8.5 The Internally Pressurised Cylinder

8.5.1 Elastic Solution

Consider the problem of a long thick hollow cylinder, with internal and external radii a and b, subjected to an internal pressure p. This can be regarded as a plane problem, with stress and strain independent of the axial direction z. The solution to the axisymmetric elastic problem is (see §4.3.5)

$$
\begin{aligned}
\sigma_{rr} &= -p\frac{b^2/r^2-1}{b^2/a^2-1} \\
\sigma_{\theta\theta} &= +p\frac{b^2/r^2+1}{b^2/a^2-1} \\
\sigma_{zz} &= +p\frac{1}{b^2/a^2-1}\times\begin{cases} 2\nu, & \text{plane strain} \\ 0, & \text{open end} \\ 1, & \text{closed end} \end{cases}
\end{aligned}
\tag{8.5.1}
$$

There are no shear stresses and these are the principal stresses. The strains are (with constant axial strain $\bar{\varepsilon}_{zz}$),

$$
\begin{aligned}
\bar{\varepsilon}_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] \quad \to \quad \sigma_{zz} = E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right) \\
\varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr}-\nu\left(\sigma_{\theta\theta}+\sigma_{zz}\right)\right] = -\nu\bar{\varepsilon}_{zz}+\frac{1+\nu}{E}\left[\left(1-\nu\right)\sigma_{rr}-\nu\sigma_{\theta\theta}\right] \\
\varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta}-\nu\left(\sigma_{rr}+\sigma_{zz}\right)\right] = -\nu\bar{\varepsilon}_{zz}+\frac{1+\nu}{E}\left[-\nu\sigma_{rr}+\left(1-\nu\right)\sigma_{\theta\theta}\right]
\end{aligned}
\tag{8.5.2}
$$

with

$$
\bar{\varepsilon}_{zz} = \frac{p}{E}\frac{1}{b^2/a^2-1}\times\begin{cases} 0, & \text{plane strain} \\ -2\nu, & \text{open end} \\ 1-2\nu, & \text{closed end} \end{cases}
\tag{8.5.3}
$$

Axial Force

The axial force in the elastic tube can be calculated through

$$
\begin{aligned}
P &= \int_0^{2\pi}\int_a^b \sigma_{zz} r\,dr\,d\theta = \int_0^{2\pi}\int_a^b \left[E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] r\,dr\,d\theta \\
&= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+\nu\int_0^{2\pi}\int_a^b\left(\sigma_{rr}+\sigma_{\theta\theta}\right) r\,dr\,d\theta \quad , \\
&= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+2\nu\pi p a^2
\end{aligned}
\tag{8.5.4}
$$

which is the result expected:

$$P = \pi a^2 p \times \begin{cases} 2\nu, & \text{plane strain} \\ 0, & \text{open end} \\ 1, & \text{closed end} \end{cases} \qquad (8.5.5)$$

i.e., consistent with the result obtained from a simple consideration of the axial stress in 8.5.1c.

8.5.2 Plastic Solution

The pressure is now increased so that the cylinder begins to deform plastically. It will be assumed that the material is isotropic and elastic perfectly-plastic and that it satisfies the Tresca criterion.

First Yield

It can be seen from 8.5.1 that $\sigma_{\theta\theta} > \sigma_{zz} \geq 0 > \sigma_{rr}$ (σ_{rr} does equal zero on the outer surface, $r = b$, but that is not relevant here, as plastic flow will begin on the inner wall), so the intermediate stress is σ_{zz}, and so the Tresca criterion reads

$$\left|\sigma_{rr} - \sigma_{\theta\theta}\right| = \sigma_{\theta\theta} - \sigma_{rr} = 2p\frac{b^2/r^2}{b^2/a^2 - 1} \equiv 2k \qquad (8.5.6)$$

This expression has its maximum value at the inner surface, $r = a$, and hence it is here that plastic flow first begins. From this, plastic deformation begins when

$$p_{flow} = k\left(1 - \frac{a^2}{b^2}\right), \qquad (8.5.7)$$

irrespective of the end conditions.

Confined Plastic Flow and Collapse

As the pressure increases above p_{flow}, the plastic region spreads out from the inner face; suppose that it reaches out to $r = c$. With the material perfectly plastic, the material in the annulus $a < r < c$ satisfies the yield condition 8.5.6 at all times. Consider now the equilibrium of this *plastic* material. Since this is an axi-symmetric problem, there is only one equilibrium equation:

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0\,. \qquad (8.5.8)$$

It follows that

$$\frac{d\sigma_{rr}}{dr} - \frac{2k}{r} = 0 \quad \rightarrow \quad \sigma_{rr} = 2k\ln r + C_1 . \tag{8.5.9}$$

The constant of integration can be obtained from the pressure boundary condition at $r = a$, leading to

$$\sigma_{rr} = -p + 2k\ln(r/a) \qquad (a \le r \le c) \;\; \textbf{Plastic} \tag{8.5.10}$$

The stresses in the elastic region are again given by the elastic stress solution 8.5.1, only with *a* replaced by *c* and the pressure *p* is now replaced by the pressure exerted by the plastic region at $r = c$, i.e. $p - 2k\ln(c/a)$.

The precise location of the boundary *c* can be obtained by noting that the elastic stresses must satisfy the yield criterion at $r = c$. Since in the elastic region,

$$\sigma_{\theta\theta} - \sigma_{rr} = 2\left(p - 2k\ln(c/a)\right)\frac{b^2/r^2}{b^2/c^2 - 1} \qquad (c \le r \le b) \;\; \textbf{Elastic} \tag{8.5.11}$$

one has from $\left(\sigma_{\theta\theta} - \sigma_{rr}\right)_{r=c} = 2k$ that

$$(1 - c^2/b^2) + 2\ln(c/a) = \frac{p}{k} \tag{8.5.12}$$

Fig. 8.5.1 shows a plot of Eqn. 8.5.12.

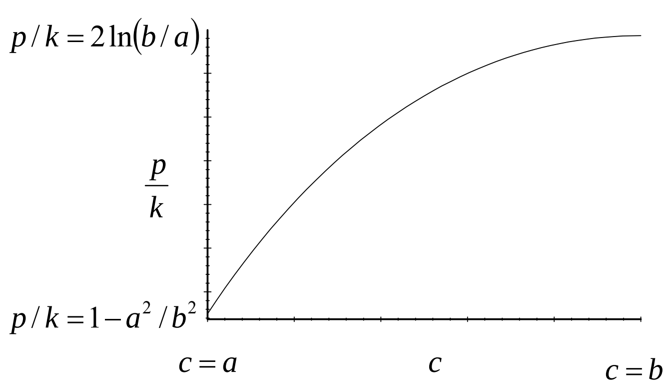

Figure 8.5.1: Extent of the plastic region $r = c$ during confined plastic flow

The complete cylinder will become plastic when *c* reaches *b*, or when the pressure reaches the **collapse pressure** (or **ultimate pressure**)

$$p_U = 2k\ln(b/a) . \tag{8.5.13}$$

This problem illustrates a number of features of elastic-plastic problems in general. First, **confined plastic flow** occurs. This is where the plastic region is surrounded by an elastic region, and so the plastic strains are of the same order as the elastic strains. It is only when the pressure reaches the collapse pressure does catastrophic failure occur.

Stress Field

As discussed after Eqn. 8.5.10, the stresses in the elastic region are derived from Eqns. 8.5.1 with the pressure given by $p - 2k\ln(c/a)$ and a replaced with c:

$$\begin{aligned} \sigma_{rr} &= -k\frac{c^2}{b^2}\left(\frac{b^2}{r^2} - 1\right) \\ \sigma_{\theta\theta} &= +k\frac{c^2}{b^2}\left(\frac{b^2}{r^2} + 1\right), \qquad c \le r \le b \quad \textbf{Elastic} \\ \sigma_{zz} &= +2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.14}$$

For the plastic region, the radial and hoop stresses can be obtained from 8.5.10 and 8.5.6. The Tresca flow rule, 8.4.33, implies that $d\varepsilon_{zz}^p = 0$ and so the axial strain $\bar{\varepsilon}_{zz}$ is purely elastic. Thus the elastic Hooke's Law relation $\bar{\varepsilon}_{zz} = \left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right]/E$ holds also in the plastic region, and

$$\begin{aligned} \sigma_{rr} &= -k\left(1 - \frac{c^2}{b^2} + 2\ln\frac{c}{r}\right) \\ \sigma_{\theta\theta} &= +k\left(1 + \frac{c^2}{b^2} - 2\ln\frac{c}{r}\right) \qquad , \quad a \le r \le c \quad \textbf{Plastic} \\ \sigma_{zz} &= +2k\nu\left(\frac{c^2}{b^2} - 2\ln\frac{c}{r}\right) + E\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.15}$$

Note that, as in the fully elastic solution, the radial and tangential stresses are independent of the end conditions, i.e. they are independent of $\bar{\varepsilon}_{zz}$. Unlike the elastic case, the axial stress is not constant in the plastic zone, and in fact it is tensile and compressive in different regions of the cylinder, depending on the end-conditions.

Axial Force and Strain

The total axial force can be obtained by integrating the axial stresses over the elastic zone and the plastic zone:

$$\begin{aligned} P &= \int_0^{2\pi}\int_a^c \sigma_{zz}^p r\,dr\,d\theta + \int_0^{2\pi}\int_c^b \sigma_{zz}^e r\,dr\,d\theta \\ &= 2\pi\left\{-4k\nu\ln\frac{c}{r}\int_a^c \sigma_{zz}^p r\,dr + \left(2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz}\right)\int_a^c \sigma_{zz}^p r\,dr + \left(2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz}\right)\int_c^b r\,dr\right\} \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2 - a^2\right) + 2k\pi\nu a^2\left[\left(1 - c^2/b^2\right) + \ln(c/a)\right] \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2 - a^2\right) + 2\nu\pi p a^2 \end{aligned} \tag{8.5.16}$$

the last line coming from Eqn. 8.5.12.

A neat alternative way of deriving this result for this problem is as follows: first, the equation of equilibrium 8.5.8, which of course applies in both elastic and plastic zones, can be used to write

$$\left(\sigma_{rr}+\sigma_{\theta\theta}\right)r = r\left(\sigma_{\theta\theta}-\sigma_{rr}\right)+2r\sigma_{rr} = r^2\frac{d\sigma_{rr}}{dr}+2r\sigma_{rr} = \frac{d}{dr}\left(r^2\sigma_{rr}\right) \tag{8.5.17}$$

Using the same method as used in Eqn. 8.5.4, but this time using 8.5.17 and the boundary conditions on the radial stresses at the inner and outer walls:

$$\begin{aligned} P &= \int_0^{2\pi}\int_a^b \left[E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] r\,dr\,d\theta \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+\nu 2\pi\left[r^2\sigma_{rr}\right]_a^b \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+2\nu\pi p a^2 \end{aligned} \tag{8.5.18}$$

which is the same as 8.5.16.

This axial force is the same as Eqn. 8.5.4. In other words, although σ_{zz} in general varies in the plastic zone, the axial force is independent of the plastic zone size *c*.

For the open cylinder, $P = 0$, for plane strain, $\bar{\varepsilon}_{zz} = 0$, and for the closed cylinder, $P = pa^2$. Unsurprisingly, since 8.5.16 (or 8.5.18) is the same as the elastic version, 8.5.4, one sees that the axial strains are as for the elastic solution, Eqn. 8.5.3. In terms of *c* and *k*, and using Eqn. 8.5.12, the axial strain is

$$\bar{\varepsilon}_{zz} = \frac{k}{E}\frac{1}{b^2/a^2-1}\left[1-\frac{c^2}{b^2}+2\ln(c/a)\right]\times\begin{cases} 0, & \text{plane strain} \\ -2\nu, & \text{open end} \\ 1-2\nu, & \text{closed end} \end{cases} \tag{8.5.19}$$

This can now be substituted into Eqns. 8.5.14 and 8.5.15 to get explicit expressions for the axial stresses. As an example, the stresses are plotted in Fig. 8.5.2 for the case of $b/a = 2$, and $c/b = 0.6, 0.8$.

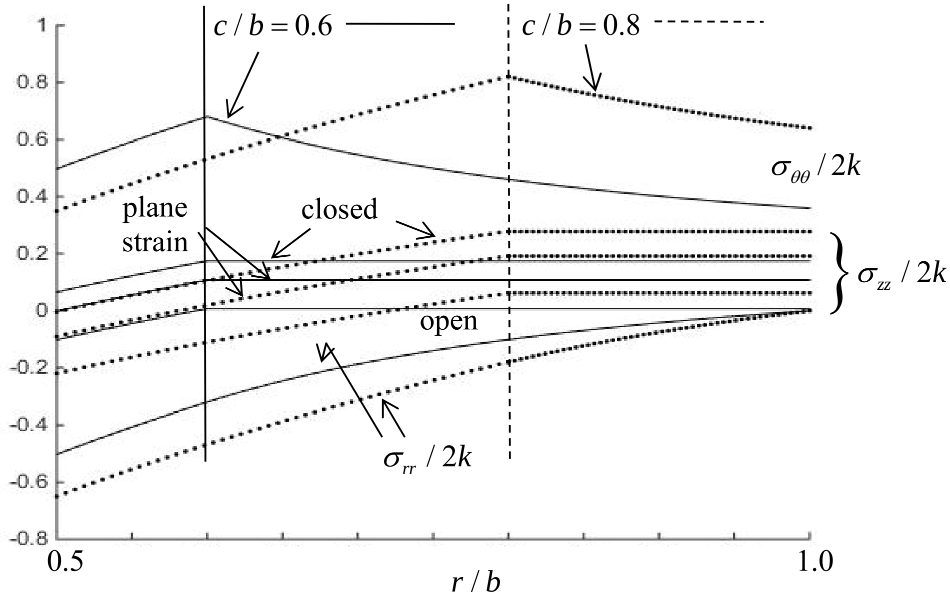

Figure 8.5.2: Stress field in the cylinder for the case of $b/a=2$ and $c/b=0.6,0.8$

Strains and Displacement

In the elastic region, the strains are given by Hooke's law

$$\begin{aligned}\varepsilon_{rr} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr}-\nu\sigma_{\theta\theta}\right]-\nu\bar{\varepsilon}_{zz}\\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta}-\nu\sigma_{rr}\right]-\nu\bar{\varepsilon}_{zz}\end{aligned} \quad (8.5.20)$$

From Eqns. 8.5.14,

$$\left.\begin{aligned}\varepsilon_{rr} &= k\frac{1+\nu}{E}\frac{c^2}{b^2}\left[(1-2\nu)-\frac{b^2}{r^2}\right]-\nu\bar{\varepsilon}_{zz}\\ \varepsilon_{\theta\theta} &= k\frac{1+\nu}{E}\frac{c^2}{b^2}\left[(1-2\nu)+\frac{b^2}{r^2}\right]-\nu\bar{\varepsilon}_{zz}\end{aligned}\right., \quad c\le r\le b \quad \textbf{Elastic} \quad (8.5.21)$$

From the definition of strain

$$\begin{aligned}\varepsilon_{rr} &= \frac{du_r}{dr}\\ \varepsilon_{\theta\theta} &= \frac{u_r}{r}\end{aligned} \quad (8.5.22)$$

Substituting Eqn. 8.5.21a into Eqn. 8.5.22a (or Eqn. 8.5.21b into Eqn. 8.5.22b) gives the displacement in the elastic region:

$$u_r = k\frac{1+\nu}{E}\frac{c^2}{b^2}\left((1-2\nu)r + \frac{b^2}{r}\right) - \nu r\bar{\varepsilon}_{zz}, \quad c \le r \le b \quad \textbf{Elastic} \tag{8.5.23}$$

Now for a Tresca material, from 8.4.33, the increments in plastic strain are $d\varepsilon_{zz}^p = 0$ and $d\varepsilon_{rr}^p = -d\varepsilon_{\theta\theta}^p$, so, from the elastic Hooke's Law,

$$\begin{aligned} \varepsilon_{rr} + \varepsilon_{\theta\theta} &= \varepsilon_{rr}^e + \varepsilon_{\theta\theta}^e \\ &= \frac{1}{E}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\sigma_{zz}\right] \\ &= \frac{1}{E}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\left(E\bar{\varepsilon}_{zz} + \nu\right)(\sigma_{rr} + \sigma_{\theta\theta})\right] \\ &= \frac{(1+\nu)(1-2\nu)}{E}(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.24}$$

Using 8.5.22 and 8.5.17, one has

$$\frac{d}{dr}(ru_r) = \frac{(1+\nu)(1-2\nu)}{E}\frac{d}{dr}\left(r^2\sigma_{rr}\right) - 2\nu r\bar{\varepsilon}_{zz}, \tag{8.5.25}$$

which integrates to

$$u_r = \frac{(1+\nu)(1-2\nu)}{E} r\sigma_{rr} - \nu r\bar{\varepsilon}_{zz} + \frac{C}{r} \tag{8.5.26}$$

Equations 8.5.24-26 are valid in both the elastic and plastic regions. The constant of integration can be obtained from the condition $\sigma_{rr} = 0$ at $r = b$, where u_r equals the elastic displacement 8.5.23, and so $C = 2k(1-\nu^2)c^2 / E$ and

$$u_r = \frac{(1+\nu)(1-2\nu)}{E} r\sigma_{rr} + 2k\frac{(1-\nu^2)c^2}{Er} - \nu r\bar{\varepsilon}_{zz}, \quad a \le r \le c \quad \textbf{Plastic} \tag{8.5.27}$$

8.5.3 Unloading

Residual Stress

Suppose that the cylinder is loaded beyond p_{flow} but not up to the collapse pressure, to a pressure p_0 say. It is then unloaded completely. After unloading the cylinder is still subjected to a stress field – these stresses which are locked into the cylinder are called **residual stresses**. If the unloading process is fully elastic, the new stresses are obtained by subtracting 8.5.1 from 8.5.14-15. Using Eqn. 8.5.7, 8.5.12 {▲Problem 1},

$$\begin{aligned}\sigma_{rr} &= -k\left(\frac{c^2}{a^2} - \frac{p_0}{p_{flow}}\right)\left(\frac{a^2}{r^2} - \frac{a^2}{b^2}\right) \\ \sigma_{\theta\theta} &= +k\left(\frac{c^2}{a^2} - \frac{p_0}{p_{flow}}\right)\left(\frac{a^2}{r^2} + \frac{a^2}{b^2}\right), \quad c \le r \le b \quad \textbf{Elastic} \\ \sigma_{zz} &= +2k\nu\left(\frac{c^2}{a^2} - \frac{p_0}{p_{flow}}\right)\frac{a^2}{b^2}\end{aligned} \tag{8.5.28}$$

$$\begin{aligned}\sigma_{rr} &= -k\left[\frac{p_0}{p_{flow}}\left(1 - \frac{a^2}{r^2}\right) - 2\ln\frac{r}{a}\right] \\ \sigma_{\theta\theta} &= -k\left[\frac{p_0}{p_{flow}}\left(1 + \frac{a^2}{r^2}\right) - 2 - 2\ln\frac{r}{a}\right], \quad a \le r \le c \quad \textbf{Plastic} \\ \sigma_{zz} &= -2k\nu\left[\frac{p_0}{p_{flow}} - 1 - 2\ln\frac{r}{a}\right]\end{aligned} \tag{8.5.29}$$

Consider as an example the case $a = 1$, $b = 2$, with $c = 1.5$, for which

$$\frac{p_{flow}}{2k} = 0.375, \qquad \frac{p_0}{2k} = 0.624 \tag{8.5.30}$$

The residual stresses are as shown in Fig. 8.5.3.

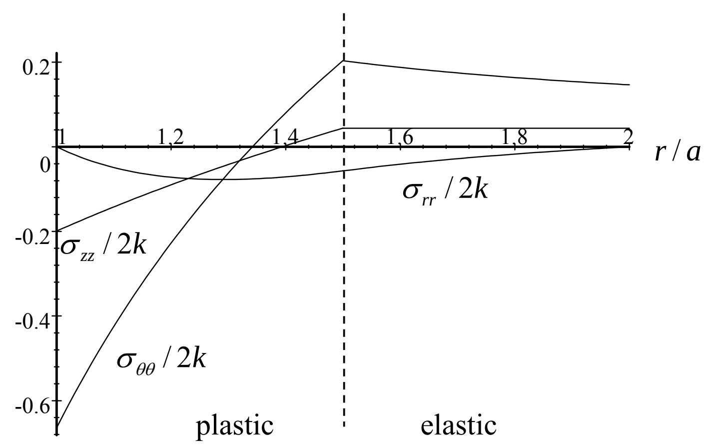

Figure 8.5.3: Residual stresses in the unloaded cylinder for the case of $a = 1, b = 2, c = 1.5$

Note that the axial strain, being purely elastic, is completely removed, and the axial stresses, as given by Eqns. 8.5.28c, 8.5.29c, are independent of the end condition.

There is the possibility that if the original pressure p_0 is very large, the unloading will lead to compressive yield. The maximum value of $|\sigma_{\theta\theta} - \sigma_{rr}|$ occurs at $r = a$ {▲Problem 2}:

$$\left|\sigma_{\theta\theta} - \sigma_{rr}\right|_{r=a} = 2k\left(\frac{p_0}{p_{flow}} - 1\right) \tag{8.5.31}$$

and so, neglecting any Bauschinger effect, yield will occur if $p_0 \geq 2p_{flow}$. Yielding in compression will not occur right up to the collapse pressure p_U if the wall ratio b/a is such that $p_0|_{c=b} = p_U < 2p_{flow}$, or when {▲Problem 3}

$$\ln\frac{b}{a} < 1 - \frac{a^2}{b^2} \tag{8.5.32}$$

The largest wall ratio for which the unloading is completely elastic is $b/a \approx 2.22$. For larger wall ratios, a new plastic zone will develop at the inner wall, with $\sigma_{\theta\theta} - \sigma_{rr} = -2k$.

Shakedown

When the cylinder is initially loaded, plasticity begins at a pressure $p = p_{flow}$. If it is loaded to some pressure p_0, with $p_{flow} < p_0 < 2p_{flow}$, then unloading will be completely elastic. When the cylinder is reloaded again it will remain elastic up to pressure p_0. In this way, it is possible to strengthen the cylinder by an initial loading; theoretically it is possible to increase the flow pressure by a factor of 2. This maximum possible new flow pressure is called the **shakedown pressure** $p_s = \min(2p_{flow}, p_U)$. **Shakedown** is said to have occurred when any subsequent loading/unloading cycles are purely elastic. The strengthening of the cylinder is due to the compressive residual hoop stresses at the inner wall – similar to the way a barrel can be strengthened with hoops. This method of strengthening is termed **autofrettage**, a French term meaning "self-hooping".

8.5.4 Validity of the Solution

One needs to check whether the assumption of the ordering of the principal stresses, $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$, holds through the deformation in the plastic region. It can be confirmed that the inequality $\sigma_{zz} \geq \sigma_{rr}$ always holds. For the inequality $\sigma_{\theta\theta} \geq \sigma_{zz}$, consider the inequality $\sigma_{\theta\theta} - \sigma_{zz} \geq 0$. The quantity on the left is a minimum when $r = a$, where it equals {▲Problem 4}

$$k\left[(1-2\nu)\left(1 + \frac{c^2}{b^2} - 2\ln\frac{c}{a}\right) + 2\nu - \frac{1}{b^2/a^2 - 1}\left(1 - \frac{c^2}{b^2} + 2\ln\frac{c}{a}\right)\begin{bmatrix} 0 \\ -2\nu \\ 1-2\nu \end{bmatrix}\right] \tag{8.5.33}$$

This quantity must be positive for all values of c up to the maximum value b, where it takes its minimum value, and so one must have

$$2(1-2\nu)\left(1-\ln\frac{b}{a}\right)+2\nu-\frac{1}{b^2/a^2-1}\left(2\ln\frac{b}{a}\right)\begin{bmatrix}0\\-2\nu\\1-2\nu\end{bmatrix}\geq 0 \qquad (8.5.34)$$

The solution is thus valid only for limited values of b/a. For $\nu = 0.3$, one must have $b/a < 5.42$ (closed ends), $b/a < 5.76$ (plane strain), $b/a < 6.19$ (open ends). For higher wall ratios, the axial stress becomes equal to the hoop stress. In this case, a solution based on large changes in geometry is necessary for higher pressures.

8.5.5 Problems

1. Derive Eqns. 8.5.28-29.
2. Use Eqns. 8.5.29a,b to show that the maximum value of $|\sigma_{\theta\theta}-\sigma_{rr}|$ occurs at $r = a$, where it equals $2k(p_0 / p_{flow} - 1)$, Eqn. 8.5.31.
3. Derive Eqn. 8.5.32.
4. Use Eqns. 8.5.15, 8.5.12 to derive Eqn. 8.5.33.