1.9 Cartesian Tensors

As with the vector, a (higher order) tensor is a mathematical object which represents many physical phenomena and which exists independently of any coordinate system. In what follows, a Cartesian coordinate system is used to describe tensors.

1.9.1 Cartesian Tensors

A second order tensor and the vector it operates on can be described in terms of Cartesian components. For example, $(\mathbf{a} \otimes \mathbf{b})\mathbf{c}$, with $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{c} = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, is

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = 4\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$$

Example (The Unit Dyadic or Identity Tensor)

The **identity tensor**, or **unit tensor**, **I**, which maps every vector onto itself, has been introduced in the previous section. The Cartesian representation of **I** is

$$\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 \equiv \mathbf{e}_i \otimes \mathbf{e}_i \tag{1.9.1}$$

This follows from

$$\begin{aligned}
(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{u} &= (\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u} + (\mathbf{e}_2 \otimes \mathbf{e}_2)\mathbf{u} + (\mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{u} \\
&= \mathbf{e}_1(\mathbf{e}_1 \cdot \mathbf{u}) + \mathbf{e}_2(\mathbf{e}_2 \cdot \mathbf{u}) + \mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{u}) \\
&= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \\
&= \mathbf{u}
\end{aligned}$$

Note also that the identity tensor can be written as $\mathbf{I} = \delta_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)$, in other words the Kronecker delta gives the components of the identity tensor in a Cartesian coordinate system.

■

Second Order Tensor as a Dyadic

In what follows, it will be shown that a second order tensor can always be written as a dyadic involving the Cartesian base vectors $\mathbf{e}_i$ [1].

Consider an arbitrary second-order tensor **T** which operates on **a** to produce **b**, $\mathbf{T}(\mathbf{a}) = \mathbf{b}$, or $\mathbf{T}(a_i\mathbf{e}_i) = \mathbf{b}$. From the linearity of **T**,

[1] this can be generalised to the case of non-Cartesian base vectors, which might not be orthogonal nor of unit magnitude (see §1.16)

$$a_1\mathbf{T}(\mathbf{e}_1) + a_2\mathbf{T}(\mathbf{e}_2) + a_3\mathbf{T}(\mathbf{e}_3) = \mathbf{b}$$

Just as **T** transforms **a** into **b**, it transforms the base vectors $\mathbf{e}_i$ into some other vectors; suppose that $\mathbf{T}(\mathbf{e}_1) = \mathbf{u}$, $\mathbf{T}(\mathbf{e}_2) = \mathbf{v}$, $\mathbf{T}(\mathbf{e}_3) = \mathbf{w}$, then

$$\begin{aligned}\mathbf{b} &= a_1\mathbf{u} + a_2\mathbf{v} + a_3\mathbf{w} \\ &= (\mathbf{a}\cdot\mathbf{e}_1)\mathbf{u} + (\mathbf{a}\cdot\mathbf{e}_2)\mathbf{v} + (\mathbf{a}\cdot\mathbf{e}_3)\mathbf{w} \\ &= (\mathbf{u}\otimes\mathbf{e}_1)\mathbf{a} + (\mathbf{v}\otimes\mathbf{e}_2)\mathbf{a} + (\mathbf{w}\otimes\mathbf{e}_3)\mathbf{a} \\ &= [\mathbf{u}\otimes\mathbf{e}_1 + \mathbf{v}\otimes\mathbf{e}_2 + \mathbf{w}\otimes\mathbf{e}_3]\mathbf{a}\end{aligned}$$

and so

$$\mathbf{T} = \mathbf{u}\otimes\mathbf{e}_1 + \mathbf{v}\otimes\mathbf{e}_2 + \mathbf{w}\otimes\mathbf{e}_3 \tag{1.9.2}$$

which is indeed a dyadic.

Cartesian components of a Second Order Tensor

The second order tensor **T** can be written in terms of components and base vectors as follows: write the vectors **u**, **v** and **w** in (1.9.2) in component form, so that

$$\begin{aligned}\mathbf{T} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)\otimes\mathbf{e}_1 + (\cdots)\otimes\mathbf{e}_2 + (\cdots)\otimes\mathbf{e}_3 \\ &= u_1\mathbf{e}_1\otimes\mathbf{e}_1 + u_2\mathbf{e}_2\otimes\mathbf{e}_1 + u_3\mathbf{e}_3\otimes\mathbf{e}_1 + \cdots\end{aligned}$$

Introduce nine scalars T_{ij} by letting $u_i = T_{i1}$, $v_i = T_{i2}$, $w_i = T_{i3}$, so that

$$\boxed{\begin{aligned}\mathbf{T} = {} & T_{11}\mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}\mathbf{e}_1\otimes\mathbf{e}_2 + T_{13}\mathbf{e}_1\otimes\mathbf{e}_3 \\ & + T_{21}\mathbf{e}_2\otimes\mathbf{e}_1 + T_{22}\mathbf{e}_2\otimes\mathbf{e}_2 + T_{23}\mathbf{e}_2\otimes\mathbf{e}_3 \\ & + T_{31}\mathbf{e}_3\otimes\mathbf{e}_1 + T_{32}\mathbf{e}_3\otimes\mathbf{e}_2 + T_{33}\mathbf{e}_3\otimes\mathbf{e}_3\end{aligned}}\quad \textbf{Second-order Cartesian Tensor} \tag{1.9.3}$$

These nine scalars T_{ij} are the components of the second order tensor **T** in the Cartesian coordinate system. In index notation,

$$\mathbf{T} = T_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j)$$

Thus whereas a vector has three components, a second order tensor has *nine* components. Similarly, whereas the three vectors $\{\mathbf{e}_i\}$ form a basis for the space of vectors, the nine dyads $\{\mathbf{e}_i\otimes\mathbf{e}_j\}$ form a basis for the space of tensors, i.e. all second order tensors can be expressed as a linear combination of these basis tensors.

It can be shown that the components of a second-order tensor can be obtained directly from {▲Problem 1}

$$\boxed{T_{ij} = \mathbf{e}_i\mathbf{T}\mathbf{e}_j} \qquad \textbf{Components of a Tensor} \tag{1.9.4}$$

which is the tensor expression analogous to the vector expression $u_i = \mathbf{e}_i \cdot \mathbf{u}$. Note that, in Eqn. 1.9.4, the components can be written simply as $\mathbf{e}_i \mathbf{T} \mathbf{e}_j$ (without a "dot"), since $\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \mathbf{e}_i \mathbf{T} \cdot \mathbf{e}_j$.

Example (The Stress Tensor)

Define the traction vector **t** acting on a surface element within a material to be the force acting on that element[2] divided by the area of the element, Fig. 1.9.1. Let **n** be a vector normal to the surface. The **stress** $\boldsymbol{\sigma}$ is defined to be that second order tensor which maps **n** onto **t**, according to

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}} \qquad \textbf{The Stress Tensor} \qquad (1.9.5)$$

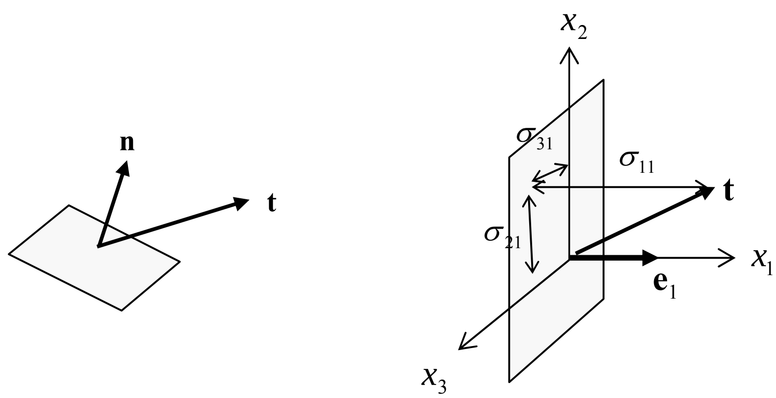

Figure 1.9.1: stress acting on a plane

If one now considers a coordinate system with base vectors $\mathbf{e}_i$, then $\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and, for example,

$$\boldsymbol{\sigma}\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3$$

Thus the components σ_{11}, σ_{21} and σ_{31} of the stress tensor are the three components of the traction vector which acts on the plane with normal $\mathbf{e}_1$.

Augustin-Louis Cauchy was the first to regard stress as a linear map of the normal vector onto the traction vector; hence the name "tensor", from the French for stress, *tension*.

■

[2] this force would be due, for example, to intermolecular forces within the material: the particles on one side of the surface element exert a force on the particles on the other side

Higher Order Tensors

The above can be generalised to tensors of order three and higher. The following notation will be used:

α, β, γ	…	0th-order tensors	("scalars")
$\mathbf{a}, \mathbf{b}, \mathbf{c}$	…	1st-order tensors	("vectors")
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	…	2nd-order tensors	("dyadics")
$\boldsymbol{\mathtt{A}}, \boldsymbol{\mathtt{B}}, \boldsymbol{\mathtt{C}}$	…	3rd-order tensors	("triadics")
$\boldsymbol{\mathsf{A}}, \boldsymbol{\mathsf{B}}, \boldsymbol{\mathsf{C}}$	…	4th-order tensors	("tetradics")

An important third-order tensor is the **permutation tensor**, defined by

$$\boldsymbol{\mathtt{E}} = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{1.9.6}$$

whose components are those of the permutation symbol, Eqns. 1.3.10-1.3.13.

A fourth-order tensor can be written as

$$\boldsymbol{\mathsf{A}} = A_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \tag{1.9.7}$$

It can be seen that a zeroth-order tensor (scalar) has $3^0 = 1$ component, a first-order tensor has $3^1 = 3$ components, a second-order tensor has $3^2 = 9$ components, so $\boldsymbol{\mathtt{A}}$ has $3^3 = 27$ components and $\boldsymbol{\mathsf{A}}$ has 81 components.

1.9.2 Simple Contraction

Tensor/vector operations can be written in component form, for example,

$$\begin{aligned} \mathbf{Ta} &= T_{ij} \left(\mathbf{e}_i \otimes \mathbf{e}_j \right) a_k \mathbf{e}_k \\ &= T_{ij} a_k \left[\left(\mathbf{e}_i \otimes \mathbf{e}_j \right) \mathbf{e}_k \right] \\ &= T_{ij} a_k \delta_{jk} \mathbf{e}_i \\ &= T_{ij} a_j \mathbf{e}_i \end{aligned} \tag{1.9.8}$$

This operation is called **simple contraction**, because the order of the tensors is contracted – to begin there was a tensor of order 2 and a tensor of order 1, and to end there is a tensor of order 1 (it is called "simple" to distinguish it from "double" contraction – see below). This is always the case – when a tensor operates on another in this way, the order of the result will be *two* less than the sum of the original orders.

An example of simple contraction of two second order tensors has already been seen in Eqn. 1.8.4a; the tensors there were simple tensors (dyads). Here is another example:

$$
\begin{aligned}
\mathbf{TS} &= T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) S_{kl}(\mathbf{e}_k \otimes \mathbf{e}_l) \\
&= T_{ij} S_{kl} \left[(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_k \otimes \mathbf{e}_l) \right] \\
&= T_{ij} S_{kl} \delta_{jk} (\mathbf{e}_i \otimes \mathbf{e}_l) \\
&= T_{ij} S_{jl} (\mathbf{e}_i \otimes \mathbf{e}_l)
\end{aligned}
\tag{1.9.9}
$$

From the above, the simple contraction of two second order tensors results in another second order tensor. If one writes $\mathbf{A} = \mathbf{TS}$, then the components of the new tensor are related to those of the original tensors through $A_{ij} = T_{ik} S_{kj}$.

Note that, in general,

$$
\begin{aligned}
\mathbf{AB} &\neq \mathbf{BA} \\
(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}) && \text{… associative} \\
\mathbf{A}(\mathbf{B} + \mathbf{C}) &= \mathbf{AB} + \mathbf{AC} && \text{… distributive}
\end{aligned}
\tag{1.9.10}
$$

The associative and distributive properties follow from the fact that a tensor is by definition a linear operator, §1.8.2; they apply to tensors of any order, for example,

$$
(\mathbf{AB})\mathbf{v} = \mathbf{A}(\mathbf{Bv}) \tag{1.9.11}
$$

To deal with tensors of any order, all one has to remember is how simple tensors operate on each other – the two vectors which are beside each other are the ones which are "dotted" together:

$$
\begin{aligned}
(\mathbf{a} \otimes \mathbf{b})\mathbf{c} &= (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \\
(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d}) \\
(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) &= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d} \otimes \mathbf{e}) \\
(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})(\mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}) &= (\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{e} \otimes \mathbf{f})
\end{aligned}
\tag{1.9.12}
$$

An example involving a higher order tensor is

$$
\begin{aligned}
\boldsymbol{\mathsf{A}} \cdot \mathbf{E} &= A_{ijkl} E_{mn} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)(\mathbf{e}_m \otimes \mathbf{e}_n) \\
&= A_{ijkl} E_{ln} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_n)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \alpha \\
\mathbf{AB} &= \mathbf{C} \\
\mathbf{Au} &= \mathbf{v} \\
\boldsymbol{\mathsf{A}}\mathbf{b} &= \mathbf{C} \\
\boldsymbol{\mathsf{AB}} &= \boldsymbol{\mathsf{C}}
\end{aligned}
$$

Note the relation (analogous to the vector relation $\mathbf{a}(\mathbf{b} \otimes \mathbf{c})\mathbf{d} = (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d})$, which follows directly from the dyad definition 1.8.3) {▲Problem 10}

$$\mathbf{A}(\mathbf{B}\otimes\mathbf{C})\mathbf{D} = (\mathbf{AB})\otimes(\mathbf{CD}) \tag{1.9.13}$$

Powers of Tensors

Integral powers of tensors are defined inductively by $\mathbf{T}^0 = \mathbf{I}$, $\mathbf{T}^n = \mathbf{T}^{n-1}\mathbf{T}$, so, for example,

$$\boxed{\mathbf{T}^2 = \mathbf{TT}} \qquad \textbf{The Square of a Tensor} \tag{1.9.14}$$

$\mathbf{T}^3 = \mathbf{TTT}$, etc.

1.9.3 Double Contraction

Double contraction, as the name implies, contracts the tensors twice as much a simple contraction. Thus, where the sum of the orders of two tensors is reduced by two in the simple contraction, the sum of the orders is reduced by four in double contraction. The double contraction is denoted by a colon (:), e.g. $\mathbf{T}:\mathbf{S}$.

First, define the double contraction of simple tensors (dyads) through

$$(\mathbf{a}\otimes\mathbf{b}):(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \tag{1.9.15}$$

So in double contraction, one takes the scalar product of four vectors which are adjacent to each other, according to the following rule:

$$(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}):(\mathbf{d}\otimes\mathbf{e}\otimes\mathbf{f}) = (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{e})(\mathbf{a}\otimes\mathbf{f})$$

For example,

$$\begin{aligned}\mathbf{T}:\mathbf{S} &= T_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j):S_{kl}(\mathbf{e}_k\otimes\mathbf{e}_l)\\ &= T_{ij}S_{kl}\left[(\mathbf{e}_i\cdot\mathbf{e}_k)(\mathbf{e}_j\cdot\mathbf{e}_l)\right]\\ &= T_{ij}S_{ij}\end{aligned} \tag{1.9.16}$$

which is, as expected, a scalar.

Here is another example, the contraction of the two second order tensors **I** (see Eqn. 1.9.1) and $\mathbf{u}\otimes\mathbf{v}$,

$$\begin{aligned}\mathbf{I}:\mathbf{u}\otimes\mathbf{v} &= (\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{u}\otimes\mathbf{v})\\ &= (\mathbf{e}_i\cdot\mathbf{u})(\mathbf{e}_i\cdot\mathbf{v})\\ &= u_i v_i\\ &= \mathbf{u}\cdot\mathbf{v}\end{aligned} \tag{1.9.17}$$

so that the scalar product of two vectors can be written in the form of a double contraction involving the Identity Tensor.

An example of double contraction involving the permutation tensor 1.9.6 is {▲Problem 11}

$$\boldsymbol{\mathsf{E}} : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \times \mathbf{v} \tag{1.9.18}$$

It can be shown that the components of a fourth order tensor are given by (compare with Eqn. 1.9.4)

$$A_{ijkl} = (\mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\mathsf{A}} : (\mathbf{e}_k \otimes \mathbf{e}_l) \tag{1.9.19}$$

In summary then,

$$\begin{aligned}
&\mathbf{A} : \mathbf{B} = \beta \\
&\boldsymbol{\mathcal{A}} : \mathbf{b} = \gamma \\
&\boldsymbol{\mathcal{A}} : \mathbf{B} = \mathbf{c} \\
&\boldsymbol{\mathsf{A}} : \mathbf{B} = \mathbf{C}
\end{aligned}$$

Note the following identities:

$$\begin{aligned}
(\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} &= \mathbf{A}(\mathbf{B} : \mathbf{C}) = (\mathbf{B} : \mathbf{C})\mathbf{A} \\
\mathbf{A} : (\mathbf{B} \otimes \mathbf{C}) &= \mathbf{C}(\mathbf{A} : \mathbf{B}) = (\mathbf{A} : \mathbf{B})\mathbf{C} \\
(\mathbf{A} \otimes \mathbf{B}) : (\mathbf{C} \otimes \mathbf{D}) &= (\mathbf{B} : \mathbf{C})(\mathbf{A} \otimes \mathbf{D}) = (\mathbf{A} \otimes \mathbf{D})(\mathbf{B} : \mathbf{C})
\end{aligned} \tag{1.9.20}$$

Note: There are many operations that can be defined and performed with tensors. The two most important operations, the ones which arise most in practice, are the simple and double contractions defined above. Other possibilities are:

(a) double contraction with two "horizontal" dots, $\mathbf{T} \cdot\cdot \mathbf{S}$, $\boldsymbol{\mathcal{A}} \cdot\cdot \mathbf{b}$, etc., which is based on the definition of the following operation as applied to simple tensors:
$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) \cdot\cdot (\mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}) \equiv (\mathbf{b} \cdot \mathbf{e})(\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \otimes \mathbf{f})$$

(b) operations involving one cross $(\times)$: $(\mathbf{a} \otimes \mathbf{b}) \times (\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \otimes \mathbf{d}) \otimes (\mathbf{b} \times \mathbf{c})$

(c) "double" operations involving the cross $(\times)$ and dot:

$$(\mathbf{a} \otimes \mathbf{b}) \mathop{}^{\times}_{\times} (\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \times \mathbf{c}) \otimes (\mathbf{b} \times \mathbf{d})$$

$$(\mathbf{a} \otimes \mathbf{b}) \mathop{}^{\times}_{\cdot} (\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \times \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$$

$$(\mathbf{a} \otimes \mathbf{b}) \mathop{}^{\cdot}_{\times} (\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \times \mathbf{d})$$

1.9.4 Index Notation

The index notation for single and double contraction of tensors of any order can easily be remembered. From the above, a single contraction of two tensors implies that the indices "beside each other" are the same[3], and a double contraction implies that a pair of indices is repeated. Thus, for example, in both symbolic and index notation:

$$\begin{aligned} \mathbf{AB} &= \mathbf{C} \qquad & A_{ijm}B_{mk} &= C_{ijk} \\ \mathbf{A} : \mathbf{B} &= \mathbf{c} \qquad & A_{ijk}B_{jk} &= c_i \end{aligned} \tag{1.9.21}$$

1.9.5 Matrix Notation

Here the matrix notation of §1.4 is extended to include second-order tensors[4]. The Cartesian components of a second-order tensor can conveniently be written as a 3×3 matrix,

$$[\mathbf{T}] = \begin{bmatrix} \mathrm{T}_{11} & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{21} & \mathrm{T}_{22} & \mathrm{T}_{13} \\ \mathrm{T}_{31} & \mathrm{T}_{32} & \mathrm{T}_{33} \end{bmatrix}$$

The operations involving vectors and second-order tensors can now be written in terms of matrices, for example,

$$\mathbf{Tu} = [\mathbf{T}][\mathbf{u}] = \begin{bmatrix} \mathrm{T}_{11} & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{21} & \mathrm{T}_{22} & \mathrm{T}_{13} \\ \mathrm{T}_{31} & \mathrm{T}_{32} & \mathrm{T}_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11}u_1 + T_{12}u_2 + T_{13}u_3 \\ T_{21}u_1 + T_{22}u_2 + T_{23}u_3 \\ T_{31}u_1 + T_{32}u_2 + T_{33}u_3 \end{bmatrix}$$

symbolic notation

"short" matrix notation

"full" matrix notation

The tensor product can be written as (see §1.4.1)

$$\mathbf{u} \otimes \mathbf{v} = [\mathbf{u}][\mathbf{v}^{\mathrm{T}}] = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} \tag{1.9.22}$$

which is consistent with the definition of the dyadic transformation, Eqn. 1.8.3.

---

[3] compare with the "beside each other rule" for matrix multiplication given in §1.4.1

[4] the matrix notation cannot be used for higher-order tensors

1.9.6 Problems

1. Use Eqn. 1.9.3 to show that the component T_{11} of a tensor $\mathbf{T}$ can be evaluated from $\mathbf{e}_1\mathbf{T}\mathbf{e}_1$, and that $T_{12} = \mathbf{e}_1\mathbf{T}\mathbf{e}_2$ (and so on, so that $T_{ij} = \mathbf{e}_i\mathbf{T}\mathbf{e}_j$).
2. Evaluate $\mathbf{aT}$ using the index notation (for a Cartesian basis). What is this operation called? Is your result equal to $\mathbf{Ta}$, in other words is this operation commutative? Now carry out this operation for two vectors, i.e. $\mathbf{a}\cdot\mathbf{b}$. Is it commutative in this case?
3. Evaluate the simple contractions $\mathbf{Ab}$ and $\mathbf{AB}$, with respect to a Cartesian coordinate system (use index notation).
4. Evaluate the double contraction $\mathbf{A}:\mathbf{B}$ (use index notation).
5. Show that, using a Cartesian coordinate system and the index notation, that the double contraction $\mathbf{A}:\mathbf{b}$ is a scalar. Write this scalar out in full in terms of the components of $\mathbf{A}$ and $\mathbf{b}$.
6. Consider the second-order tensors
$$\mathbf{D} = 3\mathbf{e}_1 \otimes \mathbf{e}_1 + 2\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3 + 5\mathbf{e}_3 \otimes \mathbf{e}_3$$
$$\mathbf{F} = 4\mathbf{e}_1 \otimes \mathbf{e}_3 + 6\mathbf{e}_2 \otimes \mathbf{e}_2 - 3\mathbf{e}_3 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$$
Compute $\mathbf{DF}$ and $\mathbf{F}:\mathbf{D}$.
7. Consider the second-order tensor
$$\mathbf{D} = 3\mathbf{e}_1 \otimes \mathbf{e}_1 - 4\mathbf{e}_1 \otimes \mathbf{e}_2 + 2\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 .$$
Determine the image of the vector $\mathbf{r} = 4\mathbf{e}_1 + 2\mathbf{e}_2 + 5\mathbf{e}_3$ when $\mathbf{D}$ operates on it.
8. Write the following out in full – are these the components of scalars, vectors or second order tensors?
   (a) B_{ii}
   (b) C_{kkj}
   (c) B_{mn}
   (d) $a_i b_j A_{ij}$
9. Write $(\mathbf{a}\otimes\mathbf{b}):(\mathbf{c}\otimes\mathbf{d})$ in terms of the components of the four vectors. What is the order of the resulting tensor?
10. Verify Eqn. 1.9.13.
11. Show that $\mathbf{E}:(\mathbf{u}\otimes\mathbf{v}) = \mathbf{u}\times\mathbf{v}$ – see (1.9.6, 1.9.18). [Hint: use the definition of the cross product in terms of the permutation symbol, (1.3.14), and the fact that $\varepsilon_{ijk} = -\varepsilon_{kji}$.]