1.10 Special Second Order Tensors & Properties of Second Order Tensors

In this section will be examined a number of special second order tensors, and special properties of second order tensors, which play important roles in tensor analysis. Many of the concepts will be familiar from Linear Algebra and Matrices. The following will be discussed:

- The Identity tensor
- Transpose of a tensor
- Trace of a tensor
- Norm of a tensor
- Determinant of a tensor
- Inverse of a tensor
- Orthogonal tensors
- Rotation Tensors
- Change of Basis Tensors
- Symmetric and Skew-symmetric tensors
- Axial vectors
- Spherical and Deviatoric tensors
- Positive Definite tensors

1.10.1 The Identity Tensor

The linear transformation which transforms every tensor into itself is called the **identity tensor**. This special tensor is denoted by **I** so that, for example,

$$\mathbf{Ia} = \mathbf{a} \quad \text{for any vector a}$$

In particular, $\mathbf{Ie}_1 = \mathbf{e}_1, \mathbf{Ie}_2 = \mathbf{e}_2, \mathbf{Ie}_3 = \mathbf{e}_3$, from which it follows that, for a Cartesian coordinate system, $I_{ij} = \delta_{ij}$. In matrix form,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.10.1}$$

1.10.2 The Transpose of a Tensor

The **transpose** of a second order tensor **A** with components A_{ij} is the tensor $\mathbf{A}^{\mathrm{T}}$ with components A_{ji}; so the transpose swaps the indices,

$$\boxed{\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{A}^{\mathrm{T}} = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j} \quad \textbf{Transpose of a Second-Order Tensor} \quad (1.10.2)$$

In matrix notation,

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \qquad [\mathbf{A}^{\mathrm{T}}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Some useful properties and relations involving the transpose are {▲Problem 2}:

$$\begin{aligned} \left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} &= \mathbf{A} \\ (\alpha\mathbf{A} + \beta\mathbf{B})^{\mathrm{T}} &= \alpha\mathbf{A}^{\mathrm{T}} + \beta\mathbf{B}^{\mathrm{T}} \\ (\mathbf{u} \otimes \mathbf{v})^{\mathrm{T}} &= \mathbf{v} \otimes \mathbf{u} \\ \mathbf{Tu} = \mathbf{uT}^{\mathrm{T}}, \quad \mathbf{uT} &= \mathbf{T}^{\mathrm{T}}\mathbf{u} \\ (\mathbf{AB})^{\mathrm{T}} &= \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \\ \mathbf{A} : \mathbf{B} &= \mathbf{A}^{\mathrm{T}} : \mathbf{B}^{\mathrm{T}} \\ (\mathbf{u} \otimes \mathbf{v})\mathbf{A} &= \mathbf{u} \otimes (\mathbf{A}^{\mathrm{T}}\mathbf{v}) \\ \mathbf{A} : (\mathbf{BC}) = \left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right) &: \mathbf{C} = \left(\mathbf{AC}^{\mathrm{T}}\right) : \mathbf{B} \end{aligned} \tag{1.10.3}$$

A formal definition of the transpose which does not rely on any particular coordinate system is as follows: the transpose of a second-order tensor is that tensor which satisfies the identity[1]

$$\mathbf{u} \cdot \mathbf{Av} = \mathbf{v} \cdot \mathbf{A}^{\mathrm{T}}\mathbf{u} \tag{1.10.4}$$

for all vectors **u** and **v**. To see that Eqn. 1.10.4 implies 1.10.2, first note that, for the present purposes, a convenient way of writing the components A_{ij} of the second-order tensor **A** is $(\mathbf{A})_{ij}$. From Eqn. 1.9.4, $(\mathbf{A})_{ij} = \mathbf{e}_i \cdot \mathbf{Ae}_j$ and the components of the transpose can be written as $\left(\mathbf{A}^{\mathrm{T}}\right)_{ij} = \mathbf{e}_i \cdot \mathbf{A}^{\mathrm{T}}\mathbf{e}_j$. Then, from 1.10.4,
$\left(\mathbf{A}^{\mathrm{T}}\right)_{ij} = \mathbf{e}_i \cdot \mathbf{A}^{\mathrm{T}}\mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{Ae}_i = (\mathbf{A})_{ji} = A_{ji}$.

1.10.3 The Trace of a Tensor

The **trace** of a second order tensor **A**, denoted by $\mathrm{tr}\mathbf{A}$, is a scalar equal to the sum of the diagonal elements of its matrix representation. Thus (see Eqn. 1.4.3)

$$\boxed{\mathrm{tr}\mathbf{A} = A_{ii}} \qquad \textbf{Trace} \tag{1.10.5}$$

A more formal definition, again not relying on any particular coordinate system, is

$$\boxed{\mathrm{tr}\mathbf{A} = \mathbf{I} : \mathbf{A}} \qquad \textbf{Trace} \tag{1.10.6}$$

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[1] as mentioned in §1.9, from the linearity of tensors, $\mathbf{uA} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{Av}$ and, for this reason, this expression is usually written simply as $\mathbf{uAv}$

and Eqn. 1.10.5 follows from 1.10.6 {▲Problem 4}. For the dyad $\mathbf{u}\otimes\mathbf{v}$ {▲Problem 5},

$$\text{tr}(\mathbf{u}\otimes\mathbf{v}) = \mathbf{u}\cdot\mathbf{v} \tag{1.10.7}$$

Another example is

$$\begin{aligned}\text{tr}(\mathbf{E}^2) &= \mathbf{I}:\mathbf{E}^2 \\ &= \delta_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j):E_{pq}E_{qr}(\mathbf{e}_p\otimes\mathbf{e}_r) \\ &= E_{iq}E_{qi}\end{aligned} \tag{1.10.8}$$

This and other important traces, and functions of the trace are listed here {▲Problem 6}:

$$\begin{aligned}\text{tr}\mathbf{A} &= A_{ii} \\ \text{tr}\mathbf{A}^2 &= A_{ij}A_{ji} \\ \text{tr}\mathbf{A}^3 &= A_{ij}A_{jk}A_{ki} \\ (\text{tr}\mathbf{A})^2 &= A_{ii}A_{jj} \\ (\text{tr}\mathbf{A})^3 &= A_{ii}A_{jj}A_{kk}\end{aligned} \tag{1.10.9}$$

Some useful properties and relations involving the trace are {▲Problem 7}:

$$\begin{aligned}\text{tr}\mathbf{A}^{\mathrm{T}} &= \text{tr}\mathbf{A} \\ \text{tr}(\mathbf{AB}) &= \text{tr}(\mathbf{BA}) \\ \text{tr}(\mathbf{A}+\mathbf{B}) &= \text{tr}\mathbf{A}+\text{tr}\mathbf{B} \\ \text{tr}(\alpha\mathbf{A}) &= \alpha\text{tr}\mathbf{A} \\ \mathbf{A}:\mathbf{B} = \text{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) &= \text{tr}(\mathbf{AB}^{\mathrm{T}}) = \text{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = \text{tr}(\mathbf{BA}^{\mathrm{T}})\end{aligned} \tag{1.10.10}$$

The double contraction of two tensors was earlier defined with respect to Cartesian coordinates, Eqn. 1.9.16. This last expression allows one to re-define the double contraction in terms of the trace, independent of any coordinate system.

Consider again the real vector space of second order tensors V^2 introduced in §1.8.5. The double contraction of two tensors as defined by 1.10.10e clearly satisfies the requirements of an inner product listed in §1.2.2. Thus this scalar quantity serves as an inner product for the space V^2:

$$\langle\mathbf{A},\mathbf{B}\rangle \equiv \mathbf{A}:\mathbf{B} = \text{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) \tag{1.10.11}$$

and generates an inner product space.

Just as the base vectors $\{\mathbf{e}_i\}$ form an orthonormal set in the inner product (vector dot product) of the space of vectors *V*, so the base dyads $\{\mathbf{e}_i\otimes\mathbf{e}_j\}$ form an orthonormal set in the inner product 1.10.11 of the space of second order tensors V^2. For example,

$$\langle \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_1 \rangle = (\mathbf{e}_1 \otimes \mathbf{e}_1) : (\mathbf{e}_1 \otimes \mathbf{e}_1) = 1 \qquad (1.10.12)$$

Similarly, just as the dot product is zero for orthogonal vectors, when the double contraction of two tensors **A** and **B** is zero, one says that the tensors are **orthogonal**,

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = 0, \qquad \mathbf{A}, \mathbf{B} \text{ orthogonal} \qquad (1.10.13)$$

1.10.4 The Norm of a Tensor

Using 1.2.8 and 1.10.11, the **norm** of a second order tensor **A**, denoted by $|\mathbf{A}|$ (or $\|\mathbf{A}\|$), is defined by

$$|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}} \qquad (1.10.14)$$

This is analogous to the norm $|\mathbf{a}|$ of a vector **a**, $\sqrt{\mathbf{a} \cdot \mathbf{a}}$.

1.10.5 The Determinant of a Tensor

The **determinant** of a second order tensor **A** is defined to be the determinant of the matrix $[\mathbf{A}]$ of components of the tensor:

$$\begin{aligned} \det \mathbf{A} &= \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \\ &= \varepsilon_{ijk} A_{i1} A_{j2} A_{k3} \\ &= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} \end{aligned} \qquad (1.10.15)$$

Some useful properties of the determinant are {▲Problem 8}

$$\begin{aligned} \det(\mathbf{AB}) &= \det \mathbf{A} \det \mathbf{B} \\ \det \mathbf{A}^{\mathrm{T}} &= \det \mathbf{A} \\ \det(\alpha \mathbf{A}) &= \alpha^3 \det \mathbf{A} \\ \det(\mathbf{u} \otimes \mathbf{v}) &= 0 \\ \varepsilon_{pqr}(\det \mathbf{A}) &= \varepsilon_{ijk} A_{ip} A_{jq} A_{kr} \\ (\mathbf{Ta} \times \mathbf{Tb}) \cdot \mathbf{Tc} &= (\det \mathbf{T})[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}] \end{aligned} \qquad (1.10.16)$$

Note that $\det \mathbf{A}$, like $\mathrm{tr}\mathbf{A}$, is independent of the choice of coordinate system / basis.

1.10.6 The Inverse of a Tensor

The **inverse** of a second order tensor **A**, denoted by $\mathbf{A}^{-1}$, is defined by

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} \qquad (1.10.17)$$

The inverse of a tensor exists only if it is **non-singular** (a **singular** tensor is one for which $\det \mathbf{A} = 0$), in which case it is said to be **invertible**.

Some useful properties and relations involving the inverse are:

$$\begin{aligned} (\mathbf{A}^{-1})^{-1} &= \mathbf{A} \\ (\alpha\mathbf{A})^{-1} &= (1/\alpha)\mathbf{A}^{-1} \\ (\mathbf{A}\mathbf{B})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ \det(\mathbf{A}^{-1}) &= (\det \mathbf{A})^{-1} \end{aligned} \qquad (1.10.18)$$

Since the inverse of the transpose is equivalent to the transpose of the inverse, the following notation is used:

$$\mathbf{A}^{-\mathrm{T}} \equiv (\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1} \qquad (1.10.19)$$

1.10.7 Orthogonal Tensors

An **orthogonal** tensor **Q** is a linear vector transformation satisfying the condition

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u} \cdot \mathbf{v} \qquad (1.10.20)$$

for all vectors **u** and **v**. Thus **u** is transformed to $\mathbf{Qu}$, **v** is transformed to $\mathbf{Qv}$ and the dot product $\mathbf{u} \cdot \mathbf{v}$ is invariant under the transformation. Thus the magnitude of the vectors and the angle between the vectors is preserved, Fig. 1.10.1.

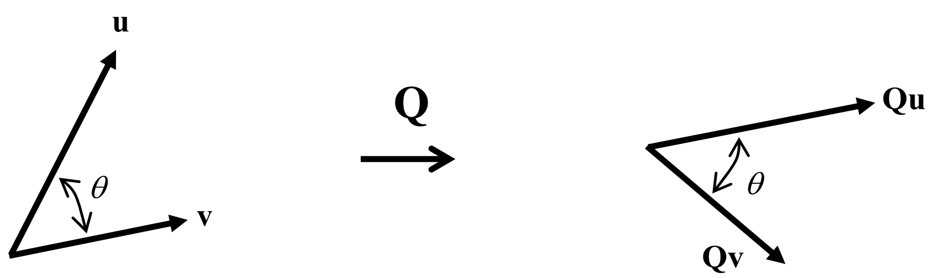

Figure 1.10.1: An orthogonal tensor

Since

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u}\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{Qv} = \mathbf{u} \cdot \left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) \cdot \mathbf{v} \qquad (1.10.21)$$

it follows that for $\mathbf{u}\cdot\mathbf{v}$ to be preserved under the transformation, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}=\mathbf{I}$, which is also used as the definition of an orthogonal tensor. Some useful properties of orthogonal tensors are {▲Problem 10}:

$$\begin{aligned} \mathbf{Q}\mathbf{Q}^{\mathrm{T}} &= \mathbf{I} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}, \qquad Q_{ik}Q_{jk} = \delta_{ij} = Q_{ki}Q_{kj} \\ \mathbf{Q}^{-1} &= \mathbf{Q}^{\mathrm{T}} \\ \det\mathbf{Q} &= \pm 1 \end{aligned} \tag{1.10.22}$$

1.10.8 Rotation Tensors

If for an orthogonal tensor, $\det\mathbf{Q}=+1$, $\mathbf{Q}$ is said to be a **proper** orthogonal tensor, corresponding to a **rotation**. If $\det\mathbf{Q}=-1$, $\mathbf{Q}$ is said to be an **improper** orthogonal tensor, corresponding to a **reflection**. Proper orthogonal tensors are also called **rotation tensors**.

1.10.9 Change of Basis Tensors

Consider a rotation tensor $\mathbf{Q}$ which rotates the base vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ into a second set, $\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3$, Fig. 1.10.2.

$$\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i \qquad i = 1, 2, 3 \tag{1.10.23}$$

Such a tensor can be termed a **change of basis tensor** from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$. The transpose $\mathbf{Q}^{\mathrm{T}}$ rotates the base vectors $\mathbf{e}'_i$ back to $\mathbf{e}_i$ and is thus **change of basis tensor** from $\{\mathbf{e}'_i\}$ to $\{\mathbf{e}_i\}$. The components of $\mathbf{Q}$ in the $\mathbf{e}_i$ coordinate system are, from 1.9.4, $Q_{ij}=\mathbf{e}_i\mathbf{Q}\mathbf{e}_j$ and so, from 1.10.23,

$$\mathbf{Q} = Q_{ij}\mathbf{e}_i\otimes\mathbf{e}_j, \qquad Q_{ij} = \mathbf{e}_i\cdot\mathbf{e}'_j, \tag{1.10.24}$$

which are the direction cosines between the axes (see Fig. 1.5.4).

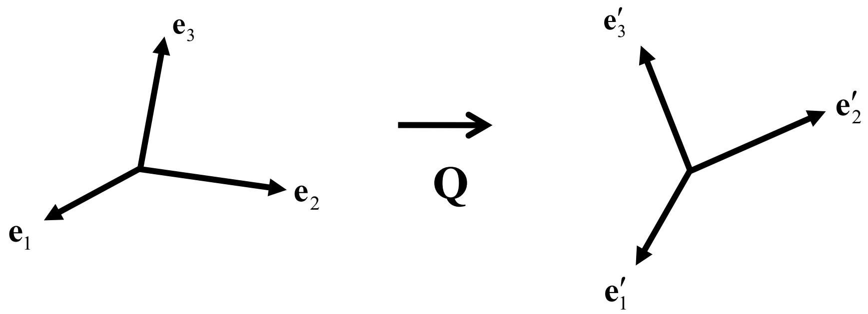

Figure 1.10.2: Rotation of a set of base vectors

The change of basis tensor can also be expressed in terms of the base vectors from *both* bases:

$$\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{e}_i , \tag{1.10.25}$$

from which the above relations can easily be derived, for example $\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i$, $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$, etc.

Consider now the operation of the change of basis tensor on a vector:

$$\mathbf{Q}\mathbf{v} = v_i(\mathbf{Q}\mathbf{e}_i) = v_i\mathbf{e}'_i \tag{1.10.26}$$

Thus $\mathbf{Q}$ transforms $\mathbf{v}$ into a second vector $\mathbf{v}'$, but this new vector has the *same components* with respect to the basis $\mathbf{e}'_i$, as $\mathbf{v}$ has with respect to the basis $\mathbf{e}_i$, $v'_i = v_i$. Note the difference between this and the coordinate transformations of §1.5: here there are two different vectors, $\mathbf{v}$ and $\mathbf{v}'$.

Example

Consider the two-dimensional rotation tensor

$$\mathbf{Q} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} (\mathbf{e}_i \otimes \mathbf{e}_j) \equiv \mathbf{e}'_i \otimes \mathbf{e}_i$$

which corresponds to a rotation of the base vectors through $\pi/2$. The vector $\mathbf{v} = [1 \quad 1]^{\mathrm{T}}$ then transforms into (see Fig. 1.10.3)

$$\mathbf{Q}\mathbf{v} = \begin{bmatrix} -1 \\ +1 \end{bmatrix} \mathbf{e}_i = \begin{bmatrix} +1 \\ +1 \end{bmatrix} \mathbf{e}'_i$$

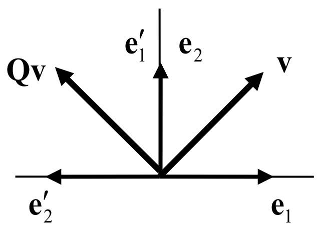

Figure 1.10.3: a rotated vector

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Similarly, for a second order tensor $\mathbf{A}$, the operation

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{Q}^{\mathrm{T}} = A_{ij}(\mathbf{Q}\mathbf{e}_i \otimes \mathbf{e}_j\mathbf{Q}^{\mathrm{T}}) = A_{ij}(\mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_j) = A_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j \tag{1.10.27}$$

results in a new tensor which has the same components with respect to the $\mathbf{e}_i'$, as $\mathbf{A}$ has with respect to the $\mathbf{e}_i$, $A_{ij}' = A_{ij}$.

1.10.10 Symmetric and Skew Tensors

A tensor $\mathbf{T}$ is said to be **symmetric** if it is identical to the transposed tensor, $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$, and **skew** (**antisymmetric**) if $\mathbf{T} = -\mathbf{T}^{\mathrm{T}}$.

Any tensor $\mathbf{A}$ can be (uniquely) decomposed into a symmetric tensor $\mathbf{S}$ and a skew tensor $\mathbf{W}$, where

$$\begin{aligned} \mathrm{sym}\mathbf{A} \equiv \mathbf{S} &= \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) \\ \mathrm{skew}\mathbf{A} \equiv \mathbf{W} &= \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right) \end{aligned} \tag{1.10.28}$$

and

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}}, \qquad \mathbf{W} = -\mathbf{W}^{\mathrm{T}} \tag{1.10.29}$$

In matrix notation one has

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}, \qquad [W] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} \tag{1.10.30}$$

Some useful properties of symmetric and skew tensors are {▲Problem 13}:

$$\begin{aligned} &\mathbf{S} : \mathbf{B} = \mathbf{S} : \mathbf{B}^{\mathrm{T}} = \mathbf{S} : \tfrac{1}{2}\left(\mathbf{B} + \mathbf{B}^{\mathrm{T}}\right) \\ &\mathbf{W} : \mathbf{B} = -\mathbf{W} : \mathbf{B}^{\mathrm{T}} = \mathbf{W} : \tfrac{1}{2}\left(-\mathbf{B}^{\mathrm{T}}\right) \\ &\mathbf{S} : \mathbf{W} = 0 \\ &\mathrm{tr}(\mathbf{SW}) = 0 \\ &\mathbf{v} \cdot \mathbf{Wv} = 0 \\ &\det \mathbf{W} = 0 \quad (\text{has no inverse}) \end{aligned} \tag{1.10.31}$$

where $\mathbf{v}$ and $\mathbf{B}$ denote any arbitrary vector and second-order tensor respectively.

Note that symmetry and skew-symmetry are tensor properties, independent of coordinate system.

1.10.11 Axial Vectors

A skew tensor **W** has only three independent coefficients, so it behaves "like a vector" with three components. Indeed, a skew tensor can always be written in the form

$$\mathbf{Wu} = \boldsymbol{\omega} \times \mathbf{u} \tag{1.10.32}$$

where **u** is any vector and $\boldsymbol{\omega}$ characterises the **axial** (or **dual**) vector of the skew tensor **W**. The components of **W** can be obtained from the components of $\boldsymbol{\omega}$ through

$$\begin{aligned} W_{ij} &= \mathbf{e}_i \cdot \mathbf{W}\mathbf{e}_j = \mathbf{e}_i \cdot \left(\boldsymbol{\omega} \times \mathbf{e}_j\right) = \mathbf{e}_i \cdot \left(\omega_k \mathbf{e}_k \times \mathbf{e}_j\right) \\ &= \mathbf{e}_i \cdot \left(\omega_k \varepsilon_{kjp} \mathbf{e}_p\right) = \varepsilon_{kji}\omega_k \\ &= -\varepsilon_{ijk}\omega_k \end{aligned} \tag{1.10.33}$$

If one knows the components of **W**, one can find the components of $\boldsymbol{\omega}$ by inverting this equation, whence {▲Problem 14}

$$\boldsymbol{\omega} = -W_{23}\mathbf{e}_1 + W_{13}\mathbf{e}_2 - W_{12}\mathbf{e}_3 \tag{1.10.34}$$

Example (of an Axial Vector)

Decompose the tensor

$$\mathbf{T} = \left[T_{ij}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

into its symmetric and skew parts. Also find the axial vector for the skew part. Verify that $\mathbf{Wa} = \boldsymbol{\omega} \times \mathbf{a}$ for $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_3$.

<u>Solution</u>

One has

$$\mathbf{S} = \frac{1}{2}\left[\mathbf{T} + \mathbf{T}^{\mathrm{T}}\right] = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\mathbf{W} = \frac{1}{2}\left[\mathbf{T} - \mathbf{T}^{\mathrm{T}}\right] = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The axial vector is

$$\boldsymbol{\omega} = -W_{23}\mathbf{e}_1 + W_{13}\mathbf{e}_2 - W_{12}\mathbf{e}_3 = \mathbf{e}_2 + \mathbf{e}_3$$

and it can be seen that

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{or} \quad \begin{aligned} \mathbf{Wa} &= W_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_1 + \mathbf{e}_3) = W_{ij}(\delta_{j1} + \delta_{j3})\mathbf{e}_i = (W_{i1} + W_{i3})\mathbf{e}_i \\ &= (W_{11} + W_{13})\mathbf{e}_1 + (W_{21} + W_{23})\mathbf{e}_2 + (W_{31} + W_{33})\mathbf{e}_3 \\ &= \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 \end{aligned}$$

and

$$\boldsymbol{\omega} \times \mathbf{a} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$$

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The Spin Tensor

The velocity of a particle rotating in a rigid body motion is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$, where $\boldsymbol{\omega}$ is the angular velocity vector and **x** is the position vector relative to the origin on the axis of rotation (see Problem 9, §1.1). If the velocity can be written in terms of a skew-symmetric second order tensor $\mathbf{w}$, such that $\mathbf{wx} = \mathbf{v}$, then it follows from $\mathbf{wx} = \boldsymbol{\omega} \times \mathbf{x}$ that the angular velocity vector $\boldsymbol{\omega}$ is the axial vector of $\mathbf{w}$. In this context, $\mathbf{w}$ is called the **spin tensor**.

1.10.12 Spherical and Deviatoric Tensors

Every tensor **A** can be decomposed into its so-called **spherical** part and its **deviatoric** part, i.e.

$$\mathbf{A} = \mathrm{sph}\mathbf{A} + \mathrm{dev}\mathbf{A} \tag{1.10.35}$$

where

$$\begin{aligned} \mathrm{sph}\mathbf{A} &= \tfrac{1}{3}(\mathrm{tr}\mathbf{A})\mathbf{I} \\ &= \begin{bmatrix} \tfrac{1}{3}(A_{11} + A_{22} + A_{33}) & 0 & 0 \\ 0 & \tfrac{1}{3}(A_{11} + A_{22} + A_{33}) & 0 \\ 0 & 0 & \tfrac{1}{3}(A_{11} + A_{22} + A_{33}) \end{bmatrix} \\ \mathrm{dev}\mathbf{A} &= \mathbf{A} - \mathrm{sph}\mathbf{A} \\ &= \begin{bmatrix} A_{11} - \tfrac{1}{3}(A_{11} + A_{22} + A_{33}) & A_{12} & A_{13} \\ A_{21} & A_{22} - \tfrac{1}{3}(A_{11} + A_{22} + A_{33}) & A_{23} \\ A_{31} & A_{32} & A_{33} - \tfrac{1}{3}(A_{11} + A_{22} + A_{33}) \end{bmatrix} \end{aligned} \tag{1.10.36}$$

Any tensor of the form $\alpha\mathbf{I}$ is known as a **spherical tensor**, while $\text{dev}\mathbf{A}$ is known as a deviator of **A**, or a **deviatoric tensor**.

Some important properties of the spherical and deviatoric tensors are

$$\begin{aligned} \text{tr}(\text{dev}\mathbf{A}) &= 0 \\ \text{sph}(\text{dev}\mathbf{A}) &= 0 \\ \text{dev}\mathbf{A} : \text{sph}\mathbf{B} &= 0 \end{aligned} \tag{1.10.37}$$

1.10.13 Positive Definite Tensors

A **positive definite** tensor **A** is one which satisfies the relation

$$\mathbf{vAv} > 0, \qquad \forall\, \mathbf{v} \neq \mathbf{o} \tag{1.10.38}$$

The tensor is called **positive semi-definite** if $\mathbf{vAv} \geq 0$.

In component form,

$$v_i A_{ij} v_j = A_{11}v_1^2 + A_{12}v_1v_2 + A_{13}v_1v_3 + A_{21}v_2v_1 + A_{22}v_2^2 + \cdots \tag{1.10.39}$$

and so the diagonal elements of the matrix representation of a positive definite tensor must always be positive.

It can be shown that the following conditions are necessary for a tensor **A** to be positive definite (although they are not sufficient):

(i) the diagonal elements of $[\mathbf{A}]$ are positive
(ii) the largest element of $[\mathbf{A}]$ lies along the diagonal
(iii) $\det\mathbf{A} > 0$
(iv) $A_{ii} + A_{jj} > 2A_{ij}$ for $i \neq j$ (no sum over i, j)

These conditions are seen to hold for the following matrix representation of an example positive definite tensor:

$$[\mathbf{A}] = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A necessary and sufficient condition for a tensor to be positive definite is given in the next section, during the discussion of the eigenvalue problem.

One of the key properties of a positive definite tensor is that, since $\det\mathbf{A} > 0$, positive definite tensors are always invertible.

An alternative definition of positive definiteness is the equivalent expression

$$\mathbf{A} : \mathbf{v} \otimes \mathbf{v} > 0 \qquad (1.10.40)$$

1.10.14 Problems

1. Show that the components of the (second-order) identity tensor are given by $I_{ij} = \delta_{ij}$.
2. Show that
   (a) $(\mathbf{u} \otimes \mathbf{v})\mathbf{A} = \mathbf{u} \otimes (\mathbf{A}^{\mathrm{T}}\mathbf{v})$
   (b) $\mathbf{A} : (\mathbf{BC}) = (\mathbf{B}^{\mathrm{T}}\mathbf{A}) : \mathbf{C} = (\mathbf{AC}^{\mathrm{T}}) : \mathbf{B}$
3. Use (1.10.4) to show that $\mathbf{I}^{\mathrm{T}} = \mathbf{I}$.
4. Show that (1.10.6) implies (1.10.5) for the trace of a tensor.
5. Show that $\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.
6. Formally derive the index notation for the functions
   $\mathrm{tr}\mathbf{A}^2, \quad \mathrm{tr}\mathbf{A}^3, \quad (\mathrm{tr}\mathbf{A})^2, \quad (\mathrm{tr}\mathbf{A})^3$
7. Show that $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$.
8. Prove (1.10.16f), $(\mathbf{Ta} \times \mathbf{Tb}) \cdot \mathbf{Tc} = (\det \mathbf{T})[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$.
9. Show that $(\mathbf{A}^{-1})^{\mathrm{T}} : \mathbf{A} = 3$. [Hint: one way of doing this is using the result from Problem 7.]
10. Use 1.10.16b and 1.10.18d to prove 1.10.22c, $\det \mathbf{Q} = \pm 1$.
11. Use the explicit dyadic representation of the rotation tensor, $\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{e}_i$, to show that the components of $\mathbf{Q}$ in the "second", $ox'_1x'_2x'_3$, coordinate system are the same as those in the first system [hint: use the rule $Q'_{ij} = \mathbf{e}'_i \cdot \mathbf{Q}\mathbf{e}'_j$]
12. Consider the tensor $\mathbf{D}$ with components (in a certain coordinate system)
$$\begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \end{bmatrix}$$
    Show that $\mathbf{D}$ is a rotation tensor (just show that $\mathbf{D}$ is proper orthogonal).
13. Show that $\mathrm{tr}(\mathbf{SW}) = 0$.
14. Multiply across (1.10.32), $W_{ij} = -\varepsilon_{ijk}\omega_k$, by ε_{ijp} to show that $\boldsymbol{\omega} = -\frac{1}{2}\varepsilon_{ijk}W_{ij}\mathbf{e}_k$. [Hint: use the relation 1.3.19b, $\varepsilon_{ijp}\varepsilon_{ijk} = 2\delta_{pk}$.]
15. Show that $\frac{1}{2}(\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a})$ is a skew tensor $\mathbf{W}$. Show that its axial vector is $\boldsymbol{\omega} = \frac{1}{2}(\mathbf{b} \times \mathbf{a})$. [Hint: first prove that $(\mathbf{b} \cdot \mathbf{u})\mathbf{a} - (\mathbf{a} \cdot \mathbf{u})\mathbf{b} = \mathbf{u} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{a}) \times \mathbf{u}$.]
16. Find the spherical and deviatoric parts of the tensor $\mathbf{A}$ for which $A_{ij} = 1$.