tensor **A** can be written in terms of the identity tensors:

$$\begin{aligned} \mathrm{sym}\mathbf{A} &= \frac{1}{2}\left(\mathsf{I} + \bar{\mathsf{I}}\right) : \mathbf{A} \\ \mathrm{skew}\mathbf{A} &= \frac{1}{2}\left(\mathsf{I} - \bar{\mathsf{I}}\right) : \mathbf{A} \end{aligned} \tag{1.12.7}$$

The deviator of **A**, 1.10.36, can be written as

$$\mathrm{dev}\mathbf{A} = \mathbf{A} - \frac{1}{3}(\mathrm{tr}\mathbf{A})\mathbf{I} = \mathbf{A} - \frac{1}{3}(\mathbf{I} : \mathbf{A})\mathbf{I} = \left(\mathsf{I} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I})\right) : \mathbf{A} \equiv \hat{\mathsf{P}} : \mathbf{A} \tag{1.12.8}$$

which defines $\hat{\mathsf{P}}$, the so-called **fourth-order projection tensor**. From Eqns. 1.10.6, 1.10.37a, it has the property that $\hat{\mathsf{P}} : \mathbf{A} : \mathbf{I} = 0$. Note also that it has the property $\hat{\mathsf{P}}^n = \hat{\mathsf{P}} : \hat{\mathsf{P}} : \cdots : \hat{\mathsf{P}} = \hat{\mathsf{P}}$. For example,

$$\begin{aligned} \hat{\mathsf{P}}^2 = \hat{\mathsf{P}} : \hat{\mathsf{P}} &= \left(\mathsf{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right) : \left(\mathsf{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right) \\ &= \mathsf{I} : \mathsf{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I} + \frac{1}{9}(\mathbf{I} \otimes \mathbf{I}) : (\mathbf{I} \otimes \mathbf{I}) = \hat{\mathsf{P}} \end{aligned} \tag{1.12.9}$$

The tensors $\left(\mathsf{I} + \bar{\mathsf{I}}\right)/2$, $\left(\mathsf{I} - \bar{\mathsf{I}}\right)/2$ in Eqn. 1.12.7 are also projection tensors, projecting the tensor **A** onto its symmetric and skew-symmetric parts.

1.12.2 Higher-Order Tensors and Symmetry

A higher order tensor possesses complete symmetry if the interchange of any indices is immaterial, for example if

$$\mathbf{A} = A_{ijk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = A_{ikj}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = A_{jik}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = \cdots$$

It is symmetric in two of its indices if the interchange of these indices is immaterial. For example the above tensor $\mathbf{A}$ is symmetric in *j* and *k* if

$$\mathbf{A} = A_{ijk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = A_{ikj}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$$

This applies also to antisymmetry. For example, the permutation tensor $\mathbf{E} = \varepsilon_{ijk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$ is completely antisymmetric, since $\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij} = \cdots$.

A fourth-order tensor $\mathbf{C}$ possesses the **minor symmetries** if

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk} \tag{1.12.10}$$

in which case it has only 36 independent components. The first equality here is for left minor symmetry, the second is for right minor symmetry.

It possesses the **major symmetries** if it *also* satisfies

$$C_{ijkl} = C_{klij} \tag{1.12.11}$$

in which case it has only 21 independent components. From 1.12.1, this can also be expressed as

$$\mathbf{A} : \mathbf{C} : \mathbf{B} = \mathbf{B} : \mathbf{C} : \mathbf{A} \tag{1.12.12}$$

for arbitrary second-order tensors **A**, **B**. Note that $\mathbf{I}, \bar{\mathbf{I}}, \mathbf{I} \otimes \mathbf{I}$ possess the major symmetries {▲Problem 2}.

1.12.3 Problems

1. Derive the relations 1.12.6.
2. Use 1.12.12 to show that $\mathbf{I}, \bar{\mathbf{I}}, \mathbf{I} \otimes \mathbf{I}$ possess the major symmetries.