8.1 Introduction to Plasticity

8.1.1 Introduction

The theory of linear elasticity is useful for modelling materials which undergo small deformations and which return to their original configuration upon removal of load. Almost all real materials will undergo some **permanent** deformation, which remains after removal of load. With metals, significant permanent deformations will usually occur when the stress reaches some critical value, called the **yield stress**, a material property.

Elastic deformations are termed **reversible;** the energy expended in deformation is stored as elastic strain energy and is completely recovered upon load removal. Permanent deformations involve the dissipation of energy; such processes are termed **irreversible**, in the sense that the original state can be achieved only by the expenditure of more energy.

The **classical theory of plasticity** grew out of the study of metals in the late nineteenth century. It is concerned with materials which initially deform elastically, but which deform **plastically** upon reaching a yield stress. In metals and other crystalline materials the occurrence of plastic deformations at the micro-scale level is due to the motion of dislocations and the migration of grain boundaries on the micro-level. In sands and other granular materials plastic flow is due both to the irreversible rearrangement of individual particles and to the irreversible crushing of individual particles. Similarly, compression of bone to high stress levels will lead to particle crushing. The deformation of micro-voids and the development of micro-cracks is also an important cause of plastic deformations in materials such as rocks.

A good part of the discussion in what follows is concerned with the plasticity of metals; this is the 'simplest' type of plasticity and it serves as a good background and introduction to the modelling of plasticity in other material-types. There are two broad groups of metal plasticity problem which are of interest to the engineer and analyst. The first involves relatively small plastic strains, often of the same order as the elastic strains which occur. Analysis of problems involving small plastic strains allows one to design structures optimally, so that they will not fail when in service, but at the same time are not stronger than they really need to be. In this sense, plasticity is seen as a material **failure**[1].

The second type of problem involves very large strains and deformations, so large that the elastic strains can be disregarded. These problems occur in the analysis of metals manufacturing and forming processes, which can involve extrusion, drawing, forging, rolling and so on. In these latter-type problems, a simplified model known as **perfect plasticity** is usually employed (see below), and use is made of special **limit theorems** which hold for such models.

Plastic deformations are normally **rate independent,** that is, the stresses induced are independent of the rate of deformation (or rate of loading). This is in marked

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[1] two other types of failure, *brittle fracture*, due to dynamic crack growth, and the *buckling* of some structural components, can be modelled reasonably accurately using elasticity theory (see, for example, Part I, §6.1, Part II, §5.3)

contrast to classical **Newtonian fluids** for example, where the stress levels are governed by the *rate* of deformation through the viscosity of the fluid.

Materials commonly known as “plastics” are not plastic in the sense described here. They, like other polymeric materials, exhibit **viscoelastic** behaviour where, as the name suggests, the material response has both elastic and viscous components. Due to their viscosity, their response is, unlike the plastic materials, **rate-dependent**. Further, although the viscoelastic materials can suffer irrecoverable deformation, they do not have any critical yield or threshold stress, which is the characteristic property of plastic behaviour. When a material undergoes plastic deformations, i.e. irrecoverable and at a critical yield stress, and these effects *are* rate dependent, the material is referred to as being **viscoplastic**.

Plasticity theory began with Tresca in 1864, when he undertook an experimental program into the extrusion of metals and published his famous yield criterion discussed later on. Further advances with yield criteria and plastic flow rules were made in the years which followed by Saint-Venant, Levy, Von Mises, Hencky and Prandtl. The 1940s saw the advent of the classical theory; Prager, Hill, Drucker and Koiter amongst others brought together many fundamental aspects of the theory into a single framework. The arrival of powerful computers in the 1980s and 1990s provided the impetus to develop the theory further, giving it a more rigorous foundation based on thermodynamics principles, and brought with it the need to consider many numerical and computational aspects to the plasticity problem.

8.1.2 Observations from Standard Tests

In this section, a number of phenomena observed in the material testing of metals will be noted. Some of these phenomena are simplified or ignored in some of the standard plasticity models discussed later on.

At issue here is the fact that any model of a component with complex geometry, loaded in a complex way and undergoing plastic deformation, must involve material parameters which can be obtained in a straight forward manner from simple laboratory tests, such as the tension test described next.

The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching (see Part I, §5.2.1). The force required to hold the specimen at a given stretch is recorded, Fig. 8.1.1. If the material is a metal, the deformation remains elastic up to a certain force level, the yield point of the material. Beyond this point, permanent plastic deformations are induced. On unloading only the elastic deformation is recovered and the specimen will have undergone a permanent elongation (and consequent lateral contraction).

In the elastic range the force-displacement behaviour for most engineering materials (metals, rocks, plastics, but not soils) is linear. After passing the elastic limit (point *A* in Fig. 8.1.1), the material “gives” and is said to undergo plastic **flow**. Further increases in load are usually required to maintain the plastic flow and an increase in displacement; this phenomenon is known as **work-hardening** or **strain-hardening**. In some cases, after an initial plastic flow and hardening, the force-displacement curve decreases, as in some soils; the material is said to be **softening**. If the specimen is unloaded from a plastic state (*B*) it will return along the path *BC* shown, parallel to the original elastic line. This is **elastic recovery**. The strain which remains upon unloading is the permanent plastic deformation. If the material is now loaded again, the force-displacement curve will re-trace the unloading path *CB* until it again reaches the plastic state. Further increases in stress will cause the curve to follow *BD*.

Two important observations concerning the above tension test (on most metals) are the following:
(1) after the onset of plastic deformation, the material will be seen to undergo negligible volume change, that is, it is **incompressible**.
(2) the force-displacement curve is more or less the same regardless of the rate at which the specimen is stretched (at least at moderate temperatures).

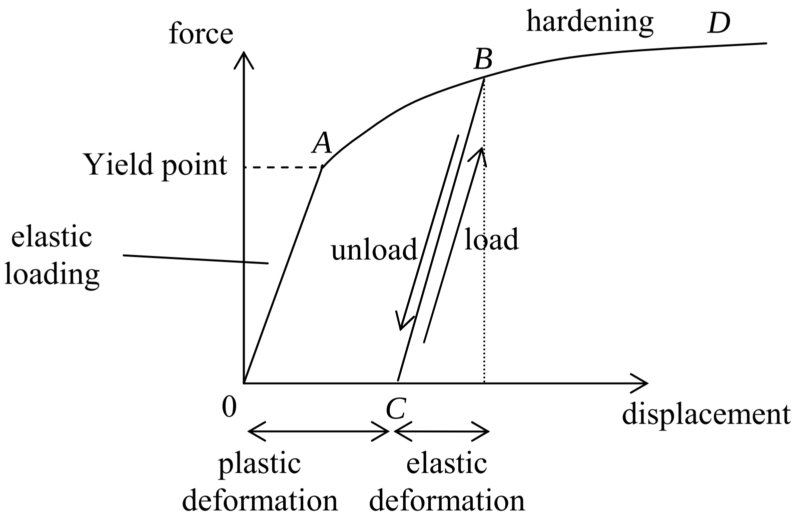


Figure 8.1.1: force/displacement curve for the tension test

Nominal and True Stress and Strain

There are two different ways of describing the force *F* which acts in a tension test. First, normalising with respect to the *original* cross sectional area of the tension test specimen A_0, one has the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{8.1.1}$$

Alternatively, one can normalise with respect to the *current* cross-sectional area *A*, leading to the **true stress**,

$$\sigma = \frac{F}{A} \tag{8.1.2}$$

in which F and A are both changing with time. For very small elongations, within the elastic range say, the cross-sectional area of the material undergoes negligible change and both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. Denoting the original specimen length by l_0 and the current length by l, one has the **engineering strain**

$$\varepsilon = \frac{l - l_0}{l_0} \tag{8.1.3}$$

Alternatively, the **true strain** is based on the fact that the "original length" is continually changing; a small change in length dl leads to a **strain increment** $d\varepsilon = dl/l$ and the total strain is *defined* as the accumulation of these increments:

$$\varepsilon_t = \int_{l_0}^{l} \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right) \tag{8.1.4}$$

The true strain is also called the **logarithmic strain** or **Hencky strain**. Again, at small deformations, the difference between these two strain measures is negligible. The true strain and engineering strain are related through

$$\varepsilon_t = \ln(1 + \varepsilon) \tag{8.1.5}$$

Using the assumption of constant volume for plastic deformation and ignoring the very small elastic volume changes, one has also {▲Problem 3}

$$\sigma = \sigma_n \frac{l}{l_0}. \tag{8.1.6}$$

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 8.1.2. The shape of the nominal stress/strain diagram, Fig. 8.1.2a, is of course the same as the graph of force versus displacement (change in length) in Fig. 8.1.1. A here denotes the point at which the maximum force the specimen can withstand has been reached. The *nominal stress* at A is called the **Ultimate Tensile Strength** (UTS) of the material. After this point, the specimen "necks", with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen ruptures, as indicated by the asterisk.

Note that, during loading into the plastic region, *the yield stress increases*. For example, if one unloads and re-loads (as in Fig. 8.1.1), the material stays elastic up until a stress higher than the original yield stress Y. In this respect, the stress-strain curve can be regarded as a yield stress versus strain curve.

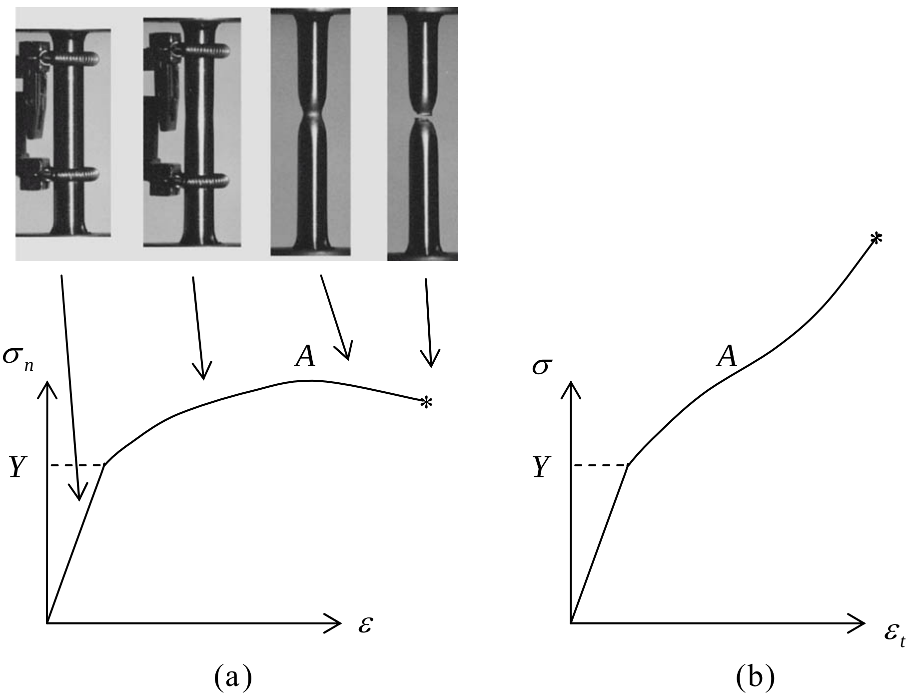


Figure 8.1.2: typical stress/strain curves; (a) engineering stress and strain, (b) true stress and strain

Compression Test

A compression test will lead to similar results as the tensile stress. The yield stress in compression will be approximately the same as (the negative of) the yield stress in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), the two curves will more or less coincide. This would indicate that the behaviour of the material under compression is broadly similar to that under tension. If one were to use the nominal stress and strain, then the two curves would not coincide; this is one of a number of good reasons for using the *true* definitions.

The Bauschinger Effect

If one takes a virgin sample and loads it in tension into the plastic range, and *then* unloads it and continues on into compression, one finds that the yield stress in compression is *not* the same as the yield strength in tension, as it would have been if the specimen had not first been loaded in tension. In fact the yield point in this case will be significantly *less* than the corresponding yield stress in tension. This reduction in yield stress is known as the **Bauschinger effect**. The effect is illustrated in Fig. 8.1.3. The solid line depicts the response of a real material. The dotted lines are two extreme cases which are used in plasticity models; the first is the **isotropic hardening** model, in which the yield stress in tension and compression are maintained equal, the second being **kinematic hardening**, in which the total elastic range is maintained constant throughout the deformation.

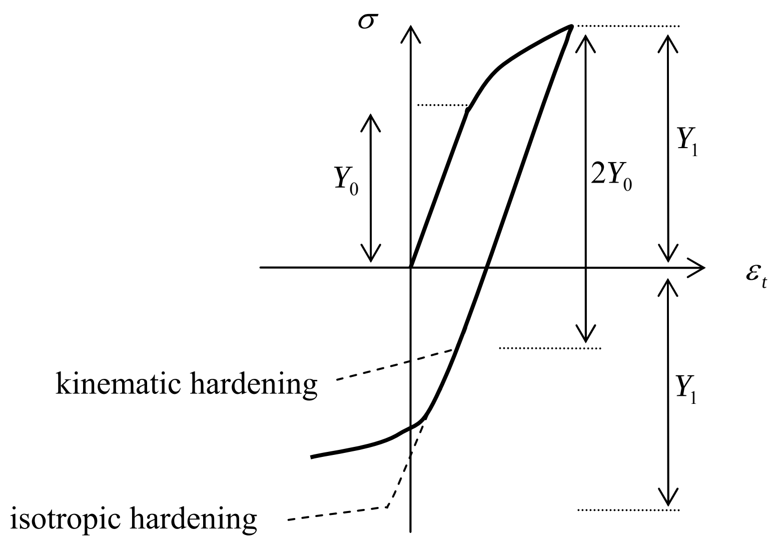


Figure 8.1.3: The Bauschinger effect

The presence of the Bauschinger effect complicates any plasticity theory. However, it is not an issue provided there are no reversals of stress in the problem under study.

Hydrostatic Pressure

Careful experiments show that, for metals, the yield behaviour is independent of hydrostatic pressure. That is, a stress state $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ has negligible effect on the yield stress of a material, right up to very high pressures. Note however that this is not true for soils or rocks.

8.1.3 Assumptions of Plasticity Theory

Regarding the above test results then, in formulating a basic plasticity theory with which to begin, the following assumptions are usually made:

(1) the response is independent of rate effects
(2) the material is incompressible in the plastic range
(3) there is no Bauschinger effect
(4) the yield stress is independent of hydrostatic pressure
(5) the material is isotropic

The first two of these will usually be very good approximations, the other three may or may not be, depending on the material and circumstances. For example, most metals can be regarded as isotropic. After large plastic deformation however, for example in rolling, the material will have become anisotropic: there will be distinct material directions and asymmetries.

Together with these, assumptions can be made on the type of hardening and on whether elastic deformations are significant. For example, consider the hierarchy of models illustrated in Fig. 8.1.4 below, commonly used in theoretical analyses. In (a) both the elastic and plastic curves are assumed linear. In (b) work-hardening is neglected and the yield stress is constant after initial yield. Such **perfectly-plastic** models are particularly appropriate for studying processes where the metal is worked at a high temperature – such as hot rolling – where work hardening is small. In many areas of applications the strains involved are large, e.g. in metal working processes such as extrusion, rolling or drawing, where up to 50% reduction ratios are common. In such cases the elastic strains can be neglected altogether as in the two models (c) and (d). The **rigid/perfectly-plastic** model (d) is the crudest of all – and hence in many ways the most useful. It is widely used in analysing metal forming processes, in the design of steel and concrete structures and in the analysis of soil and rock stability.

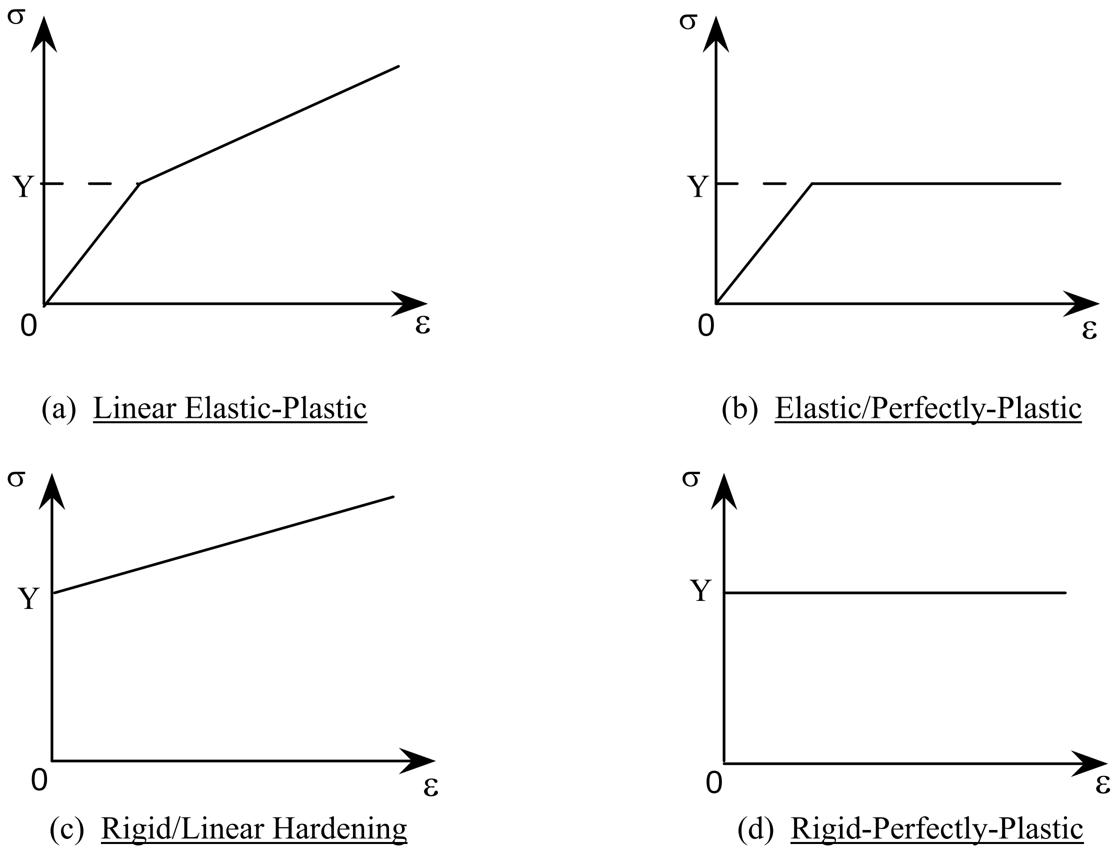


Figure 8.1.4: Simple models of elastic and plastic deformation

8.1.4 The Tangent and Plastic Modulus

Stress and strain are related through $\sigma = E\varepsilon$ in the elastic region, E being the Young's modulus, Fig. 8.1.5. The **tangent modulus** K is the slope of the stress-strain curve in the plastic region and will in general change during a deformation. At any instant of strain, the *increment* in stress $d\sigma$ is related to the *increment* in strain $d\varepsilon$ through[2]

$$d\sigma = Kd\varepsilon \tag{8.1.7}$$

[2] the symbol ε here represents the true strain (the subscript t has been dropped for clarity); as mentioned, when the strains are small, it is not necessary to specify which strain is in use since all strain measures are then equivalent

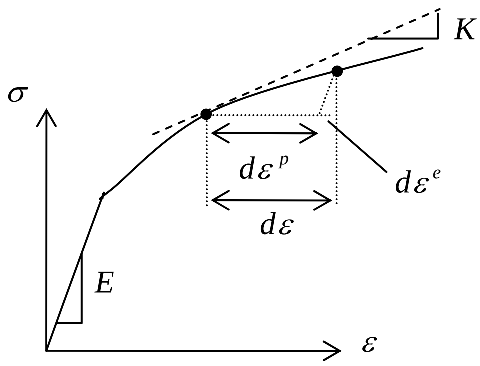


Figure 8.1.5: The tangent modulus

After yield, the strain increment consists of both elastic, ε^e, and plastic, $d\varepsilon^p$, strains:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \tag{8.1.8}$$

The stress and plastic strain increments are related by the **plastic modulus** *H*:

$$d\sigma = H\, d\varepsilon^p \tag{8.1.9}$$

and it follows that {▲Problem 4}

$$\frac{1}{K} = \frac{1}{E} + \frac{1}{H} \tag{8.1.10}$$

8.1.5 Friction Block Models

Some additional insight into the way plastic materials respond can be obtained from friction block models. The rigid perfectly plastic model can be simulated by a Coulomb friction block, Fig. 8.1.6. No strain occurs until σ reaches the yield stress *Y*. Then there is movement – although the amount of movement or plastic strain cannot be determined without more information being available. The stress cannot exceed the yield stress in this model:

$$|\sigma| \le Y \tag{8.1.11}$$

If unloaded, the block stops moving and the stress returns to zero, leaving a permanent strain, Fig. 8.1.6b.

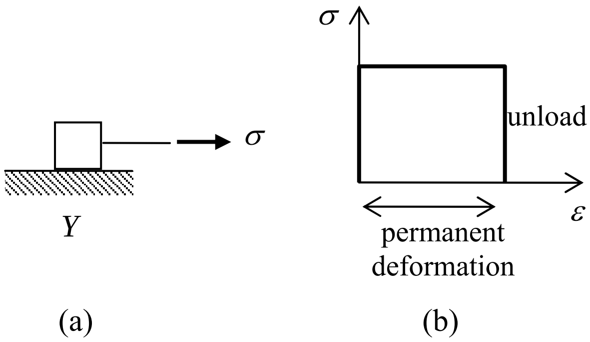


Figure 8.1.6: (a) Friction block model for the rigid perfectly plastic material, (b) response of the rigid-perfectly plastic model

The linear elastic perfectly plastic model incorporates a free spring with modulus *E* in series with a friction block, Fig. 8.1.7. The spring stretches when loaded and the block also begins to move when the stress reaches *Y*, at which time the spring stops stretching, the maximum possible stress again being *Y*. Upon unloading, the block stops moving and the spring contracts.

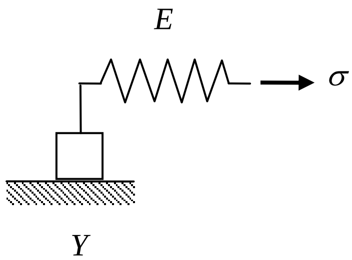


Figure 8.1.7: Friction block model for the elastic perfectly plastic material

The linear elastic plastic model with linear strain hardening incorporates a second, hardening, spring with stiffness *H*, in parallel with the friction block, Fig. 8.1.8. Once the yield stress is reached, an ever increasing stress needs to be applied in order to keep the block moving – and elastic strain continues to occur due to further elongation of the free spring. The stress is then split into the yield stress, which is carried by the moving block, and an **overstress** $\sigma - Y$ carried by the hardening spring.

Upon unloading, the block "locks" – the stress in the hardening spring remains constant whilst the free spring contracts. At zero stress, there is a negative stress taken up by the friction block, equal and opposite to the stress in the hardening spring.

The slope of the elastic loading line is *E*. For the plastic hardening line,

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \frac{\sigma - Y}{H} \qquad \rightarrow \qquad K = \frac{d\sigma}{d\varepsilon} = \frac{EH}{E+H} \tag{8.1.12}$$

It can be seen that *H* is the plastic modulus.

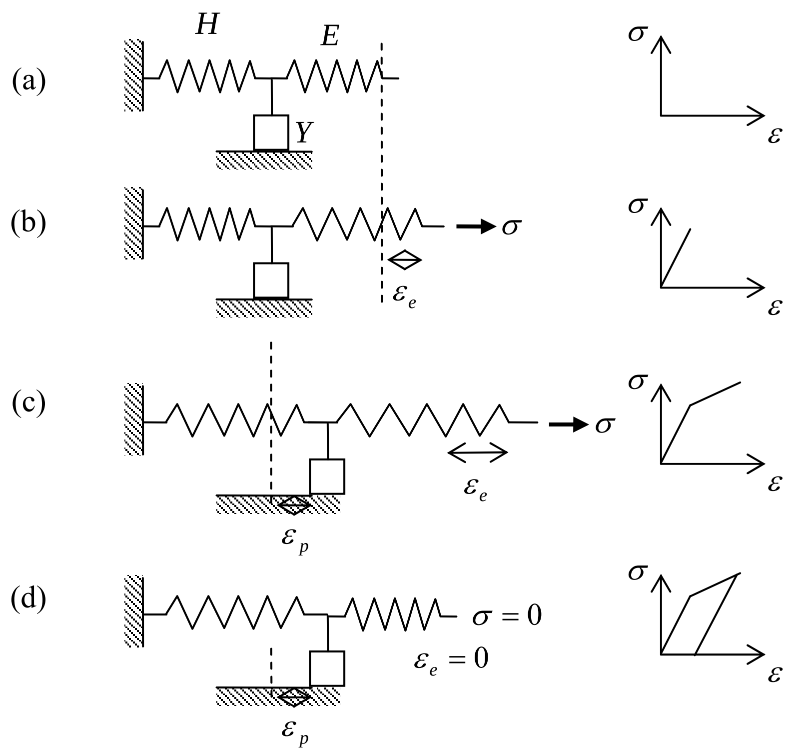


Figure 8.1.8: Friction block model for a linear elastic-plastic material with linear strain hardening; (a) stress-free, (b) elastic strain, (c) elastic and plastic strain, (d) unloading

8.1.6 Problems

1. Give two differences between plastic and viscoelastic materials.

2. A test specimen of initial length $0.01\,\text{m}$ is extended to length $0.0101\,\text{m}$. What is the percentage difference between the engineering and true strains (relative to the engineering strain)? What is this difference when the specimen is extended to length $0.015\,\text{m}$?

3. Derive the relation 8.1.6, $\sigma / \sigma_n = l / l_0$.

4. Derive Eqn. 8.1.10.

5. Which is larger, *H* or *K*? In the case of a perfectly-plastic material?

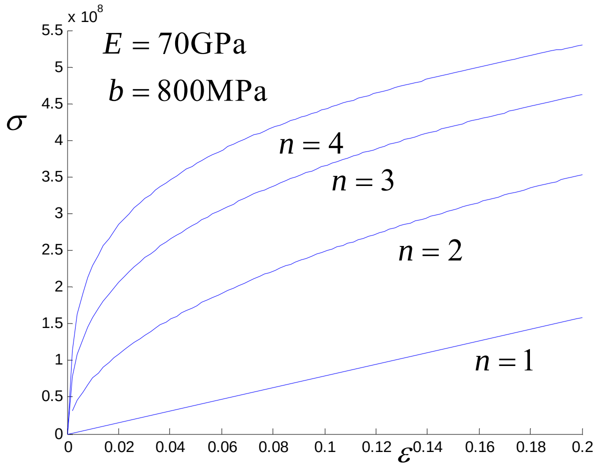
6. The **Ramberg-Osgood** model of plasticity is given by

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \left(\frac{\sigma}{b} \right)^n$$

where *E* is the Young's modulus and *b* and *n* are model constants (material parameters) obtained from a curve-fitting of the uniaxial stress-strain curve.

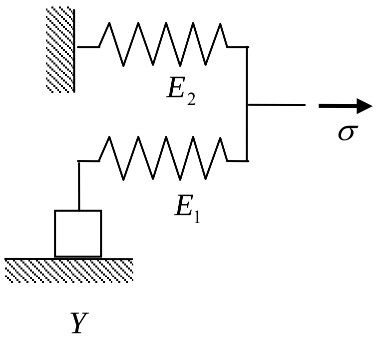
(i) Find the tangent and plastic moduli in terms of plastic strain ε^p (and the material constants).

(ii) A material with model parameters $n = 4$, $E = 70\,\text{GPa}$ and $b = 800\,\text{MPa}$ is strained in tension to $\varepsilon^p = 0.02$ and is subsequently unloaded and put into compression. Find the stress at the initiation of compressive yield assuming isotropic hardening.



[Note that the yield stress is actually zero in this model, although the plastic strain at relatively low stress levels is small for larger values of *n*.]

7. Consider the plasticity model shown below.
   (i) What is the elastic modulus?
   (ii) What is the yield stress?
   (iii) What are the tangent and plastic moduli?
   Draw a typical loading and unloading curve.



8. Draw the stress-strain diagram for a cycle of loading and unloading to the rigid - plastic model shown here. Take the maximum load reached to be $\sigma_{\max} = 4Y_1$ and $Y_2 = 2Y_1$. What is the permanent deformation after complete removal of the load? [Hint: split the cycle into the following regions: (a) $0 \le \sigma \le Y_1$, (b) $Y_1 \le \sigma \le 3Y_1$, (c) $3Y_1 \le \sigma \le 4Y_1$, then unload, (d) $4Y_1 \le \sigma \le 3Y_1$, (e) $3Y_1 \le \sigma \le 2Y_1$, (f) $2Y_1 \le \sigma \le 0$.]