Hardening

In the applications discussed in the preceding two sections, the material was assumed to be perfectly plastic. The issue of hardening (softening) materials is addressed in this section.

8.6.1 Hardening

In the one-dimensional (uniaxial test) case, a specimen will deform up to yield and then generally harden, Fig. 8.6.1. Also shown in the figure is the perfectly-plastic idealisation. In the perfectly plastic case, once the stress reaches the yield point (A), plastic deformation ensues, so long as the stress is maintained at Y. If the stress is reduced, elastic unloading occurs. In the hardening case, once yield occurs, the stress needs to be continually increased in order to drive the plastic deformation. If the stress is held constant, for example at B, no further plastic deformation will occur; at the same time, no elastic unloading will occur. Note that this condition cannot occur in the perfectly-plastic case, where there is one of plastic deformation or elastic unloading.

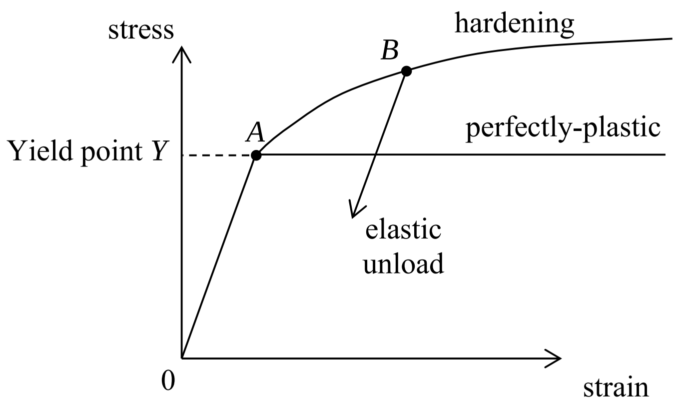

Figure 8.6.1: uniaxial stress-strain curve (for a typical metal)

These ideas can be extended to the multiaxial case, where the initial yield surface will be of the form

$$f_0(\sigma_{ij}) = 0 \tag{8.6.1}$$

In the perfectly plastic case, the yield surface remains unchanged.. In the more general case, the yield surface may change size, shape and position, and can be described by

$$f(\sigma_{ij}, \mathrm{K}_i) = 0 \tag{8.6.2}$$

Here, K_i represents one or more **hardening parameters**, which change during plastic deformation and determine the evolution of the yield surface. They may be scalars or higher-order tensors. At first yield, the hardening parameters are zero, and $f(\sigma_{ij}, 0) = f_0(\sigma_{ij})$.

The description of how the yield surface changes with plastic deformation, Eqn. 8.6.2, is called the **hardening rule**.

Strain Softening

Materials can also **strain soften**, for example soils. In this case, the stress-strain curve "turns down", as in Fig. 8.6.2. The yield surface for such a material will in general decrease in size with further straining.

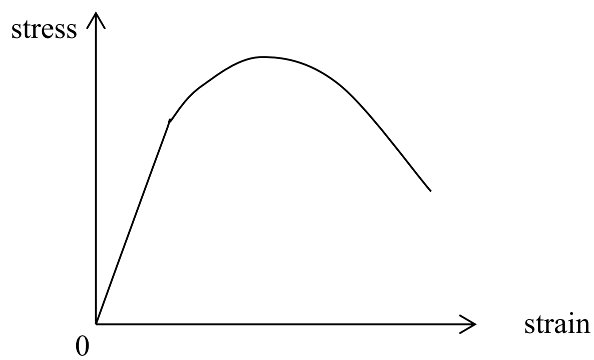

Figure 8.6.2: uniaxial stress-strain curve for a strain-softening material

8.6.2 Hardening Rules

A number of different hardening rules are discussed in this section.

Isotropic Hardening

Isotropic hardening is where the yield surface remains the same shape but expands with increasing stress, Fig. 8.6.3.

In particular, the yield function takes the form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij}) - \mathrm{K} = 0 \tag{8.6.3}$$

The shape of the yield function is specified by the initial yield function and its size changes as the hardening parameter K changes.

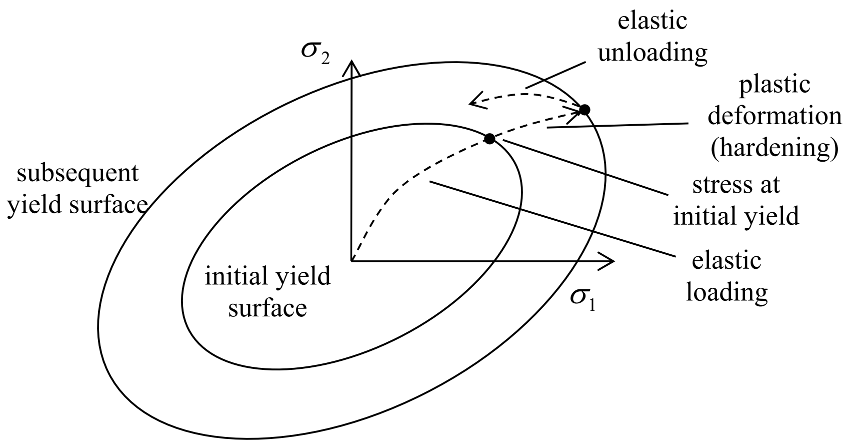

Figure 8.6.3: isotropic hardening

For example, consider the Von Mises yield surface. At initial yield, one has

$$\begin{aligned} f_0(\sigma_{ij}) &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}-Y \\ &= \sqrt{3J_2}-Y \\ &= \sqrt{\tfrac{3}{2}s_{ij}s_{ij}}-Y \end{aligned} \tag{8.6.4}$$

where Y is the yield stress in uniaxial tension. Subsequently, one has

$$f(\sigma_{ij},\mathrm{K}_i)=\sqrt{3J_2}-Y-\mathrm{K}=0 \tag{8.6.5}$$

The initial cylindrical yield surface in stress-space with radius $\sqrt{\tfrac{2}{3}}Y$ (see Fig. 8.3.11) develops with radius $\sqrt{\tfrac{2}{3}}(Y+\mathrm{K})$. The details of how the hardening parameter K actually changes with plastic deformation have not yet been specified.

As another example, consider the Drucker-Prager criterion, Eqn. 8.3.30, $f_0(\sigma_{ij})=\alpha I_1+\sqrt{J_2}-k=0$. In uniaxial tension, $I_1=Y$, $\sqrt{J_2}=Y/\sqrt{3}$, so $k=(\alpha+1/\sqrt{3})Y$. Isotropic hardening can then be expressed as

$$f(\sigma_{ij},\mathrm{K}_i)=\frac{1}{\alpha+1/\sqrt{3}}\left(\alpha I_1+\sqrt{J_2}\right)-Y-\mathrm{K}=0 \tag{8.6.6}$$

Kinematic Hardening

The isotropic model implies that, if the yield strength in tension and compression are initially the same, i.e. the yield surface is symmetric about the stress axes, they remain equal as the yield surface develops with plastic strain. In order to model the Bauschinger effect, and similar responses, where a hardening in tension will lead to a softening in a subsequent compression, one can use the **kinematic hardening** rule. This is where the yield surface remains the same shape and size but merely translates in stress space, Fig. 8.6.4.

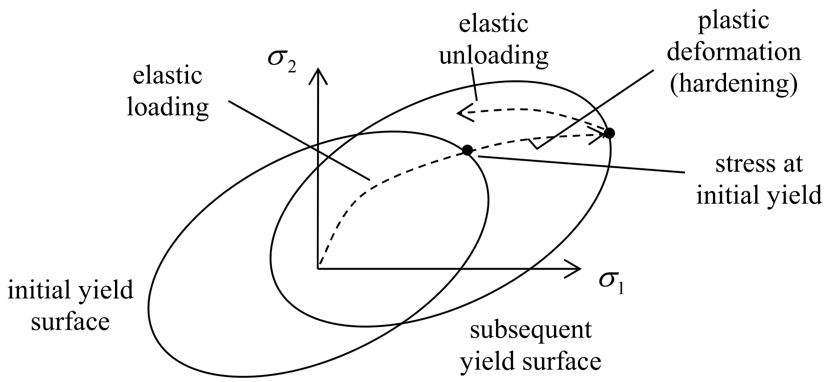

Figure 8.6.4: kinematic hardening

The yield function now takes the general form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij} - \alpha_{ij}) = 0 \tag{8.6.7}$$

The hardening parameter here is the stress α_{ij}, known as the **back-stress** or **shift-stress**; the yield surface is shifted relative to the stress-space axes by α_{ij}, Fig. 8.6.5.

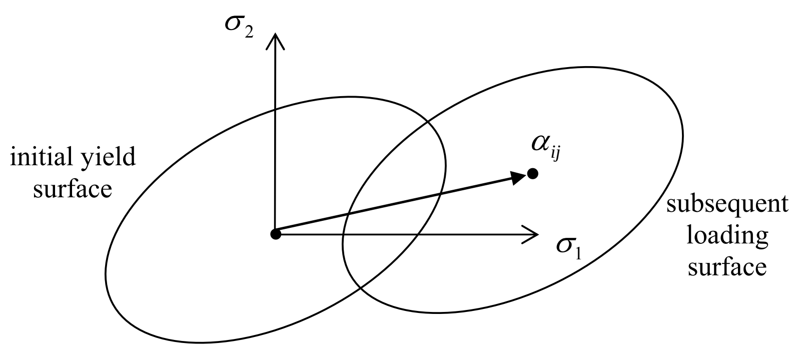

Figure 8.6.5: kinematic hardening; a shift by the back-stress

For example, again considering the Von Mises material, one has, from 8.6.4, and using the deviatoric part of $\boldsymbol{\sigma} - \boldsymbol{\alpha}$ rather than the deviatoric part of $\boldsymbol{\sigma}$,

$$f\left(\sigma_{ij}, \mathrm{K}_i\right) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)} - Y = 0 \tag{8.6.8}$$

where $\boldsymbol{\alpha}^d$ is the deviatoric part of $\boldsymbol{\alpha}$. Again, the details of how the hardening parameter α_{ij} might change with deformation will be discussed later.

Other Hardening Rules

More complex hardening rules can be used. For example, the **mixed hardening** rule combines features of both the isotropic and kinematic hardening models, and the loading function takes the general form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij} - \alpha_{ij}) - \mathrm{K} = 0 \qquad (8.6.9)$$

The hardening parameters are now the scalar K and the tensor α_{ij}.

8.6.3 The Flow Curve

In order to model plastic deformation and hardening in a complex three-dimensional geometry, one will generally have to use but the data from a simple test. For example, in the uniaxial tension test, one will have the data shown in Fig. 8.6.6a, with stress plotted against plastic strain. The idea now is to define a scalar **effective stress** $\hat{\sigma}$ and a scalar **effective plastic strain** $\hat{\varepsilon}^p$, functions respectively of the stresses and plastic strains in the loaded body. The following hypothesis is then introduced: a plot of effective stress against effective plastic strain follows the same **universal plastic stress-strain curve** as in the uniaxial case. This assumed universal curve is known as the **flow curve**.

The question now is: how should one define the effective stress and the effective plastic strain?

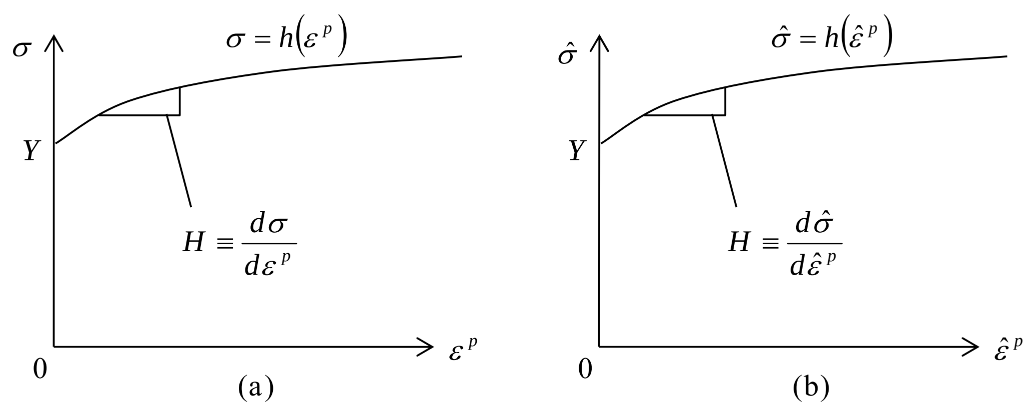

Figure 8.6.6: the flow curve; (a) uniaxial stress – plastic strain curve, (b) effective stress – effective plastic strain curve

8.6.4 A Von Mises Material with Isotropic Hardening

Consider a Von Mises material. Here, it is appropriate to define the effective stress to be

$$\hat{\sigma}(\sigma_{ij}) = \sqrt{3J_2} \qquad (8.6.10)$$

This has the essential property that, in the uniaxial case, $\hat{\sigma}(\sigma_{ij}) = Y$. (In the same way, for example, the effective stress for the Drucker-Prager material, Eqn. 8.6.6, would be $\hat{\sigma}(\sigma_{ij}) = \left(\alpha I_1 + \sqrt{J_2}\right)/(\alpha + 1/\sqrt{3})$.)

For the effective plastic strain, one possibility is to define it in the following rather intuitive, non-rigorous, way. The deviatoric stress **s** and plastic strain (increment) tensor $d\boldsymbol{\varepsilon}^p$ are of a similar character. In particular, their traces are zero, albeit for different physical reasons; $J_1 = 0$ because of independence of hydrostatic pressure, $d\varepsilon_{ii}^p = 0$ because of material incompressibility in the plastic range. For this reason, one chooses the effective plastic strain (increment) $d\hat{\varepsilon}^p$ to be a similar function of $d\varepsilon_{ij}^p$ as $\hat{\sigma}$ is of the s_{ij}. Thus, in lieu of $\hat{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$, one chooses $d\hat{\varepsilon}^p = C\sqrt{d\varepsilon_{ij}^p d\varepsilon_{ij}^p}$. One can determine the constant C by ensuring that the expression reduces to $d\hat{\varepsilon}^p = d\varepsilon_1^p$ in the uniaxial case. Considering this uniaxial case, $d\varepsilon_{11}^p = d\varepsilon_1^p$, $d\varepsilon_{22}^p = d\varepsilon_{33}^p = -\frac{1}{2} d\varepsilon_1^p$, one finds that

$$
\begin{aligned}
d\hat{\varepsilon}^p &= \sqrt{\tfrac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p} \\
&= \frac{\sqrt{2}}{3}\sqrt{\left(d\varepsilon_1^p - d\varepsilon_2^p\right)^2 + \left(d\varepsilon_2^p - d\varepsilon_3^p\right)^2 + \left(d\varepsilon_3^p - d\varepsilon_1^p\right)^2}
\end{aligned}
\tag{8.6.11}
$$

Let the hardening in the uniaxial tension case be described using a relationship of the form (see Fig. 8.6.6)

$$\sigma = h\left(\varepsilon^p\right) \tag{8.6.12}$$

The slope of this flow curve is the plastic modulus, Eqn. 8.1.9,

$$H = \frac{d\sigma}{d\varepsilon^p} \tag{8.6.13}$$

The effective stress and effective plastic strain for any conditions are now assumed to be related through

$$\hat{\sigma} = h\left(\hat{\varepsilon}^p\right) \tag{8.6.14}$$

and the effective plastic modulus is given by

$$H = \frac{d\hat{\sigma}}{d\hat{\varepsilon}^p} \tag{8.6.15}$$

Isotropic Hardening

Assuming isotropic hardening, the yield surface is given by Eqn. 8.6.5, and with the definition of the effective stress, Eqn. 8.6.10,

$$f(\sigma_{ij}, \mathrm{K}_i) = \hat{\sigma} - Y - \mathrm{K} = 0 \tag{8.6.16}$$

Differentiating with respect to the effective plastic strain,

$$H = \frac{\partial \hat{\sigma}}{\partial \hat{\varepsilon}^p} = \frac{\partial \mathrm{K}}{\partial \hat{\varepsilon}^p} \tag{8.6.17}$$

One can now see how the hardening parameter evolves with deformation: K here is a function of the effective plastic strain, and its functional dependence on the effective plastic strain is given by the plastic modulus H of the universal flow curve.

Loading Histories

Each material particle undergoes a plastic strain history. One such path is shown in Fig. 8.6.7. At point q, the plastic strain is $\varepsilon_i^p(q)$. The effective plastic strain at q must be evaluated through an integration over the complete history of deformation:

$$\hat{\varepsilon}^p(q) = \int_0^q d\hat{\varepsilon}^p = \int_0^q \sqrt{\tfrac{2}{3} d\varepsilon_i^p d\varepsilon_i^p} \tag{8.6.18}$$

Note that the effective plastic strain at q is not simply $\sqrt{\tfrac{2}{3}\varepsilon_i^p(q)\varepsilon_i^p(q)}$, hence the definition of an effective plastic strain *increment* in Eqn. 8.6.11.

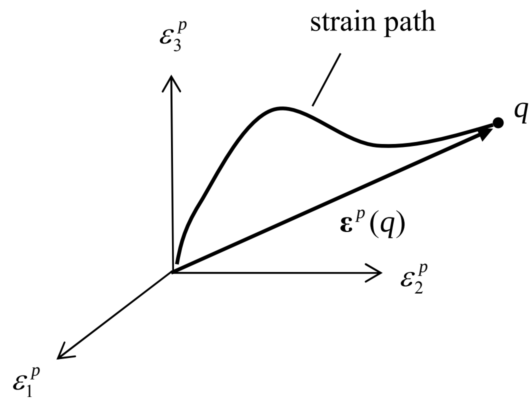

Figure 8.6.7: plastic strain space

Prandtl-Reuss Relations in terms of Effective Parameters

Using the Prandtl-Reuss (Levy-Mises) flow rule 8.4.1, and the definitions 8.6.10-11 for effective stress and effective plastic strain, one can now express the plastic multiplier as{▲Problem 1}

$$d\lambda = \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}} \tag{8.6.19}$$

and the plastic strain increments, Eqn. 8.4.6, now read

$$\begin{aligned}
d\varepsilon_{xx}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{xx} - \tfrac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{yy} - \tfrac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{zz} - \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{xy} \\
d\varepsilon_{yz}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{yz} \\
d\varepsilon_{zx}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{zx}
\end{aligned} \tag{8.6.20}$$

or

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}} s_{ij} . \tag{8.6.21}$$

Knowledge of the plastic modulus, Eqn. 8.6.15, now makes equations 8.6.21 complete.

Note here that the plastic modulus in the Prandtl-Reuss equations is conveniently expressible in a simple way in terms of the effective stress and plastic strain increment, Eqn. 8.6.19. It will be shown in the next section that this is no coincidence, and that the Prandtl-Reuss flow-rule is indeed naturally associated with the Von-Mises criterion.

8.6.5 Application: Combined Tension/Torsion of a thin walled tube with Isotropic Hardening

Consider again the thin-walled tube under combined tension and torsion. The Von Mises yield function in terms of the axial stress σ and the shear stress τ is, as in §8.3.1, $f_0\left(\sigma_{ij}\right) = \sqrt{\sigma^2 + 3\tau^2} - Y = 0$. This defines the ellipse of Fig. 8.3.2. Subsequent yield surfaces are defined by

$$\begin{aligned}
f\left(\sigma_{ij}, \mathrm{K}_i\right) &= \sqrt{\sigma^2 + 3\tau^2} - Y - K \\
&= f_0\left(\sigma_{ij}\right) - K \\
&= \hat{\sigma} - \left(Y + K\right) \\
&= 0
\end{aligned} \tag{8.6.22}$$

Whereas the initial yield surface is the ellipse with major and minor axes Y and $Y/\sqrt{3}$, subsequent yield ellipses have axes $Y + K$ and $(Y + K)/\sqrt{3}$, Fig. 8.6.8.

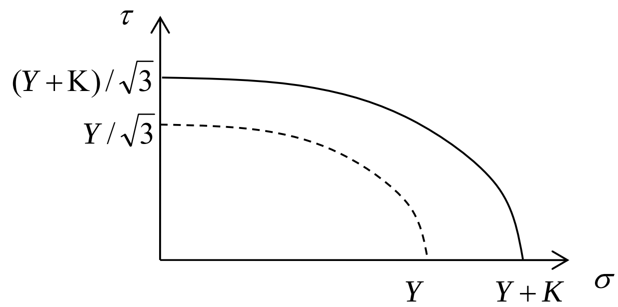

Figure 8.6.8: expansion of the yield locus (ellipse) for a thin-walled tube under isotropic hardening

The Prandtl-Reuss equations in terms of effective stress and effective plastic strain, 8.6.20-21, reduce to

$$\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{E}d\sigma + \frac{d\hat{\varepsilon}^p}{\hat{\sigma}}\sigma \\
d\varepsilon_{yy} = d\varepsilon_{zz} &= -\frac{\nu}{E}d\sigma - \frac{1}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}}\sigma \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}}\tau
\end{aligned} \qquad (8.6.23)$$

Consider the case where the material is brought to first yield through tension only, in which case the Von Mises condition reduces to $\sigma = Y$. Let the material then be subjected to a twist whilst maintaining the axial stress constant. The expansion of the yield surface is then as shown in Fig. 8.6.9.

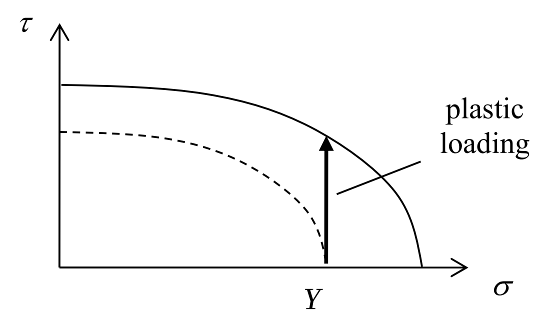

Figure 8.6.9: expansion of the yield locus for a thin-walled tube under constant axial loading

Introducing the plastic modulus, then, one has

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}Y \\ d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{1}{2}\frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}Y \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}\tau \end{aligned} \tag{8.6.24}$$

Using $\hat{\sigma} = \sqrt{Y^2 + 3\tau^2}$,

$$\begin{aligned} d\varepsilon_{xx} &= \frac{Y}{H}\frac{\tau d\tau}{\tau^2 + Y^2/3} \\ d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{1}{2}\frac{Y}{H}\frac{\tau d\tau}{\tau^2 + Y^2/3} \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{1}{H}\frac{\tau^2 d\tau}{\tau^2 + Y^2/3} \end{aligned} \tag{8.6.25}$$

These equations can now be integrated. If the material is **linear hardening**, so H is constant, then they can be integrated exactly using

$$\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln\left(x^2+a^2\right), \qquad \int \frac{x^2}{x^2+a^2}dx = x - a\arctan\left(\frac{x}{a}\right) \tag{8.6.26}$$

leading to {▲Problem 2}

$$\begin{aligned} \frac{E}{Y}\varepsilon_{xx} &= 1 + \frac{1}{2}\frac{E}{H}\ln\left(1+3\frac{\tau^2}{Y^2}\right) \\ \frac{E}{Y}\varepsilon_{yy} &= \frac{E}{Y_0}\varepsilon_{zz} = -\frac{E}{4H}\ln\left(1+3\frac{\tau^2}{Y^2}\right) \\ \frac{E}{Y}\varepsilon_{xy} &= (1+\nu)\left(\frac{\tau}{Y}\right) + \frac{3}{2}\frac{E}{H}\left[\frac{\tau}{Y} - \frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}\frac{\tau}{Y}\right)\right] \end{aligned} \tag{8.6.27}$$

Results are presented in Fig. 8.6.10 for the case of $\nu = 0.3$, $E/H = 10$. The axial strain grows logarithmically and is eventually dominated by the faster-growing shear strain.

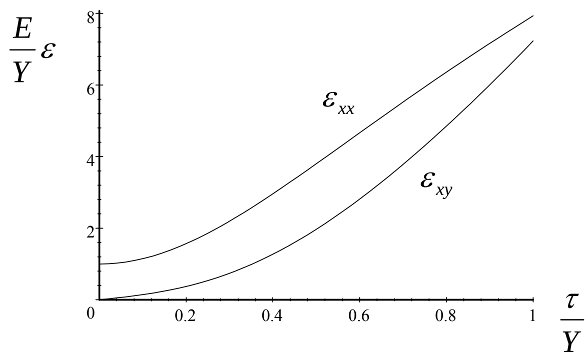

Figure 8.6.10: Stress-strain curves for thin-walled tube with isotropic linear strain hardening

8.6.6 Kinematic Hardening Rules

A typical uniaxial kinematic hardening curve is shown in Fig. 8.6.11a (see Fig. 8.1.3). During cyclic loading, the elastic zone always remains at $2Y$. Depending on the stress history, one can even have the situation shown in Fig. 8.6.11b, where yielding occurs upon unloading, even though the stress is still tensile.

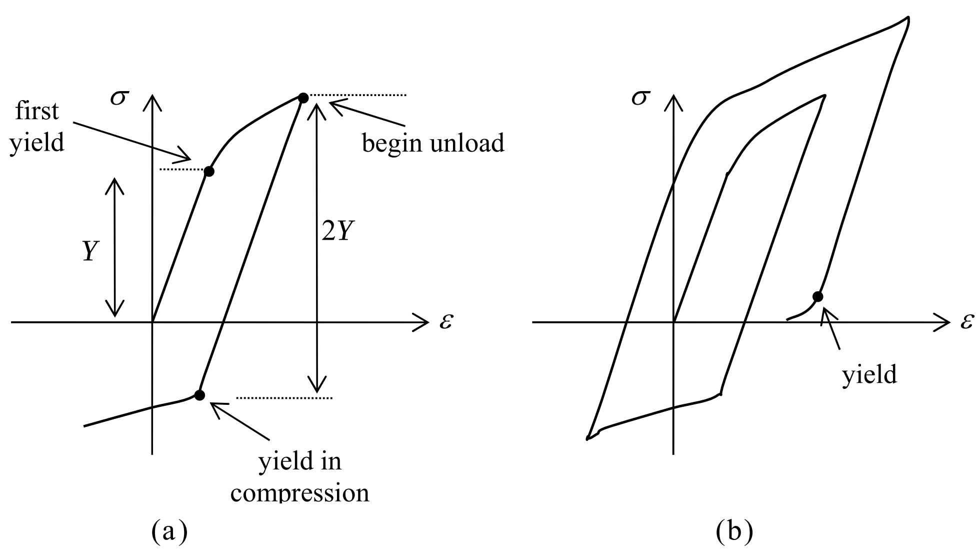

Figure 8.6.11: Kinematic Hardening; (a) load-unload, (b) cyclic loading

The multiaxial yield function for a kinematic hardening Von Mises is given by Eqn. 8.6.8,

$$f\left(\sigma_{ij}, \mathrm{K}_i\right) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)} - Y = 0$$

The deviatoric shift stress α_{ij}^d describes the shift in the centre of the Von Mises cylinder, as viewed in the π-plane, Fig. 8.6.12. This is a generalisation of the uniaxial case, in that the radius of the Von Mises cylinder remains constant, just as the elastic zone in the uniaxial case remains constant (at $2Y$).

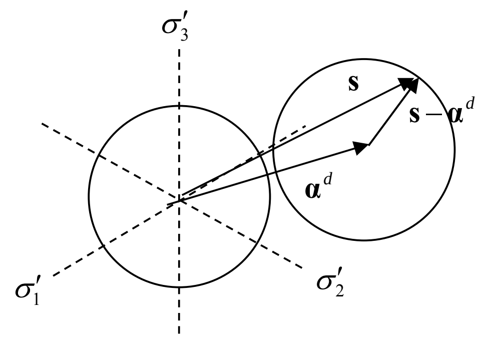

Figure 8.6.12: The Von Mises cylinder shifted in the π-plane

One needs to specify, by specifying the evolution of the hardening paremter $\boldsymbol{\alpha}$, how the yield surface shifts with deformation. In the multiaxial case, one has the added complication that the direction in which the yield surface shifts in stress space needs to be specified. The simplest model is the **linear kinematic** (or **Prager's**) **hardening rule**. Here, the back stress is assumed to depend on the plastic strain according to

$$\alpha_{ij} = c\varepsilon_{ij}^p \quad \text{or} \quad d\alpha_{ij} = cd\varepsilon_{ij}^p \tag{8.6.28}$$

where c is a material parameter, which might change with deformation. Thus the yield surface is translated in the same direction as the plastic strain increment. This is illustrated in Fig. 8.6.13, where the principal directions of stress and plastic strain are superimposed.

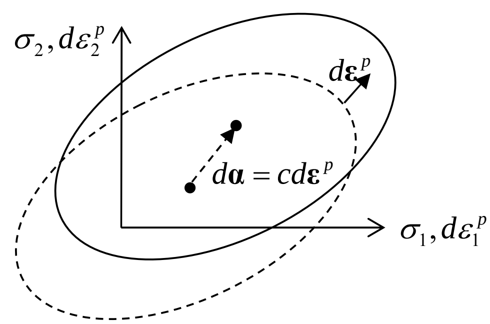

Figure 8.6.13: Linear kinematic hardening rule

One can use the uniaxial (possibly cyclic) curve to again define a universal plastic modulus H. Using the effective plastic strain, one can relate the constant c to H. This will be discussed in §8.8, where a more general formulation will be used.

Ziegler's hardening rule is

$$d\alpha_{ij} = da\left(\varepsilon_{ij}^p\right)\left(\sigma_{ij} - \alpha_{ij}\right) \tag{8.6.29}$$

where a is some scalar function of the plastic strain. Here, then, the loading function translates in the direction of $\sigma_{ij} - \alpha_{ij}$, Fig. 8.6.14.

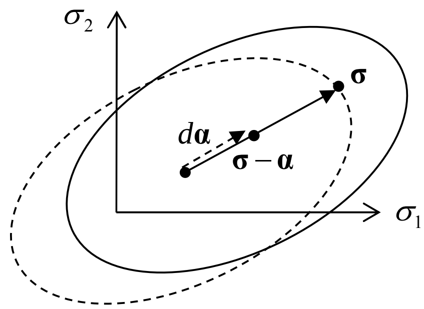

Figure 8.6.14: Ziegler's kinematic hardening rule

8.6.7 Strain Hardening and Work Hardening

In the models considered above, the hardening parameters have been functions of the plastic strains. For example, in the Von Mises isotropic hardening model, the hardening parameter K is a function of the effective plastic strain, $\hat{\varepsilon}^p$. Hardening expressed in this way is called **strain hardening**.

Another means of generalising the uniaxial results to multiaxial conditions is to use the **plastic work** (per unit volume), also known as the **plastic dissipation**,

$$dW^p = \sigma_{ij} d\varepsilon_{ij}^p \tag{8.6.30}$$

The total plastic work is the area under the stress – plastic strain curve of Fig. 8.6.6a,

$$W^p = \int \sigma_{ij} d\varepsilon_{ij}^p \tag{8.6.31}$$

A plot of stress against the plastic work can therefore easily be generated, as in Fig. 8.6.15.

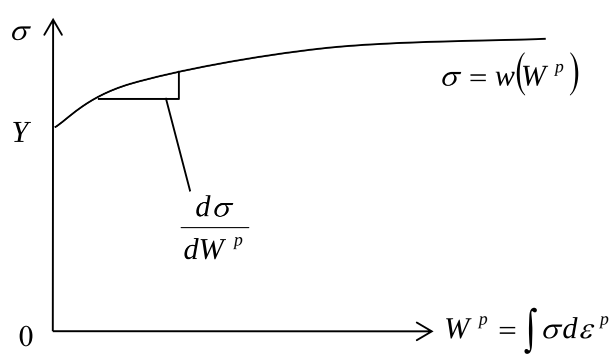

Figure 8.6.15: uniaxial stress – plastic work curve (for a typical metal)

The stress is now expressed in the form (compare with Eqn. 8.6.12)

$$\sigma = w(W^p) = w\left(\int \sigma d\varepsilon^p\right) \tag{8.6.32}$$

Again defining an effective stress $\hat{\sigma}$, the universal flow curve to be used for arbitrary loading conditions is then (compare with Eqn. 8.6.14)

$$\hat{\sigma} = w(W^p) \tag{8.6.33}$$

where now W^p is the plastic work during the multiaxial deformation. This is known as a **work hardening** formulation.

Equivalence of Strain and Work Hardening for the Isotropic Hardening Von Mises Material

Consider the Prandtl-Reuss flow rule, Eqn. 8.4.1, $d\varepsilon_i^p = s_i d\lambda$ (other flow rules will be examined more generally in §8.7). In this case, working with principal stresses, the plastic work increment is (see Eqns. 8.2.7-10)

$$\begin{aligned} dW^p &= \sigma_i d\varepsilon_i^p \\ &= \sigma_i s_i d\lambda \\ &= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] d\lambda \end{aligned} \tag{8.6.34}$$

Using the Von Mises effective stress 8.6.10, and Eqn. 8.6.19,

$$\begin{aligned} dW^p &= \tfrac{2}{3}\hat{\sigma}^2 d\lambda \\ &= \hat{\sigma}\, d\hat{\varepsilon}^p \end{aligned} \tag{8.6.35}$$

where $\hat{\varepsilon}^p$ is the very same effective plastic strain as used in the strain hardening isotropic model, Eqn. 8.6.11. Although true for the Von Mises yield condition, this will not be so in general.

8.6.8 Problems

1. Staring with the definition of the effective plastic strain, Eqn. 8.6.11, and using Eqn. 8.4.1, $d\varepsilon_i^p = d\lambda s_i$, derive Eqns. 8.6.19, $d\lambda = \dfrac{3}{2}\dfrac{d\hat{\varepsilon}^p}{\hat{\sigma}}$

2. Integrate Eqns. 8.6.25 and use the initial (first yield) conditions to get Eqns. 8.6.27.

3. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of an isotropic hardening Von Mises metal with uniaxial yield stress Y. The strain-hardening is linear with plastic modulus *H*. The tube is loaded, keeping the ratio $\sigma / \tau = \sqrt{3}$ at all times throughout the elasto-plastic deformation, until $\sigma = Y$.
   (i) Show that the stresses and strains at first yield are given by
   $$\sigma^Y = \frac{1}{\sqrt{2}}Y, \quad \tau^Y = \frac{1}{\sqrt{6}}Y, \quad \varepsilon_{xx}^Y = \frac{1}{\sqrt{2}}\frac{Y}{E}, \quad \varepsilon_{xy}^Y = \frac{1+\nu}{\sqrt{6}}\frac{Y}{E}$$
   (ii) The Prandtl-Reuss equations in terms of the effective stress and effective plastic strain are given by Eqns. 8.6.23. Eliminate τ from these equations (using $\sigma / \tau = \sqrt{3}$).
   (iii) Eliminate the effective plastioc strain using the plastic modulus.
   (iv) The effective stress is defined as $\hat{\sigma} = \sqrt{\sigma^2 + 3\tau^2}$ (see Eqn. 8.6.22). Eliminate the effective stress to obtain
   $$d\varepsilon_{xx} = \frac{1}{E}d\sigma + \frac{1}{H}d\sigma$$
   $$d\varepsilon_{xy} = \frac{1}{\sqrt{3}}\frac{1+\nu}{E}d\sigma + \frac{\sqrt{3}}{2}\frac{1}{H}d\sigma$$
   (v) Integrate the differential equations and evaluate any constants of integration
   (vi) Hence, show that the strains at the final stress values $\sigma = Y$, $\tau = Y / \sqrt{3}$ are given by
   $$\frac{E}{Y}\varepsilon_{xx} = 1 + \frac{E}{H}\left(1 - \frac{1}{\sqrt{2}}\right)$$
   $$\frac{E}{Y}\varepsilon_{xy} = \frac{1+\nu}{\sqrt{3}} + \frac{\sqrt{3}}{2}\frac{E}{H}\left(1 - \frac{1}{\sqrt{2}}\right)$$
   (vii) Sketch the initial yield (elliptical) locus and the final yield locus in (σ, τ) space and the loading path.
   (viii) Plot σ against ε_{xx}.