7.1 Vectors, Tensors and the Index Notation

The equations governing three dimensional mechanics problems can be quite lengthy. For this reason, it is essential to use a short-hand notation called the **index notation**[1]. Consider first the notation used for vectors.

7.1.1 Vectors

Vectors are used to describe physical quantities which have both a magnitude and a direction associated with them. Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow. Analytically, in what follows, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **b**.

The **dot product** of two vectors **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is a scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta . \tag{7.1.1}$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$.

Cartesian Coordinate System

So far the short discussion has been in **symbolic notation**[2], that is, no reference to 'axes' or 'components' or 'coordinates' is made, implied or required. Vectors exist independently of any coordinate system. The symbolic notation is very useful, but there are many circumstances in which use of the component forms of vectors is more helpful – or essential. To this end, introduce the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ having the properties

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \tag{7.1.2}$$

so that they are mutually perpendicular, and

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \tag{7.1.3}$$

so that they are unit vectors. Such a set of orthogonal unit vectors is called an **orthonormal** set, Fig. 7.1.1. This set of vectors forms a **basis,** by which is meant that any other vector can be written as a **linear combination** of these vectors, i.e. in the form

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{7.1.4}$$

where a_1, a_2 and a_3 are scalars, called the **Cartesian components** or **coordinates** of **a** along the given three directions. The unit vectors are called **base vectors** when used for

[1] or **indicial** or **subscript** or **suffix** notation
[2] or **absolute** or **invariant** or **direct** or **vector** notation

this purpose. The components a_1, a_2 and a_3 are measured along lines called the x_1, x_2 and x_3 **axes**, drawn through the base vectors.

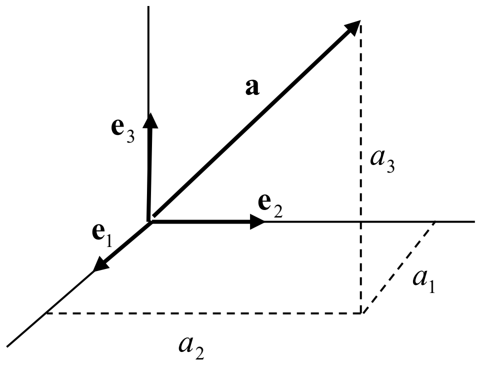

Figure 7.1.1: an orthonormal set of base vectors and Cartesian coordinates

Note further that this orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is **right-handed**, by which is meant $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (or $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ or $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$).

In the index notation, the expression for the vector **a** in terms of the components a_1, a_2, a_3 and the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \tag{7.1.5}$$

This can be simplified further by using Einstein's **summation convention**, whereby the summation sign is dropped and it is understood that for a repeated index (i in this case) a summation over the range of the index (3 in this case[3]) is implied. Thus one writes $\mathbf{a} = a_i\mathbf{e}_i$. This can be further shortened to, simply, a_i.

The dot product of two vectors **u** and **v**, referred to this coordinate system, is

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \cdot (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1v_1(\mathbf{e}_1 \cdot \mathbf{e}_1) + u_1v_2(\mathbf{e}_1 \cdot \mathbf{e}_2) + u_1v_3(\mathbf{e}_1 \cdot \mathbf{e}_3) \\
&\quad + u_2v_1(\mathbf{e}_2 \cdot \mathbf{e}_1) + u_2v_2(\mathbf{e}_2 \cdot \mathbf{e}_2) + u_2v_3(\mathbf{e}_2 \cdot \mathbf{e}_3) \\
&\quad + u_3v_1(\mathbf{e}_3 \cdot \mathbf{e}_1) + u_3v_2(\mathbf{e}_3 \cdot \mathbf{e}_2) + u_3v_3(\mathbf{e}_3 \cdot \mathbf{e}_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned} \tag{7.1.6}$$

The dot product of two vectors written in the index notation reads

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_iv_i} \qquad \textbf{Dot Product} \tag{7.1.7}$$

[3] 2 in the case of a two-dimensional space/analysis

The repeated index i is called a **dummy index**, because it can be replaced with any other letter and the sum is the same; for example, this could equally well be written as $\mathbf{u} \cdot \mathbf{v} = u_j v_j$ or $u_k v_k$.

Introduce next the **Kronecker delta symbol** δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{7.1.8}$$

Note that $\delta_{11} = 1$ but, using the index notation, $\delta_{ii} = 3$. The Kronecker delta allows one to write the expressions defining the orthonormal basis vectors (7.1.2, 7.1.3) in the compact form

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \qquad \textbf{Orthonormal Basis Rule} \tag{7.1.9}$$

Example

Recall the equations of motion, Eqns. 1.1.9, which in full read

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 &= \rho a_1 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 &= \rho a_2 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= \rho a_3 \end{aligned} \tag{7.1.10}$$

The index notation for these equations is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.1.11}$$

Note the dummy index j. The index i is called a **free index**; if one term has a free index i, then, to be consistent, all terms must have it. One free index, as here, indicates three separate equations.

7.1.2 Matrix Notation

The symbolic notation $\mathbf{v}$ and index notation $v_i \mathbf{e}_i$ (or simply v_i) can be used to denote a vector. Another notation is the **matrix notation**: the vector **v** can be represented by a 3×1 matrix (a **column vector**):

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Matrices will be denoted by square brackets, so a shorthand notation for this matrix/vector would be $[\mathbf{v}]$. The elements of the matrix $[\mathbf{v}]$ can be written in the index notation v_i.

Note the distinction between a vector and a 3×1 matrix: the former is a mathematical object independent of any coordinate system, the latter is a representation of the vector in a particular coordinate system – matrix notation, as with the index notation, relies on a particular coordinate system.

As an example, the dot product can be written in the matrix notation as

$$[\mathbf{u}^{\mathrm{T}}][\mathbf{v}] = [u_1 \quad u_2 \quad u_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

"short"
matrix notation

"full"
matrix notation

Here, the notation $[\mathbf{u}^{\mathrm{T}}]$ denotes the 1×3 matrix (the **row vector**). The result is a 1×1 matrix, $u_i v_i$.

The matrix notation for the Kronecker delta δ_{ij} is the identity matrix

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, for example, in both index and matrix notation:

$$\delta_{ij} u_j = u_i \qquad [\mathbf{I}][\mathbf{u}] = [\mathbf{u}] \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{7.1.12}$$

Matrix – Matrix Multiplication

When discussing vector transformation equations further below, it will be necessary to multiply various matrices with each other (of sizes 3×1, 1×3 and 3×3). It will be helpful to write these matrix multiplications in the short-hand notation.

First, it has been seen that the dot product of two vectors can be represented by $[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]$ or $u_i v_i$. Similarly, the matrix multiplication $[\mathbf{u}][\mathbf{v}^{\mathrm{T}}]$ gives a 3×3 matrix with element form $u_i v_j$ or, in full,

$$\begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

This operation is called the **tensor product** of two vectors, written in symbolic notation as $\mathbf{u} \otimes \mathbf{v}$ (or simply **uv**).

Next, the matrix multiplication

$$[\mathbf{Q}][\mathbf{u}] \equiv \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

is a 3×1 matrix with elements $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij} u_j$. The elements of $[\mathbf{Q}][\mathbf{u}]$ are the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$, which can be expressed as $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}])_i \equiv u_j Q_{ij}$.

The expression $[\mathbf{u}][\mathbf{Q}]$ is meaningless, but $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ {▲Problem 4} is a 1×3 matrix with elements $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}])_i \equiv u_j Q_{ji}$.

This leads to the following rule:

1. if a vector pre-multiplies a matrix $[\mathbf{Q}]$ $\rightarrow$ the vector is the transpose $[\mathbf{u}^{\mathrm{T}}]$
2. if a matrix $[\mathbf{Q}]$ pre-multiplies the vector $\rightarrow$ the vector is $[\mathbf{u}]$
3. if summed indices are "beside each other", as the *j* in $u_j Q_{ji}$ or $Q_{ij} u_j$
   $\rightarrow$ the matrix is $[\mathbf{Q}]$
4. if summed indices are not beside each other, as the *j* in $u_j Q_{ij}$
   $\rightarrow$ the matrix is the transpose, $[\mathbf{Q}^{\mathrm{T}}]$

Finally, consider the multiplication of 3×3 matrices. Again, this follows the "beside each other" rule for the summed index. For example, $[\mathbf{A}][\mathbf{B}]$ gives the 3×3 matrix {▲Problem 8} $([\mathbf{A}][\mathbf{B}])_{ij} = A_{ik} B_{kj}$, and the multiplication $[\mathbf{A}^{\mathrm{T}}][\mathbf{B}]$ is written as $([\mathbf{A}^{\mathrm{T}}][\mathbf{B}])_{ij} = A_{ki} B_{kj}$. There is also the important identity

$$([\mathbf{A}][\mathbf{B}])^{\mathrm{T}} = [\mathbf{B}^{\mathrm{T}}][\mathbf{A}^{\mathrm{T}}] \tag{7.1.13}$$

Note also the following:

(i) if there is no free index, as in $u_i v_i$, there is one element

(ii) if there is one free index, as in $u_j Q_{ji}$, it is a 3×1 (or 1×3) matrix

(iii) if there are two free indices, as in $A_{ki} B_{kj}$, it is a 3×3 matrix

7.1.3 Vector Transformation Rule

Introduce two Cartesian coordinate systems with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ and common origin *o*, Fig. 7.1.2. The vector $\mathbf{u}$ can then be expressed in two ways:

$$\mathbf{u} = u_i \mathbf{e}_i = u'_i \mathbf{e}'_i \tag{7.1.14}$$

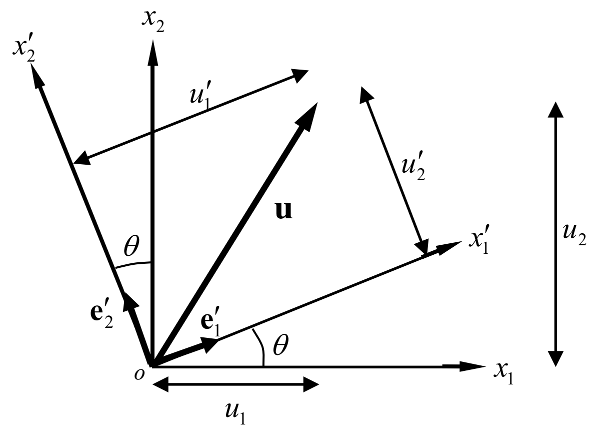

Figure 7.1.2: a vector represented using two different coordinate systems

Note that the x'_i coordinate system is obtained from the x_i system by a *rotation* of the base vectors. Fig. 7.1.2 shows a rotation θ about the x_3 axis (the sign convention for rotations is positive counterclockwise).

Concentrating for the moment on the two dimensions $x_1 - x_2$, from trigonometry (refer to Fig. 7.1.3),

$$\begin{aligned} \mathbf{u} &= u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 \\ &= \big[|OB| - |AB|\big]\mathbf{e}_1 + \big[|BD| + |CP|\big]\mathbf{e}_2 \\ &= \big[\cos\theta\, u'_1 - \sin\theta\, u'_2\big]\mathbf{e}_1 + \big[\sin\theta\, u'_1 + \cos\theta\, u'_2\big]\mathbf{e}_2 \end{aligned} \tag{7.1.15}$$

and so

$$\begin{aligned} u_1 &= \cos\theta\, u'_1 - \sin\theta\, u'_2 \\ u_2 &= \sin\theta\, u'_1 + \cos\theta\, u'_2 \end{aligned} \tag{7.1.16}$$

vector components in first coordinate system

vector components in second coordinate system

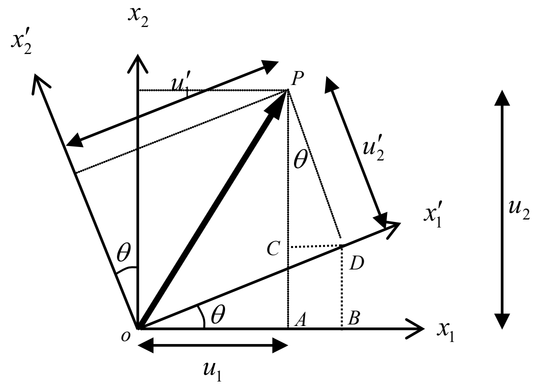

Figure 7.1.3: geometry of the 2D coordinate transformation

In matrix form, these transformation equations can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \tag{7.1.17}$$

The 2×2 matrix is called the **transformation matrix** or **rotation matrix** $[\mathbf{Q}]$. By pre-multiplying both sides of these equations by the inverse of $[\mathbf{Q}]$, $[\mathbf{Q}^{-1}]$, one obtains the transformation equations transforming from $[u_1 \quad u_2]^{\mathrm{T}}$ to $[u_1' \quad u_2']^{\mathrm{T}}$:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.1.18}$$

It can be seen that the components of $[\mathbf{Q}]$ are the **directions cosines**, i.e. the cosines of the angles between the coordinate directions:

$$Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j' \tag{7.1.19}$$

It is straight forward to show that, in the full three dimensions, Fig. 7.1.4, the components in the two coordinate systems are also related through

$$\boxed{\begin{array}{lll} u_i = Q_{ij} u_j' & \ldots & [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \\ u_i' = Q_{ji} u_j & \ldots & [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{array}} \quad \textbf{Vector Transformation Rule} \tag{7.1.20}$$

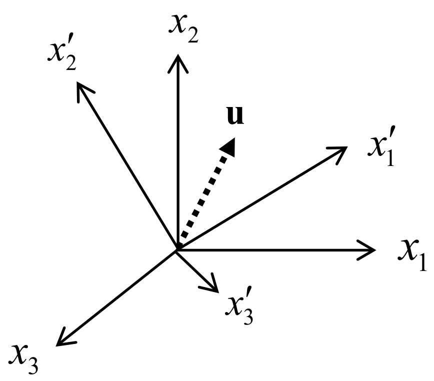

Figure 7.1.4: two different coordinate systems in a 3D space

Orthogonality of the Transformation Matrix $[\mathbf{Q}]$

From 7.1.20, it follows that

$$\begin{aligned} u_i &= Q_{ij} u_j' & \cdots \quad [\mathbf{u}] &= [\mathbf{Q}][\mathbf{u}'] \\ &= Q_{ij} Q_{kj} u_k & \cdots \quad &= [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{aligned} \tag{7.1.21}$$

and so

$$Q_{ij} Q_{kj} = \delta_{ik} \qquad \cdots \qquad [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}] = [\mathbf{I}] \tag{7.1.22}$$

A matrix such as this for which $[\mathbf{Q}^{\mathrm{T}}] = [\mathbf{Q}^{-1}]$ is called an **orthogonal matrix**.

Example

Consider a Cartesian coordinate system with base vectors $\mathbf{e}_i$. A coordinate transformation is carried out with the new basis given by

$$\begin{aligned} \mathbf{e}_1' &= a_1^{(1)} \mathbf{e}_1 + a_2^{(1)} \mathbf{e}_2 + a_3^{(1)} \mathbf{e}_3 \\ \mathbf{e}_2' &= a_1^{(2)} \mathbf{e}_1 + a_2^{(2)} \mathbf{e}_2 + a_3^{(2)} \mathbf{e}_3 \\ \mathbf{e}_3' &= a_1^{(3)} \mathbf{e}_1 + a_2^{(3)} \mathbf{e}_2 + a_3^{(3)} \mathbf{e}_3 \end{aligned}$$

What is the transformation matrix?

<u>Solution</u>

The transformation matrix consists of the direction cosines $Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j'$, so

$$[\mathbf{Q}] = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & a_1^{(3)} \\ a_2^{(1)} & a_2^{(2)} & a_2^{(3)} \\ a_3^{(1)} & a_3^{(2)} & a_3^{(3)} \end{bmatrix}$$

■

7.1.4 Tensors

The concept of the **tensor** is discussed in detail in Book III, where it is indispensable for the description of large-strain deformations. For small deformations, it is not so necessary; the main purpose for introducing the tensor here (in a rather non-rigorous way) is that it helps to deepen one's understanding of the concept of stress.

A **second-order tensor**[4] **A** may be *defined* as an operator that acts on a vector **u** generating another vector **v**, so that $\mathbf{T}(\mathbf{u}) = \mathbf{v}$, or

$$\boxed{\mathbf{Tu} = \mathbf{v}} \qquad \textbf{Second-order Tensor} \qquad (7.1.23)$$

The second-order tensor **T** is a **linear operator**, by which is meant

$$\begin{aligned} \mathbf{T}(\mathbf{a} + \mathbf{b}) &= \mathbf{Ta} + \mathbf{Tb} && \ldots \text{ distributive} \\ \mathbf{T}(\alpha\mathbf{a}) &= \alpha(\mathbf{Ta}) && \ldots \text{ associative} \end{aligned}$$

for scalar α. In a Cartesian coordinate system, the tensor **T** has nine components and can be represented in the matrix form

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

The rule 7.1.23, which is expressed in symbolic notation, can be expressed in the index and matrix notation when **T** is referred to particular axes:

$$u_i = T_{ij} v_j \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad (7.1.24)$$

Again, one should be careful to distinguish between a tensor such as **T** and particular matrix representations of that tensor. The relation 7.1.23 is a **tensor relation**, relating vectors and a tensor and is valid in all coordinate systems; the matrix representation of this tensor relation, Eqn. 7.1.24, is to be sure valid in all coordinate systems, but the entries in the matrices of 7.1.24 depend on the coordinate system chosen.

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[4] to be called simply a tensor in what follows

Note also that the transformation formulae for vectors, Eqn. 7.1.20, is not a tensor relation; although 7.1.20 looks similar to the tensor relation 7.1.24, the former relates the components of a vector to the components of the *same* vector in different coordinate systems, whereas (by definition of a tensor) the relation 7.1.24 relates the components of a vector to those of a different vector in the same coordinate system.

For these reasons, the notation $u_i = Q_{ij}u'$ in Eqn. 7.1.20 is more formally called **element form**, the Q_{ij} being elements of a matrix rather than components of a tensor. This distinction between element form and index notation should be noted, but the term "index notation" is used for both tensor and matrix-specific manipulations in these notes.

Example

Recall the strain-displacement relations, Eqns. 1.2.19, which in full read

$$\begin{aligned} &\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ &\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \quad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \end{aligned} \tag{7.1.25}$$

The index notation for these equations is

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.1.26}$$

This expression has two free indices and as such indicates nine separate equations. Further, with its two subscripts, ε_{ij}, the strain, is a tensor. It can be expressed in the matrix notation

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \partial u_1/\partial x_1 & \frac{1}{2}\left(\partial u_1/\partial x_2 + \partial u_2/\partial x_1\right) & \frac{1}{2}\left(\partial u_1/\partial x_3 + \partial u_3/\partial x_1\right) \\ \frac{1}{2}\left(\partial u_2/\partial x_1 + \partial u_1/\partial x_2\right) & \partial u_2/\partial x_2 & \frac{1}{2}\left(\partial u_2/\partial x_3 + \partial u_3/\partial x_2\right) \\ \frac{1}{2}\left(\partial u_3/\partial x_1 + \partial u_1/\partial x_3\right) & \frac{1}{2}\left(\partial u_3/\partial x_2 + \partial u_2/\partial x_3\right) & \partial u_3/\partial x_3 \end{bmatrix}$$

7.1.5 Tensor Transformation Rule

Consider now the tensor definition 7.1.23 expressed in two different coordinate systems:

$$\begin{aligned} u_i &= T_{ij}v_j \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad \text{in } \{x_i\} \\ u'_i &= T'_{ij}v'_j \qquad [\mathbf{u}'] = [\mathbf{T}'][\mathbf{v}'] \qquad \text{in } \{x'_i\} \end{aligned} \tag{7.1.27}$$

From the vector transformation rule 7.1.20,

$$\begin{aligned} u_i' &= Q_{ji}u_j \qquad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \\ v_i' &= Q_{ji}v_j \qquad [\mathbf{v}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \end{aligned} \tag{7.1.28}$$

Combining 7.1.27-28,

$$Q_{ji}u_j = T_{ij}'Q_{kj}v_k \qquad [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] = [\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.29}$$

and so

$$Q_{mi}Q_{ji}u_j = Q_{mi}T_{ij}'Q_{kj}v_k \qquad [\mathbf{u}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.30}$$

(Note that $Q_{mi}Q_{ji}u_j = \delta_{mj}u_j = u_m$.) Comparing with 7.1.24, it follows that

$$\boxed{\begin{aligned} T_{ij} &= Q_{ip}Q_{jq}T_{pq}' \qquad \ldots \quad [\mathbf{T}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}] \\ T_{ij}' &= Q_{pi}Q_{qj}T_{pq} \qquad \ldots \quad [\mathbf{T}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{T}][\mathbf{Q}] \end{aligned}} \qquad \textbf{Tensor Transformation Rule} \tag{7.1.31}$$

7.1.6 Problems

1. Write the following in index notation: $|\mathbf{v}|$, $\mathbf{v}\cdot\mathbf{e}_1$, $\mathbf{v}\cdot\mathbf{e}_k$.
2. Show that $\delta_{ij}a_ib_j$ is equivalent to $\mathbf{a}\cdot\mathbf{b}$.
3. Evaluate or simplify the following expressions:
   (a) δ_{kk} (b) $\delta_{ij}\delta_{ij}$ (c) $\delta_{ij}\delta_{jk}$
4. Show that $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ is a 1×3 matrix with elements u_jQ_{ji} (write the matrices out in full)
5. Show that $([\mathbf{Q}][\mathbf{u}])^{\mathrm{T}} = [\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$
6. Are the three elements of $[\mathbf{Q}][\mathbf{u}]$ the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$?
7. What is the index notation for $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$?
8. Write out the 3×3 matrices $[\mathbf{A}]$ and $[\mathbf{B}]$ in full, i.e. in terms of A_{11}, A_{12}, etc. and verify that $[\mathbf{AB}]_{ij} = A_{ik}B_{kj}$ for $i=2,\ j=1$.
9. What is the index notation for
   (a) $[\mathbf{A}][\mathbf{B}^{\mathrm{T}}]$
   (b) $[\mathbf{v}^{\mathrm{T}}][\mathbf{A}][\mathbf{v}]$ (there is no ambiguity here, since $([\mathbf{v}^{\mathrm{T}}][\mathbf{A}])[\mathbf{v}] = [\mathbf{v}^{\mathrm{T}}]([\mathbf{A}][\mathbf{v}])$)
   (c) $[\mathbf{B}^{\mathrm{T}}][\mathbf{A}][\mathbf{B}]$
10. The angles between the axes in two coordinate systems are given in the table below.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

Construct the corresponding transformation matrix $[\mathbf{Q}]$ and verify that it is orthogonal.

11. Consider a two-dimensional problem. If the components of a vector **u** in one coordinate system are

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

what are they in a second coordinate system, obtained from the first by a positive rotation of 30°? Sketch the two coordinate systems and the vector to see if your answer makes sense.

12. Consider again a two-dimensional problem with the same change in coordinates as in Problem 11. The components of a 2D tensor in the first system are

$$\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

What are they in the second coordinate system?