1.4 Matrices and Element Form

1.4.1 Matrix – Matrix Multiplication

In the next section, §1.5, regarding vector transformation equations, it will be necessary to multiply various matrices with each other (of sizes 3×1, 1×3 and 3×3). It will be helpful to write these matrix multiplications in a short-hand element form and to develop some short "rules" which will be beneficial right through this chapter.

First, it has been seen that the dot product of two vectors can be represented by $[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]$, or $u_i v_i$. Similarly, the matrix multiplication $[\mathbf{u}][\mathbf{v}^{\mathrm{T}}]$ gives a 3×3 matrix with element form $u_i v_j$ or, in full,

$$\begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

This type of matrix represents the **tensor product** of two vectors, written in symbolic notation as $\mathbf{u}\otimes\mathbf{v}$ (or simply **uv**). Tensor products will be discussed in detail in §1.8 and §1.9.

Next, the matrix multiplication

$$[\mathbf{Q}][\mathbf{u}] \equiv \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1.4.1}$$

is a 3×1 matrix with elements $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij} u_j$ {▲Problem 1}. The elements of $[\mathbf{Q}][\mathbf{u}]$ are the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$, which in element form reads $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}])_i \equiv u_j Q_{ij}$.

The expression $[\mathbf{u}][\mathbf{Q}]$ is meaningless, but $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ {▲Problem 2} is a 1×3 matrix with elements $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}])_i \equiv u_j Q_{ji}$.

This leads to the following rule:

1. if a vector pre-multiplies a matrix $[\mathbf{Q}]$ $\rightarrow$ it is the transpose $[\mathbf{u}^{\mathrm{T}}]$
2. if a matrix $[\mathbf{Q}]$ pre-multiplies the vector $\rightarrow$ it is $[\mathbf{u}]$
3. if summed indices are "beside each other", as the *j* in $u_j Q_{ji}$ or $Q_{ij} u_j$
   $\rightarrow$ the matrix is $[\mathbf{Q}]$
4. if summed indices are not beside each other, as the *j* in $u_j Q_{ij}$
   $\rightarrow$ the matrix is the transpose, $[\mathbf{Q}^{\mathrm{T}}]$

Finally, consider the multiplication of 3×3 matrices. Again, this follows the "beside each other" rule for the summed index. For example, $[\mathbf{A}][\mathbf{B}]$ gives the 3×3 matrix {▲Problem 6} $([\mathbf{A}][\mathbf{B}])_{ij} = A_{ik}B_{kj}$, and the multiplication $[\mathbf{A}^{\mathrm{T}}][\mathbf{B}]$ is written as $([\mathbf{A}^{\mathrm{T}}][\mathbf{B}])_{ij} = A_{ki}B_{kj}$. There is also the important identity

$$([\mathbf{A}][\mathbf{B}])^{\mathrm{T}} = [\mathbf{B}^{\mathrm{T}}][\mathbf{A}^{\mathrm{T}}] \tag{1.4.2}$$

Note also the following (which applies to both the index notation and element form):

(i) if there is no free index, as in $u_i v_i$, there is one element (representing a scalar)

(ii) if there is one free index, as in $u_j Q_{ji}$, it is a 3×1 (or 1×3) matrix (representing a vector)

(iii) if there are two free indices, as in $A_{ki}B_{kj}$, it is a 3×3 matrix (representing, as will be seen later, a second-order tensor)

1.4.2 The Trace of a Matrix

Another important notation involving matrices is the **trace** of a matrix, defined to be the sum of the diagonal terms, and denoted by

$$\boxed{\mathrm{tr}[\mathbf{A}] = A_{11} + A_{22} + A_{33} \equiv A_{ii}} \qquad \textbf{The Trace} \tag{1.4.3}$$

1.4.3 Problems

1. Show that $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij}u_j$. To do this, multiply the matrix and the vector in Eqn. 1.4.1 and write out the resulting vector in full; Show that the three elements of the vector are $Q_{1j}u_j$, $Q_{2j}u_j$ and $Q_{3j}u_j$.
2. Show that $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ is a 1×3 matrix with elements $u_j Q_{ji}$ (write the matrices out in full).
3. Show that $([\mathbf{Q}][\mathbf{u}])^{\mathrm{T}} = [\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$.
4. Are the three elements of $[\mathbf{Q}][\mathbf{u}]$ the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$?
5. What is the element form for the matrix representation of $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$?
6. Write out the 3×3 matrices **A** and **B** in full, i.e. in terms of A_{11}, A_{12}, etc. and verify that $[\mathbf{AB}]_{ij} = A_{ik}B_{kj}$ for $i=2,\ j=1$.
7. What is the element form for
   (i) $[\mathbf{A}][\mathbf{B}^{\mathrm{T}}]$
   (ii) $[\mathbf{v}^{\mathrm{T}}][\mathbf{A}][\mathbf{v}]$ (there is no ambiguity here, since $([\mathbf{v}^{\mathrm{T}}][\mathbf{A}])[\mathbf{v}] = [\mathbf{v}^{\mathrm{T}}]([\mathbf{A}][\mathbf{v}])$)
   (iii) $[\mathbf{B}^{\mathrm{T}}][\mathbf{A}][\mathbf{B}]$
8. Show that $\delta_{ij}A_{ij} = \mathrm{tr}[\mathbf{A}]$.
9. Show that $\det[\mathbf{A}] = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{ijk}A_{1i}A_{j2}A_{k3}$.