6.1 Plate Theory

6.1.1 Plates

A **plate** is a flat structural element for which the thickness is small compared with the surface dimensions. The thickness is usually constant but may be variable and is measured normal to the **middle surface** of the plate, Fig. 6.1.1

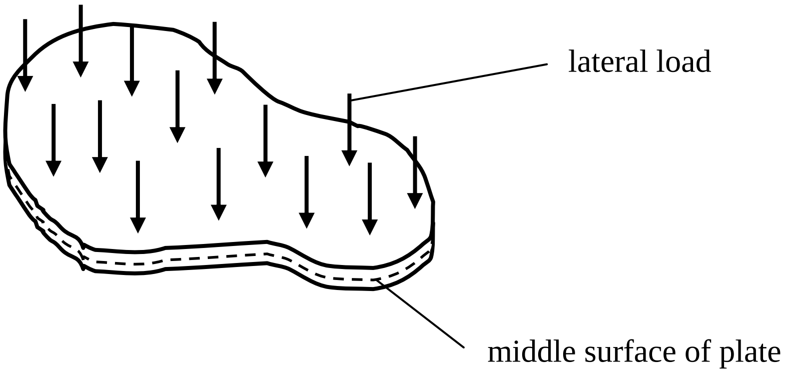


Fig. 6.1.1: A plate

6.1.2 Plate Theory

Plates subjected only to in-plane loading can be solved using two-dimensional plane stress theory[1] (see Book I, §3.5). On the other hand, **plate theory** is concerned mainly with **lateral loading**.

One of the differences between plane stress and plate theory is that in the plate theory the stress components are allowed to vary *through the thickness* of the plate, so that there can be bending moments, Fig. 6.1.2.

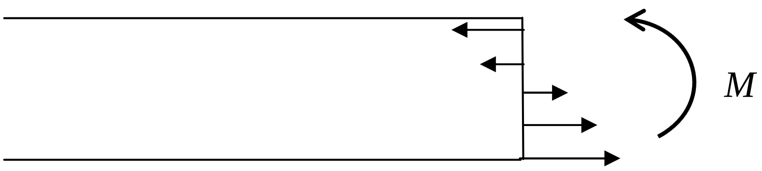


Fig. 6.1.2: Stress distribution through the thickness of a plate and resultant bending moment

Plate Theory and Beam Theory

Plate theory is an approximate theory; assumptions are made and the general three dimensional equations of elasticity are reduced. It is very like the **beam theory** (see Book

[1] although if the in-plane loads are compressive and sufficiently large, they can buckle (see §6.7)

I, §7.4) – only with an extra dimension. It turns out to be an accurate theory provided *the plate is relatively thin* (as in the beam theory) but also that *the deflections are small relative to the thickness*. This last point will be discussed further in §6.10.

Things are more complicated for plates than for the beams. For one, the plate not only bends, but torsion may occur (it can twist), as shown in Fig. 6.1.3

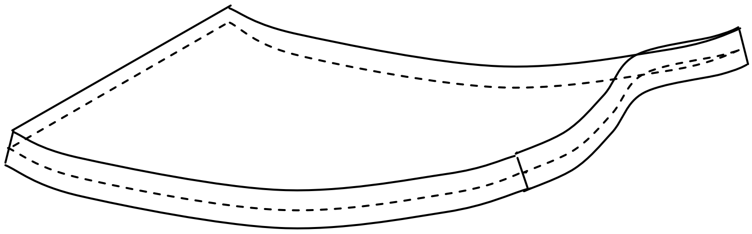


Fig. 6.1.3: torsion of a plate

Assumptions of Plate Theory

Let the plate mid-surface lie in the $x-y$ plane and the z – axis be along the thickness direction, forming a right handed set, Fig. 6.1.4.

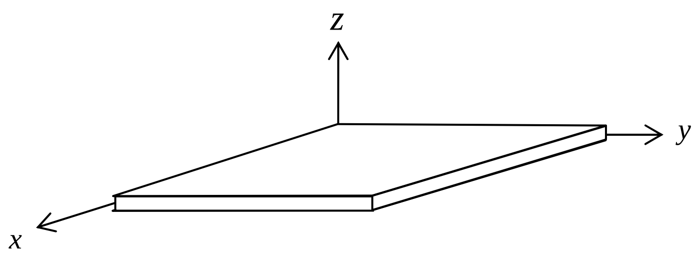


Fig. 6.1.4: Cartesian axes

The stress components acting on a typical element of the plate are shown in Fig. 6.1.5.

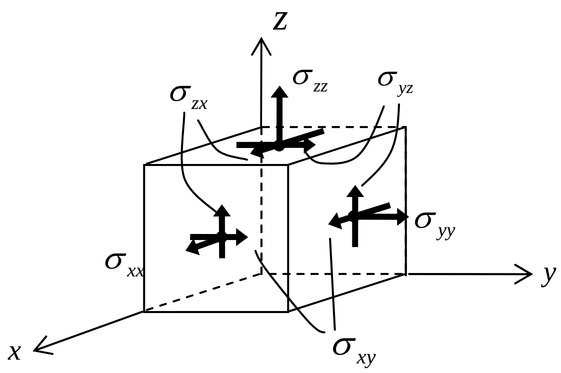


Fig. 6.1.5: stresses acting on a material element

The following assumptions are made:

(i) The mid-plane is a "neutral plane"

The middle plane of the plate remains free of in-plane stress/strain. Bending of the plate will cause material above and below this mid-plane to deform in-plane. The mid-plane plays the same role in plate theory as the neutral axis does in the beam theory.

(ii) Line elements remain normal to the mid-plane

Line elements lying perpendicular to the middle surface of the plate remain perpendicular to the middle surface during deformation, Fig. 6.1.6; this is similar the "plane sections remain plane" assumption of the beam theory.

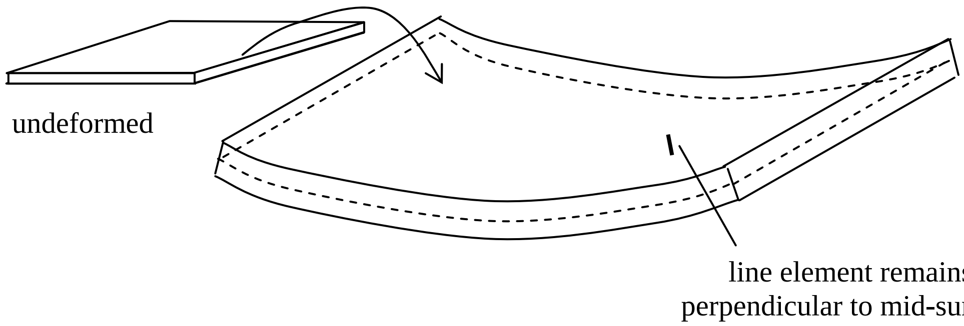


Fig. 6.1.6: deformed line elements remain perpendicular to the mid-plane

(iii) Vertical strain is ignored

Line elements lying perpendicular to the mid-surface do not change length during deformation, so that $\varepsilon_{zz} = 0$ throughout the plate. Again, this is similar to an assumption of the beam theory.

These three assumptions are the basis of the **Classical Plate Theory** or the **Kirchhoff Plate Theory**. The second assumption can be relaxed to develop a more exact theory (see §6.10).

6.1.3 Notation and Stress Resultants

The **stress resultants** are obtained by integrating the stresses through the thickness of the plate. In general there will be

moments *M*:	2 bending moments and 1 twisting moment
out-of-plane forces *V*:	2 shearing forces
in-plane forces *N*:	2 normal forces and 1 shear force

They are defined as follows:

In-plane normal forces and bending moments, Fig. 6.1.7:

$$
\begin{aligned}
N_x &= \int_{-h/2}^{+h/2} \sigma_{xx} dz, \qquad N_y = \int_{-h/2}^{+h/2} \sigma_{yy} dz \\
M_x &= -\int_{-h/2}^{+h/2} z\sigma_{xx} dz, \quad M_y = -\int_{-h/2}^{+h/2} z\sigma_{yy} dz
\end{aligned}
\tag{6.1.1}
$$

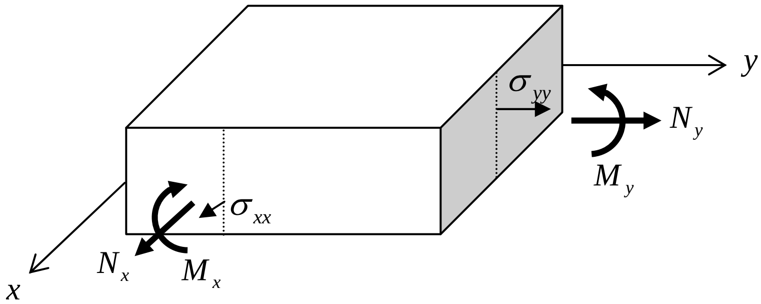


Fig. 6.1.7: in-plane normal forces and bending moments

In-plane shear force and twisting moment, Fig. 6.1.8:

$$
N_{xy} = \int_{-h/2}^{+h/2} \sigma_{xy} dz, \qquad M_{xy} = \int_{-h/2}^{+h/2} z\sigma_{xy} dz
\tag{6.1.2}
$$

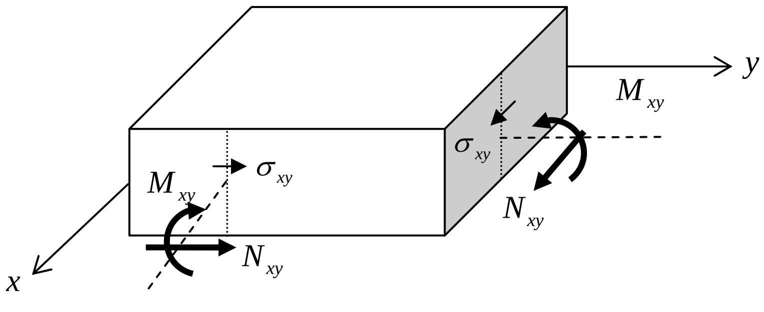


Fig. 6.1.8: in-plane shear force and twisting moment

Out-of-plane shearing forces, Fig. 6.1.9:

$$
V_x = -\int_{-h/2}^{+h/2} \sigma_{zx} dz, \qquad V_y = -\int_{-h/2}^{+h/2} \sigma_{yz} dz
\tag{6.1.3}
$$

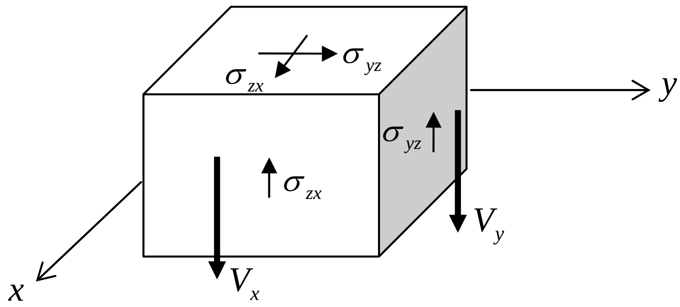


Fig. 6.1.9: out of plane shearing forces

Note that the above "forces" and "moments" are actually forces and moments *per unit length*. This allows one to have moments varying across any section – unlike in the beam theory, where the moments are for the complete beam cross-section. If one considers an element with dimensions Δx and Δy, the actual moments acting on the element are

$$M_x \Delta y, \quad M_y \Delta x, \quad M_{xy} \Delta x, \quad M_{xy} \Delta y \tag{6.1.4}$$

and the forces acting on the element are

$$V_x \Delta y, \quad V_y \Delta x, \quad N_x \Delta y, \quad N_y \Delta x, \quad N_{xy} \Delta x, \quad N_{xy} \Delta y \tag{6.1.5}$$

The in-plane forces, which are analogous to the axial forces of the beam theory, do not play a role in most of what follows. They are useful in the analysis of buckling of plates and it is necessary to consider them in more exact theories of plate bending (see later).

6.2 The Moment-Curvature Equations

6.2.1 From Beam Theory to Plate Theory

In the beam theory, based on the assumptions of plane sections remaining plane and that one can neglect the transverse strain, the strain varies linearly through the thickness. In the notation of the beam, with y positive up, $\varepsilon_{xx} = -y/R$, where R is the **radius of curvature**, R positive when the beam bends "up" (see Book I, Eqn. 7.4.16). In terms of the **curvature** $\partial^2 v/\partial x^2 = 1/R$, where v is the deflection (see Book I, Eqn. 7.4.36), one has

$$\varepsilon_{xx} = -y\frac{\partial^2 v}{\partial x^2} \tag{6.2.1}$$

The beam theory assumptions are essentially the same for the plate, leading to strains which are proportional to distance from the neutral (mid-plane) surface, z, and expressions similar to 6.2.1. This leads again to linearly varying stresses σ_{xx} and σ_{yy} (σ_{zz} is also taken to be zero, as in the beam theory).

6.2.2 Curvature and Twist

The plate is initially undeformed and flat with the mid-surface lying in the $x-y$ plane. When deformed, the mid-surface occupies the surface $w = w(x,y)$ and w is the elevation above the $x-y$ plane, Fig. 6.2.1.

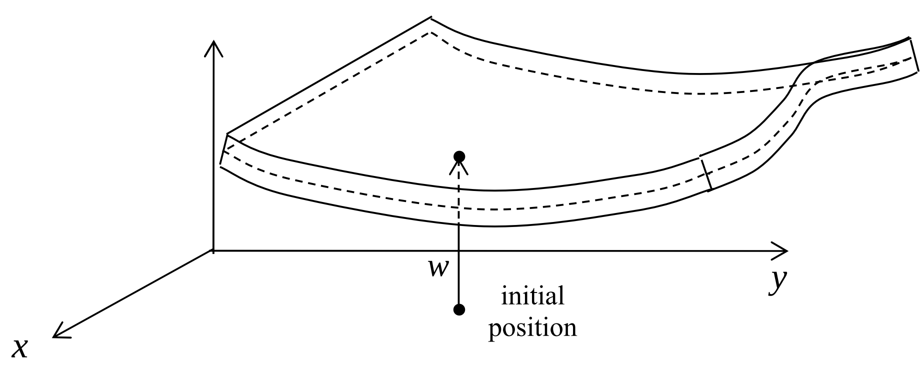


Fig. 6.2.1: Deformed Plate

The slopes of the plate along the x and y directions are $\partial w/\partial x$ and $\partial w/\partial y$.

Curvature

Recall from Book I, §7.4.11, that the curvature in the *x* direction, κ_x, is the rate of change of the slope angle ψ with respect to arc length *s*, Fig. 6.2.2, $\kappa_x = d\psi / ds$. One finds that

$$\kappa_x = \frac{\partial^2 w / \partial x^2}{\left[1 + \left(\partial w / \partial x\right)^2\right]^{3/2}} \tag{6.2.2}$$

Also, the radius of curvature R_x, Fig. 6.2.2, is the reciprocal of the curvature, $R_x = 1/\kappa_x$.

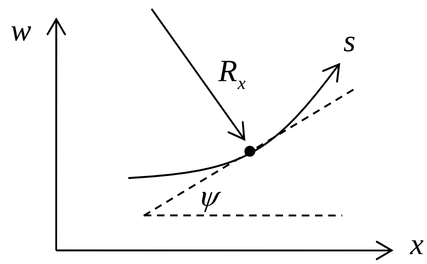


Fig. 6.2.2: Angle and arc-length used in the definition of curvature

As with the beam, *when the slope is small*, one can take $\psi \approx \tan\psi = \partial w / \partial x$ and $d\psi / ds \approx \partial \psi / \partial x$ and Eqn. 6.2.2 reduces to (and similarly for the curvature in the *y* direction)

$$\kappa_x = \frac{1}{R_x} = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = \frac{1}{R_y} = \frac{\partial^2 w}{\partial y^2} \tag{6.2.3}$$

This important assumption of small slope, $\partial w / \partial x, \partial w / \partial y << 1$, means that the theory to be developed will be valid when the deflections are small compared to the overall dimensions of the plate.

The curvatures 6.2.3 can be interpreted as in Fig. 6.2.3, as the unit increase in slope along the *x* and *y* directions.

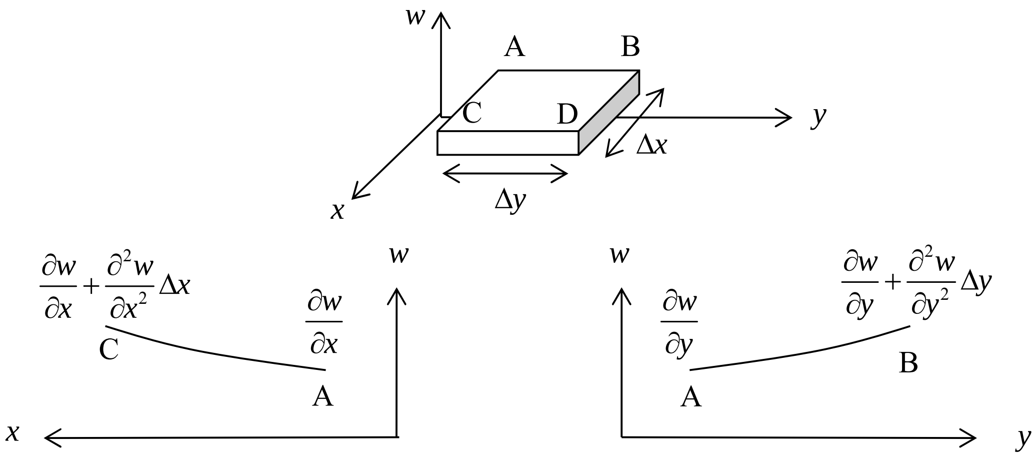


Figure 6.2.3: Physical meaning of the curvatures

Twist

Not only does a plate curve up or down, it can also twist (see Fig. 6.1.3). For example, shown in Fig. 6.2.4 is a plate undergoing a *pure* twisting (constant applied twisting moments and no bending moments).

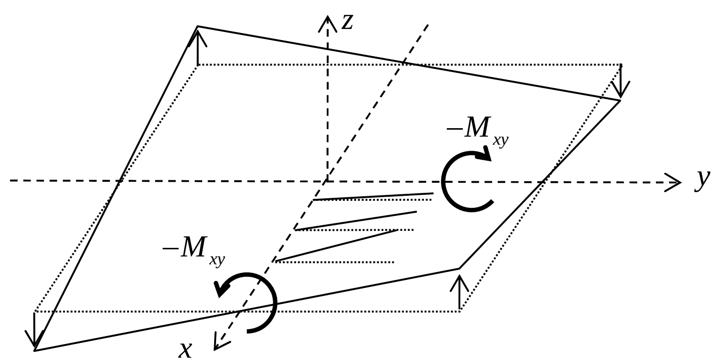


Figure 6.2.4: A twisting plate

If one takes a row of line elements lying in the *y* direction, emanating from the *x* axis, the further one moves along the *x* axis, the more they twist, Fig. 6.2.4. Some of these line elements are shown in Fig. 6.2.5 (bottom right), as veiwed looking down the *x* axis towards the origin (elements along the *y* axis are shown bottom left). If a line element at position *x* has slope $\partial w/\partial y$, the slope at $x+\Delta x$ is $\partial w/\partial y+\Delta x\partial(\partial w/\partial y)/\partial x$. This motivates the definition of the **twist**, defined analogously to the curvature, and denoted by $1/T_{xy}$; it is a measure of the "twistiness" of the plate:

$$\frac{1}{T_{xy}}=\frac{\partial^2 w}{\partial x\partial y} \tag{6.2.4}$$

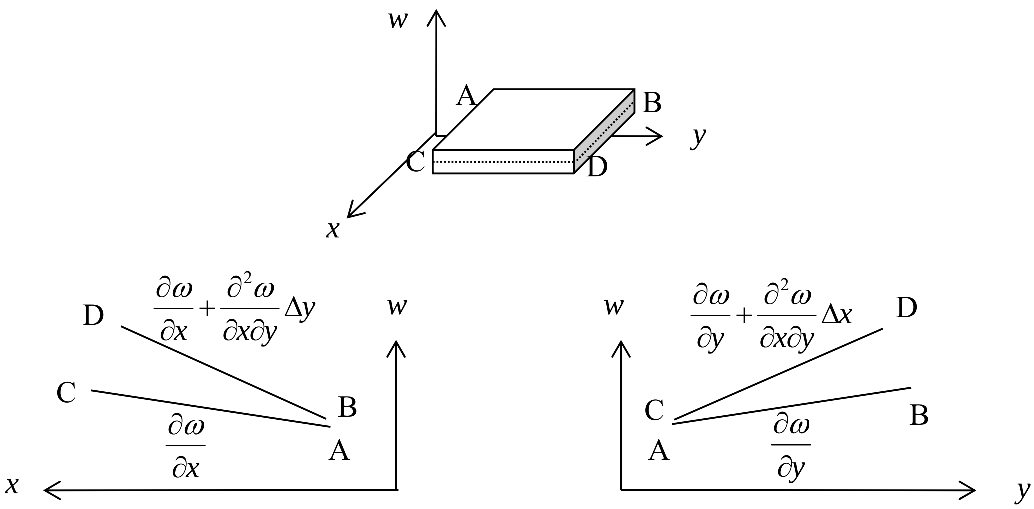


Figure 6.2.5: Physical meaning of the twist

The signs of the moments, radii of curvature and curvatures are illustrated in Fig. 6.2.6. Note that the deflection w may or may not be of the same sign as the curvature. Note also that when $M_x > 0$, $\partial^2\omega/\partial x^2 > 0$, when $M_y > 0$, $\partial^2\omega/\partial y^2 > 0$.

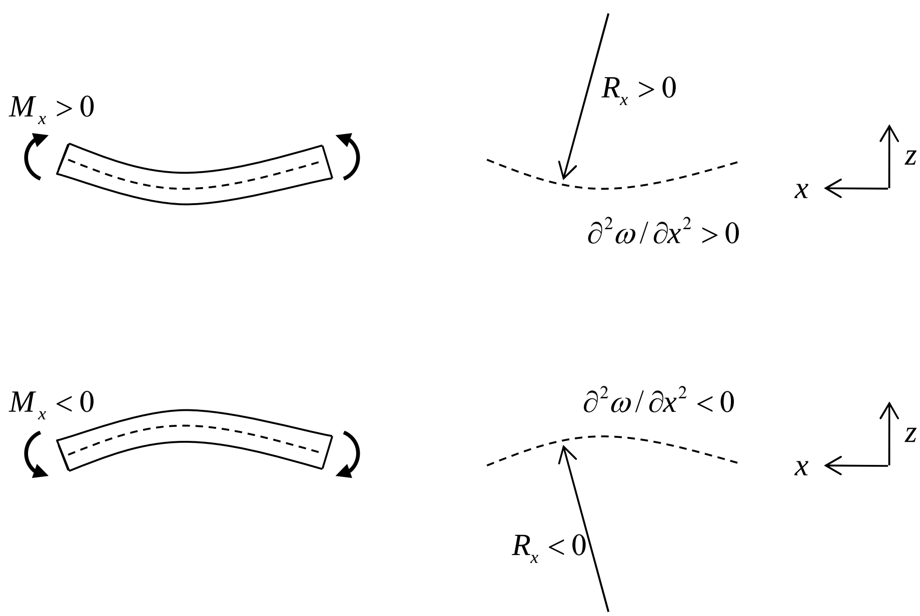


Figure 6.2.6: sign convention for curvatures and moments

On the other hand, for the twist, with the sign convention being used, when $M_{xy} > 0$, $\partial^2\omega/\partial x\partial y < 0$, as depicted in Fig. 6.2.4.

Principal Curvatures

Consider the two Cartesian coordinate systems shown in Fig. 6.2.7, the second ($t-n$) obtained from the first ($x-y$) by a positive rotation θ. The partial derivatives arising in the curvature expressions can be expressed in terms of derivatives with respect to *t* and *n* as follows: with $w = w(x, y)$, an increment in *w* is

$$\Delta w = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y \tag{6.2.5}$$

Also, referring to Fig. 6.2.7, with $\Delta n = 0$,

$$\Delta x = \Delta t \cos\theta, \qquad \Delta y = \Delta t \sin\theta \tag{6.2.6}$$

Thus

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta \tag{6.2.7}$$

Similarly, for an increment Δn, one finds that

$$\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial x}\sin\theta + \frac{\partial w}{\partial y}\cos\theta \tag{6.2.8}$$

Equations 6.2.7-8 can be inverted to get the inverse relations

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial t}\cos\theta - \frac{\partial w}{\partial n}\sin\theta \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial t}\sin\theta + \frac{\partial w}{\partial n}\cos\theta \end{aligned} \tag{6.2.9}$$

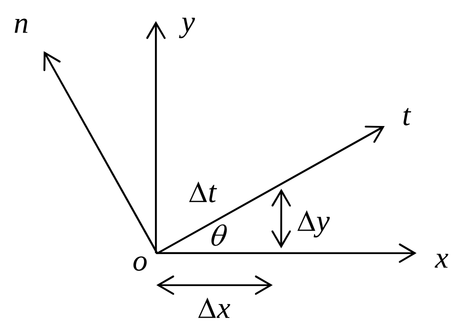


Figure 6.2.7: Two different Cartesian coordinate systems

The relationship between second derivatives can be found in the same way. For example,

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \left(\frac{\partial}{\partial t}\cos\theta - \frac{\partial}{\partial n}\sin\theta\right)\left(\frac{\partial w}{\partial t}\cos\theta - \frac{\partial w}{\partial n}\sin\theta\right) \\ &= \cos^2\theta\frac{\partial^2 w}{\partial t^2} + \sin^2\theta\frac{\partial^2 w}{\partial n^2} - \sin 2\theta\frac{\partial^2 w}{\partial t\partial n} \end{aligned} \tag{6.2.10}$$

In summary, one has

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= \cos^2\theta \frac{\partial^2 \omega}{\partial t^2} + \sin^2\theta \frac{\partial^2 \omega}{\partial n^2} - \sin 2\theta \frac{\partial^2 \omega}{\partial t \partial n} \\
\frac{\partial^2 w}{\partial y^2} &= \sin^2\theta \frac{\partial^2 \omega}{\partial t^2} + \cos^2\theta \frac{\partial^2 \omega}{\partial n^2} + \sin 2\theta \frac{\partial^2 \omega}{\partial t \partial n} \\
\frac{\partial^2 w}{\partial x \partial y} &= -\sin\theta\cos\theta \left(\frac{\partial^2 \omega}{\partial n^2} - \frac{\partial^2 \omega}{\partial t^2} \right) + \cos 2\theta \frac{\partial^2 \omega}{\partial t \partial n}
\end{aligned} \tag{6.2.11}$$

and the inverse relations

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= \cos^2\theta \frac{\partial^2 \omega}{\partial x^2} + \sin^2\theta \frac{\partial^2 \omega}{\partial y^2} + \sin 2\theta \frac{\partial^2 \omega}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial n^2} &= \sin^2\theta \frac{\partial^2 \omega}{\partial x^2} + \cos^2\theta \frac{\partial^2 \omega}{\partial y^2} - \sin 2\theta \frac{\partial^2 \omega}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial t \partial n} &= \sin\theta\cos\theta \left(\frac{\partial^2 \omega}{\partial y^2} - \frac{\partial^2 \omega}{\partial x^2} \right) + \cos 2\theta \frac{\partial^2 \omega}{\partial x \partial y}
\end{aligned} \tag{6.2.12}$$

or[1]

$$\begin{aligned}
\frac{1}{R_t} &= \cos^2\theta \frac{1}{R_x} + \sin^2\theta \frac{1}{R_y} + \sin 2\theta \frac{1}{T_{xy}} \\
\frac{1}{R_n} &= \sin^2\theta \frac{1}{R_x} + \cos^2\theta \frac{1}{R_y} - \sin 2\theta \frac{1}{T_{xy}} \\
\frac{1}{T_{tn}} &= \sin\theta\cos\theta \left(\frac{1}{R_y} - \frac{1}{R_x} \right) + \cos 2\theta \frac{1}{T_{xy}}
\end{aligned} \tag{6.2.13}$$

These equations which transform between curvatures in different coordinate systems have the same structure as the stress transformation equations (and the strain transformation equations), Book I, Eqns. 3.4.8. As with principal stresses/strains, there will be some angle θ for which the twist is zero; at this angle, one of the curvatures will be the minimum and one will be the maximum at that point in the plate. These are called the **principal curvatures**. Similarly, just as the sum of the normal stresses is an invariant (see Book I, Eqn. 3.5.1), the sum of the curvatures is an invariant[2]:

$$\frac{1}{R_x} + \frac{1}{R_y} = \frac{1}{R_t} + \frac{1}{R_n} \tag{6.2.14}$$

If the principal curvatures are equal, the curvatures are the same at all angles, the twist is always zero and so the plate deforms locally into the surface of a sphere.

[1] these equations are valid for any continuous surface; Eqns. 6.2.12 are restricted to nearly-flat surfaces.
[2] this is known as Euler's theorem for curvatures

6.2.3 Strains in a Plate

The strains arising in a plate are next examined. A comprehensive strain-state will be first examined and this will then be simpilfied down later to various approximate solutions. Consider a line element parallel to the x axis, of length Δx. Let the element displace as shown in Fig. 6.2.8. Whereas w was used in the previous section on curvatures to denote displacement of the mid-surface, here, for the moment, let $w(x, y, z)$ be the general vertical displacement of any particle in the plate. Let u and v be the corresponding displacements in the x and y directions. Denote the original and deformed length of the element by dS and ds respectively.

The unit change in length of the element (that is, the exact normal strain) is, using Pythagoras' theorem,

$$\varepsilon_{xx} = \frac{ds - dS}{dS} = \frac{|p'q'| - |pq|}{|pq|} = \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \qquad (6.2.15)$$

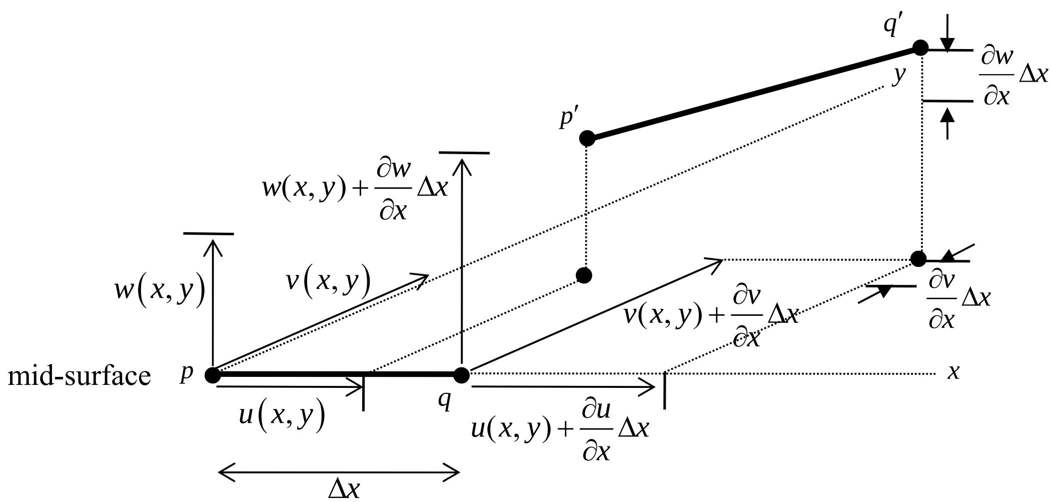


Figure 6.2.8: deformation of a material fibre in the *x* direction

In the plate theory, it will be assumed that the displacement gradients are small:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial z}, \quad \frac{\partial w}{\partial z},$$

of order $\varepsilon << 1$ say, so that squares and products of these terms may be neglected.

However, for the moment, the squares and products of the slopes will be retained, as they may be significant, i.e. of the same order as the strains, under certain circumstances:

$$\left(\frac{\partial w}{\partial x}\right)^2, \quad \left(\frac{\partial w}{\partial y}\right)^2, \quad \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Eqn. 6.2.15 now reduces to

$$\varepsilon_{xx} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \tag{6.2.16}$$

With $\sqrt{1+x} \approx 1 + x/2$ for $x << 1$, one has (and similarly for the other normal strains)

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned} \tag{6.2.17}$$

Consider next the angle change for line elements initially lying parallel to the axes, Fig. 6.2.9. Let θ be the angle $\angle r'p'q'$, so that $\gamma = \pi/2 - \theta$ is the change in the initial right angle $\angle rpq$.

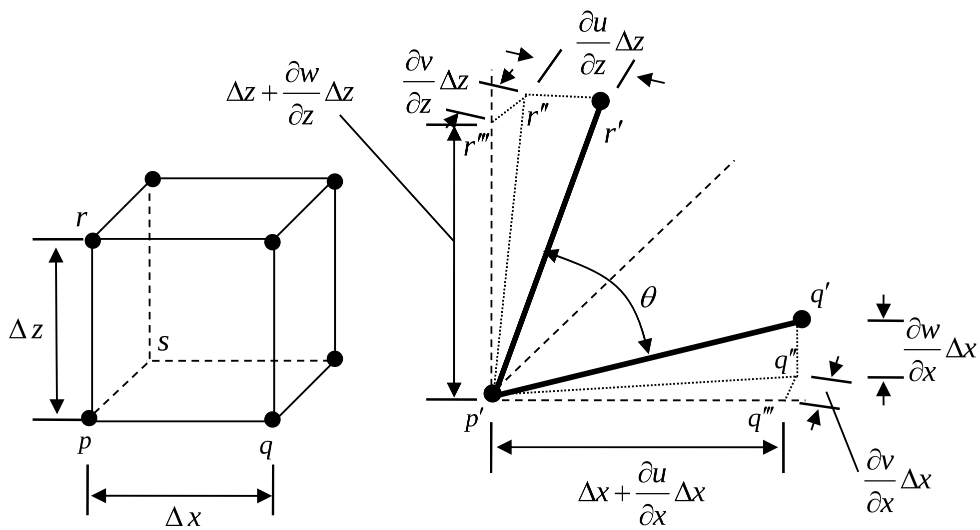


Figure 6.2.9: the deformation of Fig. 6.2.8, showing shear strains

Taking the dot product of the of the vector elements $\overrightarrow{p'q'}$ and $\overrightarrow{p'r'}$:

$$\cos\theta = \frac{|p'q'''||r''r'| + |q'''q''||r'''r''| + |q''q'||p'r'''|}{|p'q'||p'r'|}$$
$$= \frac{\left(\Delta x + \frac{\partial u}{\partial x}\Delta x\right)\left(\frac{\partial u}{\partial z}\Delta z\right) + \left(\frac{\partial v}{\partial x}\Delta x\right)\left(\frac{\partial v}{\partial z}\Delta z\right) + \left(\frac{\partial w}{\partial x}\Delta x\right)\left(\Delta z + \frac{\partial w}{\partial z}\Delta z\right)}{\Delta x\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\,\Delta z\sqrt{\left(1+\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}} \quad (6.2.18)$$

Again, with the displacement gradients $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial z$, $\partial v/\partial z$, $\partial w/\partial z$ of order $\varepsilon << 1$ (and the squares $(\partial w/\partial x)^2$ at most of order $\varepsilon << 1$),

$$\cos\theta = \frac{\Delta x\left(\frac{\partial u}{\partial z}\Delta z\right) + \left(\frac{\partial v}{\partial x}\Delta x\right)\left(\frac{\partial v}{\partial z}\Delta z\right) + \left(\frac{\partial w}{\partial x}\Delta x\right)\Delta z}{\Delta x \Delta z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} \quad (6.2.19)$$

For small γ, $\gamma \approx \sin\gamma = \cos\theta$, so (and similarly for the other shear strains)

$$\begin{aligned}
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned} \quad (6.2.20)$$

The normal strains 6.2.17 and the shear strains 6.2.20 are non-linear. They are the starting point for the various different plate theories.

Von Kármán Strains

Introduce now the assumptions of the classical plate theory. The assumption that line elements normal to the mid-plane remain inextensible implies that

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \quad (6.2.21)$$

This implies that $w = w(x, y)$ so that all particles at a given (x, y) through the thickness of the plate experinece the same vertical displacement. The assumption that line elements perpendicular to the mid-plane remain normal to the mid-plane after deformation then implies that $\varepsilon_{xz} = \varepsilon_{yz} = 0$.

The strains now read

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\varepsilon_{zz} &= 0 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= 0 \\
\varepsilon_{yz} &= 0
\end{aligned} \tag{6.2.22}$$

These are known as the **Von Kármán strains**.

Membrane Strains and Bending Strains

Since $\varepsilon_{xz} = 0$ and $w = w(x, y)$, one has from Eqn. 6.2.20b,

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} \quad \to \quad u(x, y, z) = -z\frac{\partial w}{\partial x} + u_0(x, y) \tag{6.2.23}$$

It can be seen that the function $u_0(x, y)$ is the displacement in the mid-plane. In terms of the mid-surface displacements u_0, v_0, w_0, then,

$$u = u_0 - z\frac{\partial w_0}{\partial x}, \quad v = v_0 - z\frac{\partial w_0}{\partial y}, \quad w = w_0 \tag{6.2.24}$$

and the strains 6.2.22 may be expressed as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right) - z\frac{\partial^2 w_0}{\partial x \partial y}
\end{aligned} \tag{6.2.25}$$

The first terms are the usual small-strains, for the mid-surface. The second terms, involving squares of displacement gradients, are non-linear, and need to be considered when the plate bending is fairly large (when the rotations are about 10 – 15 degrees). These first two terms together are called the **membrane strains**. The last terms, involving second derivatives, are the **flexural** (**bending**) **strains**. They involve the curvatures.

When the bending is not too large (when the rotations are below about 10 degrees), one has (dropping the subscript "0" from *w*)

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\\ \varepsilon_{yy} &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{\partial^2 w}{\partial x \partial y}\end{aligned} \tag{6.2.26}$$

Some of these strains are illustrated in Figs. 6.2.10 and 6.2.11; the physical meaning of ε_{xx} is shown in Fig. 6.2.10 and some terms from ε_{xy} are shown in Fig. 6.2.11.

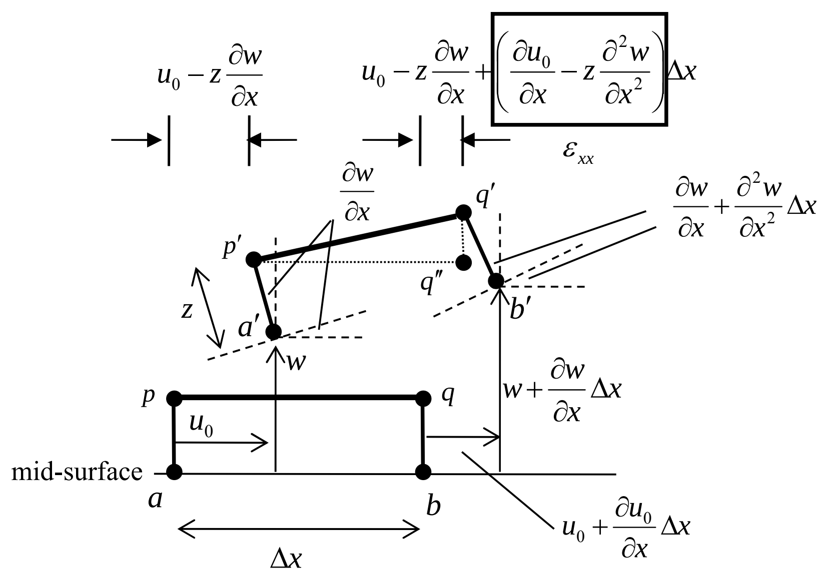


Figure 6.2.10: deformation of material fibres in the *x* direction

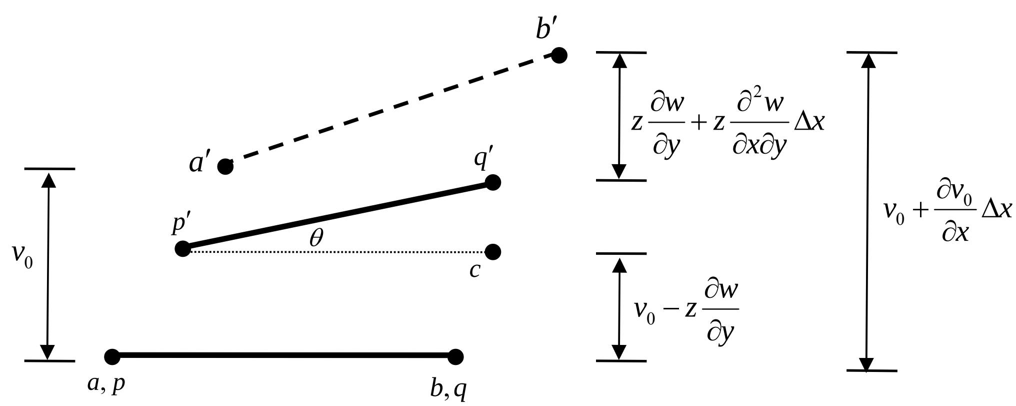


Figure 6.2.11: the deformation of 6.2.10 viewed "from above"; a', b' are the deformed positions of the mid-surface points a, b

Finally, when the mid-surface strains are neglected, according to the final assumption of the classical plate theory, one has

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2}, \qquad \varepsilon_{xy} = -z\frac{\partial^2 w}{\partial x \partial y} \tag{6.2.27}$$

In summary, when the plate bends "up", the curvature is positive, and points "above" the mid-surface experience negative normal strains and points "below" experience positive normal strains; there is zero shear strain. On the other hand, when the plate undergoes a positive pure twist, so the twisting moment is negative, points "above" the mid-surface experience negative shear strain and points "below" experience positive shear strain; there is zero normal strain. A pure shearing of the plate in the $x-y$ plane is illustrated in Fig. 6.2.12.

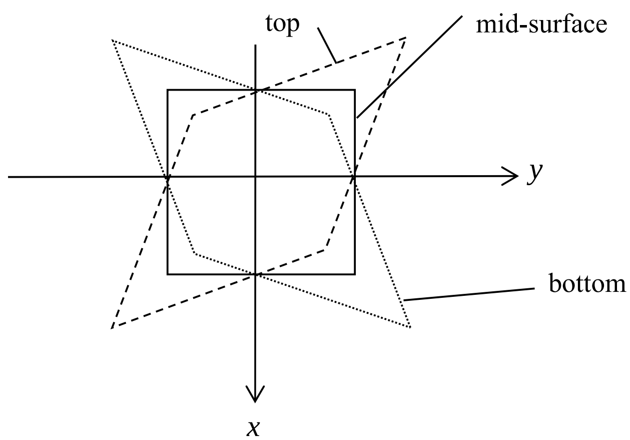


Figure 6.2.12: Shearing of the plate due to a positive twist (neative twisting moment)

Compatibility

Note that the strain fields arising in the plate satisfy the 2D compatibility relation Eqn. 1.3.1:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{6.2.28}$$

This can be seen by substituting Eqn. 6.25 (or Eqns 6.26-27) into Eqn. 6.2.28.

6.2.4 The Moment-Curvature equations

Now that the strains have been related to the curvatures, the moment-curvature relations, which play a central role in plate theory, can be derived.

Stresses and the Curvatures/Twist in a Linear Elastic Plate

From Hooke's law, taking $\sigma_{zz} = 0$,

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}, \quad \varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}, \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} \tag{6.2.29}$$

so, from 6.2.27, and solving 6.2.29a-b for the normal stresses,

$$\begin{aligned} \sigma_{xx} &= -\frac{E}{1-\nu^2} z \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ \sigma_{yy} &= -\frac{E}{1-\nu^2} z \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ \sigma_{xy} &= -\frac{E}{1+\nu} z \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{6.2.30}$$

The Moment-Curvature Equations

Substituting Eqns. 6.2.30 into the definitions of the moments, Eqns. 6.1.1, 6.1.2, and integrating, one has

$$\boxed{\begin{aligned} M_x &= D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ M_y &= D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ M_{xy} &= -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{aligned}} \tag{6.2.31}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{6.2.32}$$

Equations 6.2.31 are the **moment-curvature equations** for a plate. The moment-curvature equations are analogous to the beam moment-deflection equation $\partial^2 v / \partial x^2 = M / EI$. The factor D is called the **plate stiffness** or **flexural rigidity** and plays the same role in the plate theory as does the flexural rigidity term EI in the beam theory.

Stresses and Moments

From 6.30-6.31, the stresses and moments are related through

$$\sigma_{xx} = -\frac{M_x z}{h^3/12}, \quad \sigma_{yy} = -\frac{M_y z}{h^3/12}, \quad \sigma_{xy} = +\frac{M_{xy} z}{h^3/12} \tag{6.2.33}$$

Note the similarity of these relations to the beam formula $\sigma = -My/I$ with $I = h^3/12$ times the width of the beam.

6.2.5 Principal Moments

It was seen how the curvatures in different directions are related, through Eqns. 6.2.11-12. It comes as no surprise, examining 6.2.31, that the moments are related in the same way.

Consider a small differential element of a plate, Fig. 6.2.13a, subjected to stresses σ_{xx}, σ_{yy}, σ_{xy}, and corresponding moments M_x, M_y, M_{xy} given by 6.1.1-2. On any perpendicular planes rotated from the orginal $x-y$ axes by an angle θ, one can find the new stresses σ_{tt}, σ_{nn}, σ_{tn}, Fig. 6.2.13b (see Fig. 6.2.5), through the stress transformatrion equations (Book I, Eqns. 3.4.8). Then

$$\begin{aligned} M_t &= -\int z\sigma_{tt}dz = \cos^2\theta\left[-\int z\sigma_{xx}dz\right] + \sin^2\theta\left[-\int z\sigma_{yy}dz\right] + \sin 2\theta\left[-\int z\sigma_{xy}dz\right] \\ &= \cos^2\theta M_x + \sin^2\theta M_y - \sin 2\theta M_{xy} \end{aligned} \tag{6.2.34}$$

and similarly for the other moments, leading to

$$\begin{aligned} M_t &= \cos^2\theta M_x + \sin^2\theta M_y - \sin 2\theta M_{xy} \\ M_n &= \sin^2\theta M_x + \cos^2\theta M_y + \sin 2\theta M_{xy} \\ M_{tn} &= -\cos\theta\sin\theta\left(M_y - M_x\right) + \cos 2\theta M_{xy} \end{aligned} \tag{6.2.35}$$

Also, there exist principal planes, upon which the shear stress is zero (right through the thickness). The moments acting on these planes, M_1 and M_2, are called the **principal moments**, and are the greatest and least bending moments which occur at the element. On these planes, the twisting moment is zero.

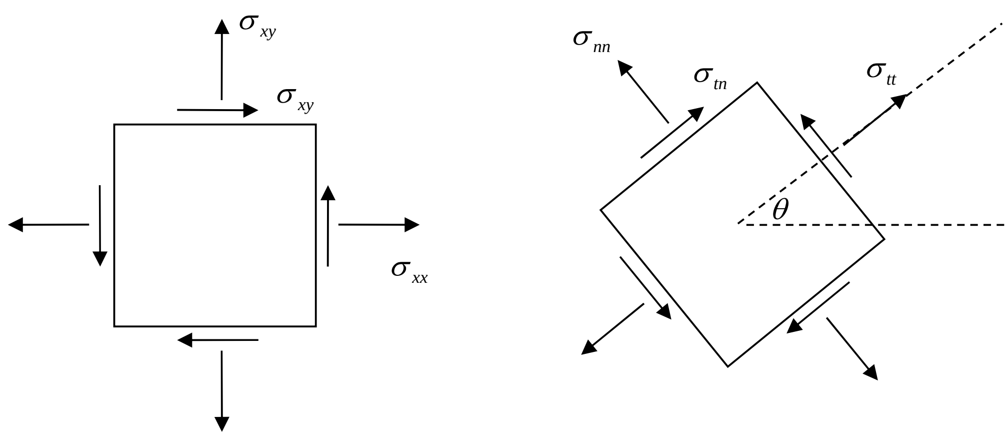


Figure 6.2.13: Plate Element; (a) stresses acting on element, (b) rotated element

Moments in Different Coordinate Systems

From the moment-curvature equations 6.2.31, {▲Problem 1}

$$
\begin{aligned}
M_t &= D\left(\frac{\partial^2 \omega}{\partial t^2} + \nu \frac{\partial^2 \omega}{\partial n^2}\right) \\
M_n &= D\left(\frac{\partial^2 \omega}{\partial n^2} + \nu \frac{\partial^2 \omega}{\partial t^2}\right) \\
M_{tn} &= -D(1-\nu)\frac{\partial^2 \omega}{\partial t \partial n}
\end{aligned}
\qquad (6.2.36)
$$

showing that the moment-curvature relations 6.2.31 hold in all Cartesian coordinate systems.

6.2.6 Problems

1. Use the curvature transformation relations 6.2.11 and the moment transformation relations 6.2.35 to derive the moment-curvature relations 6.2.36.

6.3 Plates subjected to Pure Bending and Twisting

6.3.1 Pure Bending of an Elastic Plate

Consider a plate subjected to bending moments $M_x = M_1 > 0$ and $M_y = M_2 > 0$, with no other loading, as shown in Fig. 6.3.1.

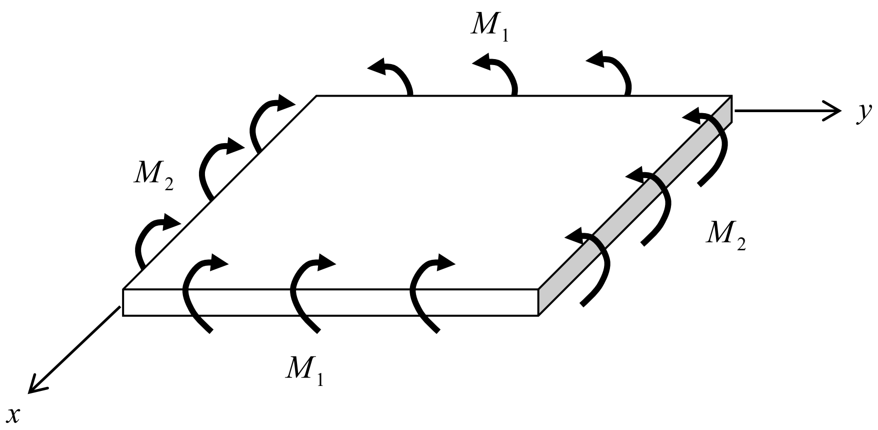


Figure 6.3.1: A plate under Pure Bending

From equilibrium considerations, these moments act at all points within the plate – they are constant throughout the plate. Thus, from the moment-curvature equations 6.2.31, one has the set of coupled partial differential equations

$$\frac{M_1}{D} = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad \frac{M_2}{D} = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad 0 = \frac{\partial^2 w}{\partial x \partial y} \tag{6.3.1}$$

Solving for the derivatives,

$$\frac{\partial^2 w}{\partial x^2} = \frac{M_1 - \nu M_2}{D(1-\nu^2)}, \quad \frac{\partial^2 w}{\partial y^2} = \frac{M_2 - \nu M_1}{D(1-\nu^2)}, \quad \frac{\partial^2 w}{\partial x \partial y} = 0 \tag{6.3.2}$$

Integrating the first two equations twice gives[1]

$$w = \frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)} x^2 + f_1(y)x + f_2(y), \quad w = \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)} y^2 + g_1(x)y + g_2(x) \tag{6.3.3}$$

and integrating the third shows that two of these four unknown functions are constants:

$$\frac{\partial w}{\partial x} = G(x), \quad \frac{\partial w}{\partial y} = F(y) \qquad \to \qquad f_1(y) = A, \;\; g_1(x) = B \tag{6.3.4}$$

[1] this analysis is similar to that used to evaluate displacements in plane elastostatic problems, §1.2.4

Equating both expressions for w in 6.3.3 gives

$$\frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)}x^2 + Ax - g_2(x) = \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)}y^2 + By - f_2(y) \tag{6.3.5}$$

For this to hold, both sides here must be a constant, $-C$ say. It follows that

$$w = \frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)}x^2 + \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)}y^2 + Ax + By + C \tag{6.3.6}$$

The three unknown constants represent an arbitrary rigid body motion. To obtain values for these, one must fix three degrees of freedom in the plate. If one supposes that the deflection w and slopes $\partial w / \partial x, \partial w / \partial y$ are zero at the origin $x = y = 0$ (so the origin of the axes are at the plate-centre), then $A = B = C = 0$; all deformation will be measured relative to this reference. It follows that

$$w = \frac{M_2}{2D(1-\nu^2)}\left[\left[(M_1/M_2) - \nu\right]x^2 + \left[1 - \nu(M_1/M_2)\right]y^2\right] \tag{6.3.7}$$

Once the deflection w is known, all other quantities in the plate can be evaluated – the strain from 6.2.27, the stress from Hooke's law or directly from 6.2.30, and moments and forces from 6.1.1-3.

In the special case of equal bending moments, with $M_1 = M_2 = M_o$ say, one has

$$w = \frac{M_o}{2D(1+\nu)}\left(x^2 + y^2\right) \tag{6.3.8}$$

This is the equation of a sphere. In fact, from the relationship between the curvatures and the radius of curvature R,

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{M_o}{D(1+\nu)} \quad \rightarrow \quad R = \frac{D(1+\nu)}{M_o} = \text{constant} \tag{6.3.9}$$

and so the mid-surface of the plate in this case deforms into the surface of a sphere with radius given by 6.3.9, as illustrated in Fig. 6.3.2.

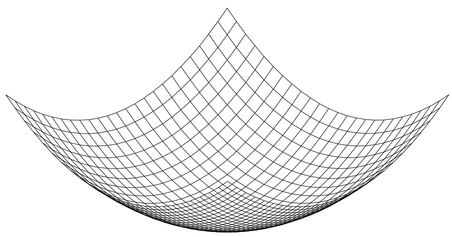
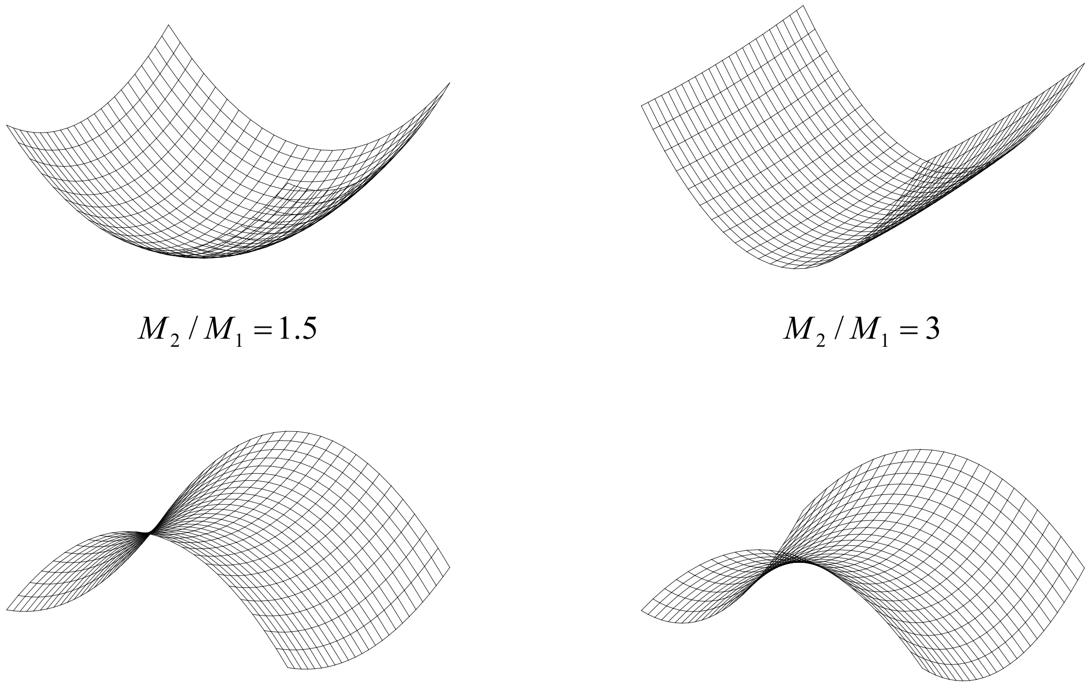


Figure 6.3.2: Deformed plate under Pure Bending with equal moments

The character of the deformed plate is plotted in Fig. 6.3.3 for various ratios M_2 / M_1 (for $\nu = 0.3$).



$M_2 / M_1 = 1.5$ $\quad$ $M_2 / M_1 = 3$

$M_2 / M_1 = -1.5$ $\quad$ $M_2 / M_1 = -3$

Figure 6.3.3: Bending of a Plate

When the curvatures $\partial^2 w / \partial x^2$ and $\partial^2 w / \partial y^2$ are of the same sign[2], the deformation is called **synclastic**. When the curvatures are of opposite sign, as in the lower plots of Fig. 6.3.3, the deformation is said to be **anticlastic**.

Note that when there is only one moment, $M_y = 0$ say, there is still curvature in both directions. In this case, one can solve the moment-curvature equations to get

$$\frac{\partial^2 w}{\partial x^2} = \frac{M_x}{\left(1-\nu^2\right)D}, \quad \frac{\partial^2 w}{\partial y^2} = -\nu \frac{\partial^2 w}{\partial x^2}, \qquad w = \frac{M_x}{2\left(1-\nu^2\right)D}\left(x^2 - \nu y^2\right) \qquad (6.3.10)$$

which is an anticlastic deformation.

In order to get a pure cylindrical deformation, $w = f(x)$ say, one needs to apply moments M_x and $M_y = \nu M_x$, in which case, from 6.3.6,

[2] or principal curvatures in the case of a more complex general loading

$$w = \frac{M_x}{2D} x^2 \tag{6.3.11}$$

The deformation for $M_2 / M_1 = 3$ in Fig. 6.3.3 is very close to cylindrical, since there $M_x \approx \nu M_y$ for typical values of ν.

6.3.2 Pure Torsion of an Elastic Plate

In pure torsion, one has the twisting moment $M_{xy} = M > 0$ with no other loading, Fig. 6.3.4. From the moment-curvature equations,

$$0 = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad 0 = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad -\frac{M}{D(1-\nu)} = \frac{\partial^2 w}{\partial x \partial y} \tag{6.3.12}$$

so that

$$\frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad \frac{\partial^2 w}{\partial x \partial y} = -\frac{M}{D(1-\nu)} \tag{6.3.13}$$

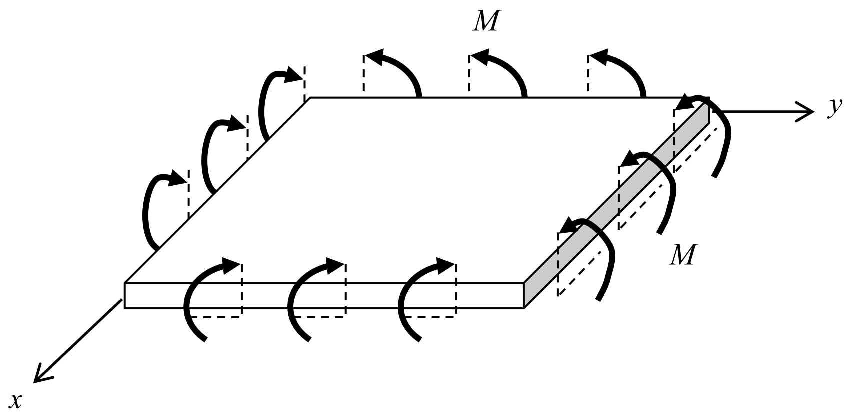


Figure 6.3.4: Twisting of a Plate

Using the same arguments as before, integrating these equations leads to

$$w = -\frac{M}{D(1-\nu)} xy \tag{6.3.14}$$

The middle surface is deformed as shown in Fig. 6.3.5, for a negative M_{xy}. Note that there is no deflection along the lines $x = 0$ or $y = 0$.

The principal curvatures will occur at 45° to the axes (see Eqns. 6.2.13):

$$\frac{1}{R_1} = +\frac{M}{D(1-\nu)}, \qquad \frac{1}{R_2} = -\frac{M}{D(1-\nu)} \tag{6.3.15}$$

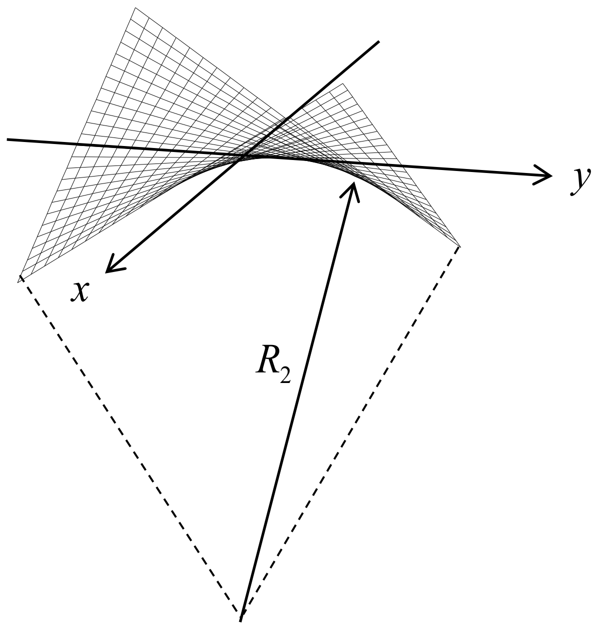


Figure 6.3.5: Deformation for a (negative) twisting moment

6.4 Equilibrium and Lateral Loading

In this section, lateral loads are considered and these lead to shearing forces V_x, V_y, in the plate.

6.4.1 The Governing Differential Equation for Lateral Loads

In general, a plate will at any location be subjected to a lateral *pressure* q, bending moments M_x, M_y, M_{xy} and out-of-plane shear forces V_x and V_y; q is the normal pressure on the upper surface of the plate:

$$\sigma_{zz}(x,y) = \begin{cases} 0, & z = -h/2 \\ -q(x,y), & z = +h/2 \end{cases} \tag{6.4.1}$$

These quantities are related to each other through force equilibrium.

Force Equilibrium

Consider a differential plate element with one corner at $(x,y) = (0,0)$, Fig. 6.4.1, subjected to moments, pressure and shear force. Taking force equilibrium in the vertical direction (neglecting a possible small variation in *q*, since this will only introduce higher order terms):

$$\sum F\Big|_z = +V_y \Delta x - \left(V_y + \frac{\partial V_y}{\partial y}\Delta y\right)\Delta x + V_x \Delta y - \left(V_x + \frac{\partial V_x}{\partial x}\Delta x\right)\Delta y - q\Delta x \Delta y = 0 \tag{6.4.2}$$

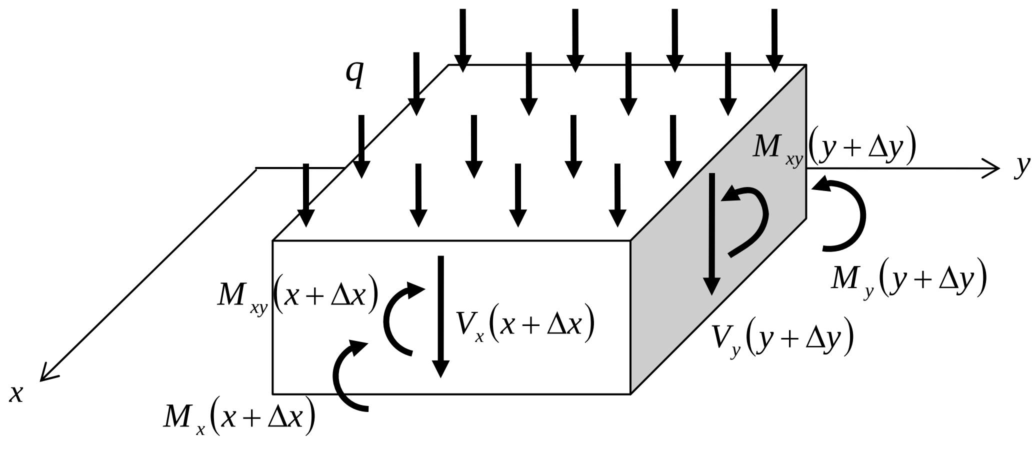


Fig. 6.4.1: a plate element subjected to moments, pressure and shear forces

Eqn. 6.4.2 gives the vertical equilibrium equation

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -q \tag{6.4.3}$$

This is analogous to the beam theory equation $p = dV / dx$ (see Book I, §7.4.3).

Next, taking moments about the *x* axis:

$$\begin{aligned}\sum M\Big|_x = M_{xy}\Delta y - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\Delta x\right)\Delta y - M_y \Delta x + \left(M_y + \frac{\partial M_y}{\partial y}\Delta y\right)\Delta x \\ - yV_y\Delta x - (y+\Delta y)\left(V_y + \frac{\partial V_y}{\partial y}\Delta y\right)\Delta x + (y + \Delta y/2)V_x\Delta y \\ - (y+\Delta y/2)\left(V_x + \frac{\partial V_x}{\partial x}\Delta x\right)\Delta y - (y+\Delta y/2)q\Delta x\Delta y = 0\end{aligned} \tag{6.4.4}$$

Using 6.4.3, this reduces to (and similarly for moments about the *x*-axis),

$$\begin{aligned}V_x &= +\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} \\ V_y &= -\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\end{aligned} \tag{6.4.5}$$

These are analogous to the beam theory equation $V = dM / dx$ (see Book I, §7.4.3).

Relations directly from the Equations of Equilibrium

The equilibrium relations 6.4.3, 6.4.5 can also be derived directly from the equations of equilibrium, Eqns. 1.1.9, which encompass the force balances:

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0\end{aligned} \tag{6.4.6}$$

Taking the first of these (which ensures equilibrium of forces in the *x* direction), multiplying by *z* and integrating over the plate thickness, gives

$$\begin{aligned}&\int_{-h/2}^{+h/2} z\frac{\partial \sigma_{xx}}{\partial x}dz + \int_{-h/2}^{+h/2} z\frac{\partial \sigma_{yx}}{\partial y}dz + \int_{-h/2}^{+h/2} z\frac{\partial \sigma_{zx}}{\partial z}dz = 0 \\ \to\ &\frac{\partial}{\partial x}\left[\int_{-h/2}^{+h/2} z\sigma_{xx}dz\right] + \frac{\partial}{\partial y}\left[\int_{-h/2}^{+h/2} z\sigma_{yx}dz\right] + \left[z\sigma_{zx}\right]_{-h/2}^{+h/2} - \int_{-h/2}^{+h/2}\sigma_{zx}dz = 0\end{aligned} \tag{6.4.7}$$

and, since the shear stress σ_{zx} must be zero over the top and bottom surfaces, one has Eqn. 6.4.5a. Applying a similar procedure to the second equilibrium equation gives Eqn.

6.4.5b. Finally, integrating directly the third equilibrium equation without multiplying across by z, one arrives at Eqn. 6.4.3.

Now, eliminating the shear forces from 6.4.3, 6.4.5 leads to the differential equation

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{6.4.8}$$

This equation is analogous to the equation $\partial^2 M / \partial x^2 = p$ in the beam theory. Finally, substituting in the moment-curvature equations 6.2.31 leads to[1]

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{q}{D}} \tag{6.4.9}$$

This is sometimes called **the equation of Sophie Germain** after the French investigator who first obtained it in 1815[2]. This partial differential equation is solved subject to the boundary conditions of the problem, i.e. the fixing conditions of the plate (see below). Again, when once an expression for $\omega(x, y)$ is obtained, the strains, stresses, forces and moments follow.

Note that the differential equation 6.4.8 with $q = 0$ is trivially satisfied in the simple pure bending and torsion problems considered earlier.

Eqn. 6.4.9 can be succinctly expressed as

$$\nabla^4 w = -\frac{q}{D} \tag{6.4.10}$$

where ∇^2 is the **Laplacian**, or "del" operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{6.4.11}$$

Note that the Laplacian operator (on w) gives the sum of the curvatures in two perpendicular directions and so it is independent of the directions chosen (see Eqn. 6.2.14).

Shear Forces in terms of Deflection

From 6.4.5 and the moment-curvature equations, one has the useful relations

[1] note that the moment curvature relations were derived for the case of pure bending; here, as in the beam theory, the possible effect of the shearing forces on the curvature is neglected. This is a valid assumption provided the thickness of the plate is small in comparison with its other dimensions. A more exact theory taking into account the effect of the shear forces on deflection can be developed
[2] Germain submitted her work to the French Academy, which was awarding a prize for anyone who could solve the problem of the vibration of plates; Lagrange was on the Academy awarding committee and corrected some of her work, deriving Eqn. 6.4.9 in its final form

$$V_x = D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \qquad V_y = D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{6.4.12}$$

6.4.2 Stresses in the Plate

The normal and in-plane shear stresses have been expressed in terms of the moments, Eqns. 6.2.33. Note that these stresses are zero over the mid-surface and attain a maximum at the outer surfaces.

Expressions for the remaining stress components can be obtained from the equations of equilibrium as follows: the first of Eqns. 6.4.6 leads, with 6.4.5a, to

$$\begin{aligned}
0 &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\
&= \frac{\partial}{\partial x}\left[-\frac{12z}{h^3}M_x\right] + \frac{\partial}{\partial y}\left[\frac{12z}{h^3}M_{xy}\right] + \frac{\partial \sigma_{zx}}{\partial z} \\
&= -\frac{12z}{h^3}\left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial \sigma_{zx}}{\partial z} \\
&= -\frac{12z}{h^3}V_x + \frac{\partial \sigma_{zx}}{\partial z}
\end{aligned} \tag{6.4.13}$$

Integrating now gives (note that V_x is independent of z)

$$\sigma_{zx} = \frac{6}{h^3}V_x z^2 + C \tag{6.4.14}$$

This shear stress must be zero at the upper and lower (free –) surfaces, at $z = \pm h/2$. This condition can be used to determine the arbitrary constant C and one finds that (see Fig. 6.1.9)

$$\sigma_{zx} = -\frac{3V_x}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{6.4.15}$$

The other shear stress, σ_{zy}, can be evaluated in a similar manner: {▲Problem 1}

$$\sigma_{zy} = -\frac{3V_y}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{6.4.16}$$

In some analyses, these shear stresses are taken to be zero, although they can be quite significant.

The only remaining stress component is σ_{zz}. This will never exceed the intensity of the external load on the plate; the lateral load itself, however, is negligibly small in comparison with the in-plane stresses set up by the bending of the plate, and for this reason it is acceptable to disregard σ_{zz}, as has been done, in the plate theory.

6.4.3 Problems

1. Derive the expression for shear stress 6.4.16.

6.5 Plate Problems in Rectangular Coordinates

In this section, a number of important plate problems will be examined using Cartesian coordinates.

6.5.1 Uniform Pressure producing Bending in One Direction

Consider first the case of a plate which bends in one direction only. From 6.3.11 the deflection and moments are

$$w = f(x), \quad M_x(x) = D\frac{d^2w}{dx^2}, \quad M_y(x) = -\nu D\frac{d^2w}{dx^2} \tag{6.5.1}$$

The differential equation 6.4.9 reads

$$\frac{d^4w}{dx^4} = -\frac{q(x)}{D} \tag{6.5.2}$$

The corresponding equation for a beam is $d^4w/dx^4 = p(x)/EI$. If $p(x)/b = -q(x)$, with b the depth of the beam, with $I = bh^3/12$, the plate will respond more stiffly than the beam by a factor of $1/(1-\nu^2)$, a factor of about 10% for $\nu = 0.3$, since

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{1}{1-\nu^2}\frac{EI}{b} \tag{6.5.3}$$

The extra stiffness is due to the constraining effect of M_y, which is not present in the beam.

6.5.2 Deflection of a Circular Plate by a Uniform Lateral Load

A solution for a circular plate problem is presented next. This problem will be examined again in the section which follows using the more natural polar coordinates.

Consider a circular plate with boundary

$$x^2 + y^2 = a^2, \tag{6.5.4}$$

clamped at its edges and subjected to a uniform lateral load q, Fig. 6.5.1.

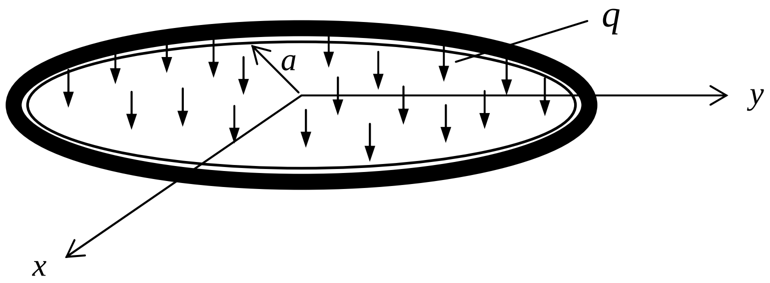


Figure 6.5.1: a clamped circular plate subjected to a uniform lateral load

The differential equation for the problem is given by 6.4.9. The boundary conditions are that the slope and deflection are zero at the boundary:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0 \qquad \text{along} \quad x^2 + y^2 = a^2 \tag{6.5.5}$$

It will be shown that the deflection

$$w = c(x^2 + y^2 - a^2)^2 \tag{6.5.6}$$

is a solution to the problem. First, this function certainly satisfies 6.5.5. Further, letting

$$f(x, y) = x^2 + y^2 - a^2, \tag{6.5.7}$$

the relevant partial derivatives are

$$\begin{aligned}
&\frac{\partial w}{\partial x} = 4cxf, \quad \frac{\partial w}{\partial y} = 4cyf \\
&\frac{\partial^2 w}{\partial x^2} = 4c\left(2x^2 + f\right), \quad \frac{\partial^2 w}{\partial x \partial y} = 8cxy, \quad \frac{\partial^2 w}{\partial y^2} = 4c\left(2y^2 + f\right) \\
&\frac{\partial^3 w}{\partial x^3} = 24cx, \quad \frac{\partial^3 w}{\partial x^2 \partial y} = 8cy, \quad \frac{\partial^3 w}{\partial x \partial y^2} = 8cx, \quad \frac{\partial^3 w}{\partial y^3} = 24cy \\
&\frac{\partial^4 w}{\partial x^4} = 24c, \quad \frac{\partial^4 w}{\partial x^2 \partial y^2} = 8c, \quad \frac{\partial^4 w}{\partial y^4} = 24c
\end{aligned} \tag{6.5.8}$$

Substituting these into the differential equation now yields

$$c = -\frac{q}{64D} \tag{6.5.9}$$

so the deflection is

$$w = -\frac{q}{64D}(x^2 + y^2 - a^2)^2 \tag{6.5.10}$$

This is plotted in Fig. 6.5.2. The maximum deflection occurs at the plate centre, where

$$w_{\max} = -\frac{qa^4}{64D}. \tag{6.5.11}$$

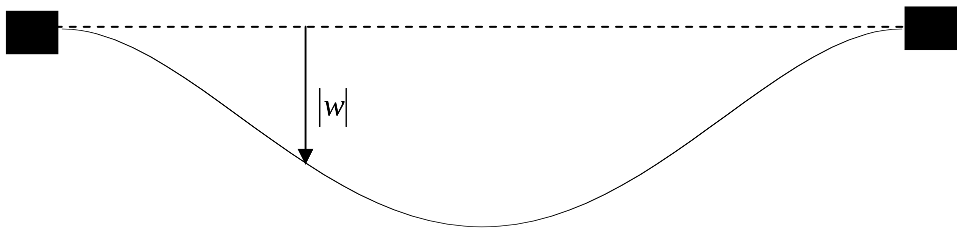


Figure 6.5.2: mid-plane deflection of the clamped circular plate

The curvature $\partial^2 w / \partial x^2$ along a radial line $y = 0$ is displayed in Fig. 6.5.3. The curvature is positive toward the centre of the plate (the plate curves upward) and is negative towards the edge of the plate (the plate curves downward).

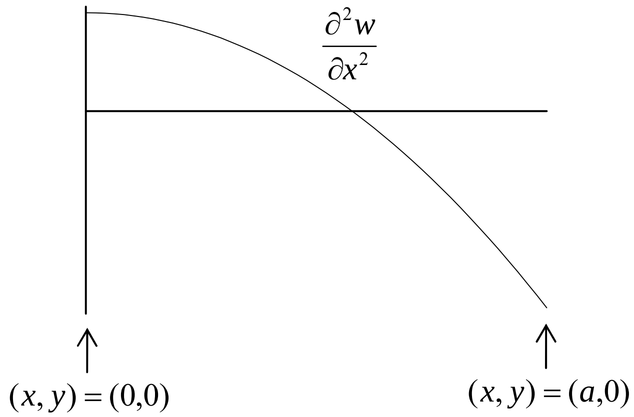


Figure 6.5.3: curvature in the clamped circular plate

The moments occurring in the plate are, from the moment-curvature equations 6.2.31 and 6.5.8,

$$\begin{aligned} M_x &= -\frac{q}{16}\left[(3+\nu)x^2 + (3\nu+1)y^2 - (1+\nu)a^2\right] \\ M_y &= -\frac{q}{16}\left[(3\nu+1)x^2 + (3+\nu)y^2 - (1+\nu)a^2\right] \\ M_{xy} &= +\frac{q}{8}(1-\nu)xy \end{aligned} \tag{6.5.12}$$

The moment M_x along a radial line $y = 0$ is of the same character as the curvature displayed in Fig. 6.5.3.

The out-of-plane shear forces are, from 6.4.5,

$$V_x = -\frac{qx}{2}, \qquad V_y = -\frac{qy}{2} \tag{6.5.13}$$

At the plate centre, the expressions become

$$M_x = M_y = \frac{q}{16}(1+\nu)a^2, \qquad M_{xy} = V_x = V_y = 0 \tag{6.5.14}$$

Stresses in the Plate

From 6.5.12-13 and 6.2.33, 6.4.15-16, the stresses in the plate are

$$\begin{aligned}
\sigma_{xx} &= \frac{3qz}{4h^3}\left[(3+\nu)x^2 + (3\nu+1)y^2 - (1+\nu)a^2\right] \\
\sigma_{yy} &= \frac{3qz}{4h^3}\left[(3\nu+1)x^2 + (3+\nu)y^2 - (1+\nu)a^2\right] \\
\sigma_{xy} &= \frac{3qz}{2h^3}(1-\nu)xy \\
\sigma_{zx} &= \frac{3qx}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right] \\
\sigma_{zy} &= \frac{3qy}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right]
\end{aligned} \tag{6.5.15}$$

Converting to polar coordinates (r,θ) through

$$x = r\cos\theta,\ y = r\sin\theta \tag{6.5.16}$$

and using a stress transformation,

$$\begin{aligned}
\sigma_{rr} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\
\sigma_{\theta\theta} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\
\sigma_{r\theta} &= \cos\theta\sin\theta\left(\sigma_{yy} - \sigma_{xx}\right) + \cos 2\theta\sigma_{xy}
\end{aligned} \tag{6.5.17}$$

leads to the axisymmetric stress field {▲Problem 1}

$$\begin{aligned}
\sigma_{rr} &= \frac{3qz}{4h^3}\left[(3+\nu)r^2 - (1+\nu)a^2\right] \\
\sigma_{\theta\theta} &= \frac{3qz}{4h^3}\left[(3\nu+1)r^2 - (1+\nu)a^2\right] \\
\sigma_{r\theta} &= 0
\end{aligned} \tag{6.5.18}$$

At the plate centre,

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{3qza^2}{4h^3}(1+\nu) \tag{6.5.19}$$

At the plate edge $r = a$,

$$\sigma_{rr} = \frac{3qza^2}{2h^3}, \qquad \sigma_{\theta\theta} = \frac{3qza^2}{2h^3}\nu \tag{6.5.20}$$

For the shear stress, the traction acting on a surface parallel to the $x - y$ plane can be expressed as (see Fig. 6.5.4)

$$\begin{aligned} \mathbf{t} &= \sigma_{zr}\mathbf{e}_r + \sigma_{z\theta}\mathbf{e}_\theta \\ &= \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y \\ &= \sigma_{zx}(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta) + \sigma_{zy}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta) \end{aligned} \tag{6.5.21}$$

where $\mathbf{e}_i$ is a unit vector in the direction *i*. Thus

$$\begin{aligned} \sigma_{zr} &= \cos\theta\sigma_{zx} + \sin\theta\sigma_{zy} = \frac{3qr}{4h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \\ \sigma_{z\theta} &= -\sin\theta\sigma_{zx} + \cos\theta\sigma_{zy} = 0 \end{aligned} \tag{6.5.22}$$

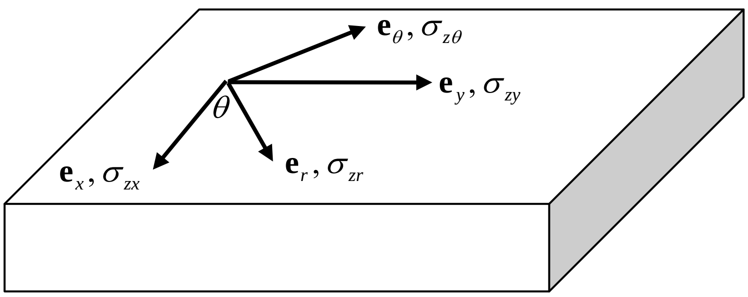


Figure 6.5.4: stress components acting on a surface

Note that the maximum stress in the plate is

$$\sigma_{\max} = \sigma_{rr}(a, h/2) = \frac{3q}{4}\left(\frac{a}{h}\right)^2 \tag{6.5.23}$$

The maximum shear stress, on the other hand, is $\sigma_{zr}(a,0) = 3q/4 \times (a/h)$. Thus the shear stress is of an order h/a smaller than the normal stress.

6.5.3 An Infinite Plate with Sinusoidal Deflection

Consider next the classic plate problem addressed by Navier in 1820. It consists of an infinite plate with an undulating "up/down" sinusoidal deflection, Fig. 6.5.5,

$$w(x,y) = w_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{6.5.24}$$

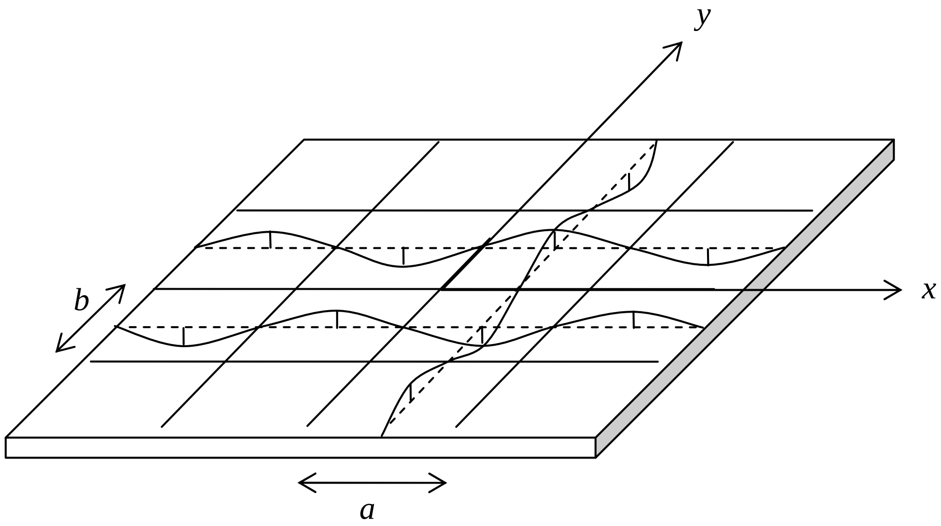


Figure 6.5.5: A plate with sinusoidal deflection

Differentiation of the deflection leads to the curvatures

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= -w_0\frac{\pi^2}{a^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\ \frac{\partial^2 w}{\partial y^2} &= -w_0\frac{\pi^2}{b^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\ \frac{\partial^2 w}{\partial xy} &= w_0\frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\end{aligned} \tag{6.5.25}$$

and hence the pressure

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial xy} + \frac{\partial^2 w}{\partial y^2} = \pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 w(x,y) \equiv -\frac{q(x,y)}{D} \tag{6.5.26}$$

The pressure thus varies like the deflection. There is no need for supports for the plate since the "up" loads balance the "down" loads.

From the moment-curvature relations,

$$M_x = -w_0 D\pi^2\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$M_y = -w_0 D\pi^2\left(\frac{\nu}{a^2}+\frac{1}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \qquad (6.5.27)$$
$$M_{xy} = -w_0 D(1-\nu)\frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

and, from 6.4.12, the shear forces are

$$V_x = -w_0 D\pi^3\frac{1}{a}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$V_y = -w_0 D\pi^3\frac{1}{b}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \qquad (6.5.28)$$

Note that both q/w and M_x/M_y are constant throughout the plate.

6.5.4 A Simply Supported Plate with Sinusoidal Deflection

Following on from the previous example, consider now a *finite* plate of dimensions a and b with the same sinusoidal deflection 6.5.24, simply supported along the edges $x=0$, $x=a$, $y=0$, $y=b$. In what follows, take w_0 in 6.5.24 to be negative, so that the plate is pushed down towards the centre.

According to 6.5.24 and 6.5.27, the deflection and slope is zero along the supported edges, as required. The vertical reactions at the supports are given by 6.5.28. However, according to Eqn. 6.5.27c, there are varying non-zero twisting moments over the ends of the plate. Thus the solution given by 6.5.24-28 is not quite the solution to the simply supported finite-plate problem, unless one can somehow apply the exact required twisting moments over the edges of the plate.

It turns out, however, that the solution 6.5.24-28 is a correct solution, except in a region close to the edges of the plate. This is explained in what follows.

Twisting Moments over "Free" Surfaces

Consider an element of material of width dy, Fig. 6.5.6. The element is subjected to a twisting moment $M_{xy}dy$, Fig. 6.5.6a. This twisting moment is due to shear stresses acting parallel to the plate surface (see Fig. 6.1.8). This system of horizontal forces can be replaced by the statically equivalent system of vertical forces shown in Fig. 6.5.6b – two *forces* of magnitude M_{xy} separate by a distance dy. Recalling Saint-Venant's principle, the difference between the statically equivalent systems of forces of Fig. 6.5.6a and 6.5.6b will lead to differences in the stress field within the plate only in a small region very close to the plate-edges.

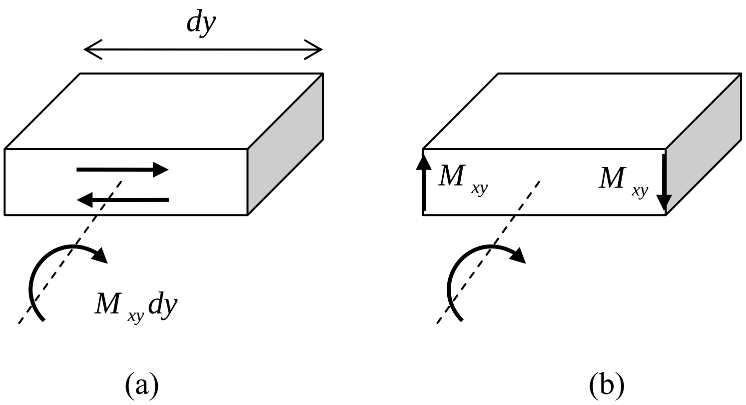


Figure 6.5.6: Equivalent systems of forces leading to the same twisting moment; (a) horizontal forces, (b) vertical forces

Consider next a distribution of twisting moment along the plate edge, Fig. 6.5.7. As can be seen, this distribution is equivalent to a distribution of shearing forces (per unit length) of magnitude

$$\overline{V}_x(y) = -\frac{\partial M_{xy}}{\partial y} \tag{6.5.29}$$

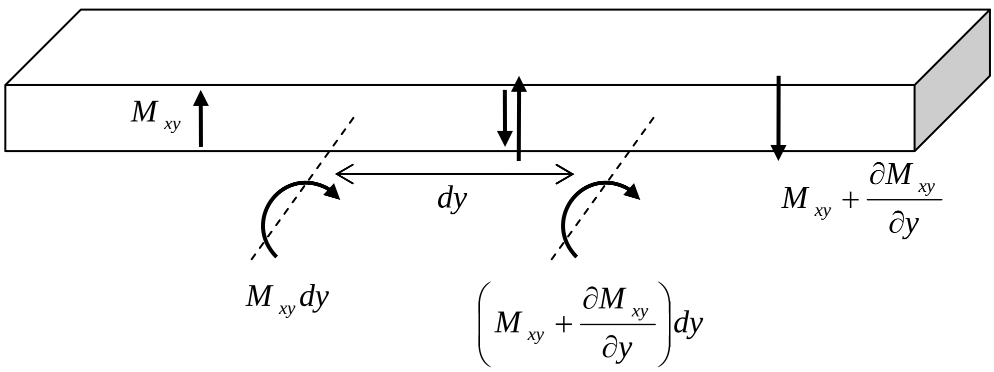


Figure 6.5.7: A distribution of twisting moments along a plate edge

The total vertical reaction along the edges can now be taken to be

$$V_x - \frac{\partial M_{xy}}{\partial y} \tag{6.5.30}$$

(and $V_y - \partial M_{xy} / \partial x$ along the other edges) and this gives a correct solution to the problem. From 6.5.27-28, these reactions are

$$
\begin{aligned}
F_{x0} &= \left(V_x - \frac{\partial M_{xy}}{\partial y} \right)_{(0,y)} = -w_0 D\pi^3 \frac{1}{a}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{b^2}\right]\sin\frac{\pi y}{b} \\
F_{xa} &= \left(V_x - \frac{\partial M_{xy}}{\partial y} \right)_{(a,y)} = +w_0 D\pi^3 \frac{1}{a}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{b^2}\right]\sin\frac{\pi y}{b} \\
F_{y0} &= \left(V_y - \frac{\partial M_{xy}}{\partial x} \right)_{(x,0)} = -w_0 D\pi^3 \frac{1}{b}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{a^2}\right]\sin\frac{\pi x}{a} \\
F_{yb} &= \left(V_y - \frac{\partial M_{xy}}{\partial x} \right)_{(x,b)} = +w_0 D\pi^3 \frac{1}{b}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{a^2}\right]\sin\frac{\pi x}{a}
\end{aligned} \tag{6.5.31}
$$

Corner Forces

Integrating 6.5.31 over the four edges, the resultant *upward* forces on the four edges (with $w_0 < 0$, they are all four upward) are

$$
\begin{aligned}
+\overline{F}_{x0} = -\overline{F}_{xa} &= -2w_0 D\pi^2 \frac{b}{a}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{b^2}\right] \\
+\overline{F}_{y0} = -\overline{F}_{yb} &= -2w_0 D\pi^2 \frac{a}{b}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1-\nu}{a^2}\right]
\end{aligned} \tag{6.5.32}
$$

and the resultant of these may be expressed as

$$
F_{\rm up} = -4w_0 D\pi^2 ab\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 + \frac{2(1-\nu)}{a^2b^2}\right] \tag{6.5.33}
$$

The resultant downward force is, using 6.5.26,

$$
\begin{aligned}
F_{\rm down} &= \int_0^b\int_0^a q(x,y)\,dxdy = -w_0 D\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 \int_0^b\int_0^a \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dxdy \\
&= -4w_0 D\pi^2 ab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2
\end{aligned} \tag{6.5.34}
$$

The difference between $F_{\rm up}$ and $F_{\rm down}$ is due to the re-distributed twisting moment, and is explained a follows: consider again Fig. 6.5.7, where the edge twisting moments have been replaced with a statically equivalent distribution of shear forces. It can be seen that there results shear forces at the ends of the plate-edge (the "corners"), where the shear forces M_{xy} have no neighbouring shear force of opposite sign with which to "cancel out". There are concentrated forces (per unit length) at the plate-corners of magnitude M_{xy}. Examining Fig. 6.5.7, which shows the edge $x = a$, the force $M_{xy}(a,0)$ is positive up whereas the force $M_{xy}(a,b)$ is positive down. There are also contributions to the corner forces at $(a,0)$ and (a,b) from the adjacent edges, shown in Fig. 6.5.8. One finds that the *downward* concentrated forces at the corner are

$$
\begin{aligned}
P_{00} &= +2M_{xy}(0,0) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\
P_{a0} &= -2M_{xy}(a,0) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\
P_{ab} &= +2M_{xy}(a,b) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\
P_{0b} &= -2M_{xy}(0,b) = -2w_0 D(1-\nu)\frac{\pi^2}{ab}
\end{aligned}
\qquad (6.5.35)
$$

Adding these to F_{down} of Eqn. 6.5.34 now gives the F_{up} of Eqn. 6.5.33.

Physically, if one applies a pressure to a simply supported plate, the plate will tend to rise at the four corners, in a twisting action. The corner forces 6.5.35 are necessary to keep the corners down and so produce the deflection 6.5.24.

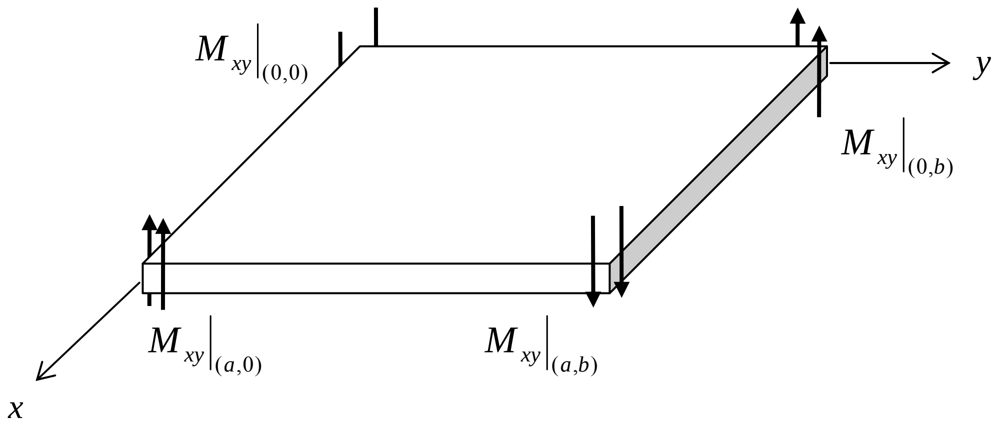


Figure 6.5.8: corner forces in the simply supported plate

The ratio of the resultant downward corner force to the downward force due to the applied pressure, F_{down}, is

$$
2(1-\nu)\frac{a^2b^2}{(a^2+b^2)^2} \qquad (6.5.36)
$$

For a square plate, this is $(1-\nu)/2$; with $\nu = 0.3$, this is 35%.

6.5.5 A Rectangular Plate Simply Supported at the Edges

The above solution can be used to solve the problem of a simply supported plate loaded by any arbitrary pressure distribution, through the use of Fourier series.

Consider again this plate, whose displacement boundary conditions are

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$$
$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{(0,y)} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{(a,y)} = 0, \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_{(x,0)} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{(x,b)} = 0 \tag{6.5.37}$$

Assume the deflection to be of the form

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.5.38}$$

with A_{mn} coefficients to be determined. It can be seen that this function satisfies the boundary conditions. Taking the derivatives of this function,

$$\frac{\partial^2 w}{\partial x^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-\frac{m^2\pi^2}{a^2}\right) A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.5.39}$$

etc., and substituting into the differential equation 6.4.9, gives

$$\pi^4 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = -q(x,y) \tag{6.5.40}$$

This can be written compactly in the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = -q(x,y) \tag{6.5.41}$$

where

$$C_{mn} = \pi^4 D A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \tag{6.5.42}$$

It remains to choose the coefficients of the series so as to satisfy the equation identically over the whole area of the plate.

One can evaluate the coefficients as one does for ordinary Fourier series, although here one has a double series and so one proceeds as follows: first, multiply both sides of (6.5.40) by $\sin(k\pi y/b)$ where *k* is an integer, and integrate over *y* between the limits $[0,b]$, so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a}\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy = -\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy \tag{6.5.43}$$

Using the orthogonality condition

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy = \begin{cases} 0, & n \neq k \\ b/2, & n = k \end{cases}, \tag{6.5.44}$$

leads to

$$\frac{b}{2}\sum_{m=1}^{\infty} C_{mk}\sin\frac{m\pi x}{a} = -\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy \tag{6.5.45}$$

Now there are functions of *x* only so, multiplying both sides by $\sin(j\pi x/a)$ and following the same procedure, one has

$$\frac{a}{2}\frac{b}{2}C_{jk} = -\int_0^a\left[\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy\right]\sin\frac{j\pi x}{a}dx \tag{6.5.46}$$

and hence the coefficients C_{mn} are (replacing the dummy subscripts j,k with m,n)

$$C_{mn} = -\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5.47}$$

Thus the coefficients A_{mn} of the original expression for the deflection $w(x,y)$, 6.5.38, are

$$A_{mn} = -\frac{1}{\pi^4 D}\frac{4}{ab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5.48}$$

It is now possible to solve for the coefficients given any loading $q(x,y)$ over the plate, and hence evaluate the deflection, moments and stresses in the plate, by taking the derivatives of the infinite series for w.

This solution is due to Navier and is called **Navier's solution** to the rectangular plate problem. A similar solution method has been used by Lévy to solve a more general problem – that of a rectangular plate simply supported on two opposite sides, and any one of the conditions free, simply-supported, or clamped, along the other two opposite sides. For example, considering a square plate, this involves using a trial function for the deflection of the form (compare with 6.5.38)

$$w(x,y) = \sum_{n=1}^{\infty} F_n(y)\sin\frac{n\pi x}{a} \tag{6.5.49}$$

and then attempting to determine the functions $F_n(y)$.

A Uniform Load

In the case of a uniform load $q(x,y) = q$, one has

$$
\begin{aligned}
A_{mn} &= -\frac{4q}{\pi^4 Dab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\int_0^a \sin\frac{m\pi x}{a}dx\int_0^b \sin\frac{n\pi y}{b}dy \\
&= -\frac{4q}{\pi^4 Dab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\left[\frac{a}{m\pi}\left(1-\cos(m\pi)\right)\right]\left[\frac{b}{n\pi}\left(1-\cos(n\pi)\right)\right] \\
&= \begin{cases} -\dfrac{16q}{\pi^6 Dmn}\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^{-2} & (m, n = 1, 3, 5, \cdots) \\ 0 & (m, n = 0, 2, 4, \cdots) \end{cases}
\end{aligned}
\qquad (6.5.50)
$$

The resulting series in 6.5.50 converges rapidly.

The deflection at the centre of the plate is then

$$
\begin{aligned}
w &= \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} A_{mn}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2} \\
&= -\frac{16qb^4}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn}\left(\frac{m^2}{(a/b)^2}+n^2\right)^{-2}\left(-1\right)^{(m+n)/2-1}
\end{aligned}
\qquad (6.5.51)
$$

For a square plate,

$$
\begin{aligned}
w &= -\frac{16qa^4}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn}\left(m^2+n^2\right)^{-2}\left(-1\right)^{(m+n)/2-1} \\
&= -\frac{qa^4}{D}\times 0.0040624
\end{aligned}
\qquad (6.5.52)
$$

Denoting the area a^2 by *A*, this is $\omega = -0.0041qA^2/D$. This can be compared with the clamped circular plate; denoting the area there, πa^2, by *A*, the maximum deflection, Eqn. 6.5.11, gives $\omega = -0.0016qA^2/D$.

Corner Forces

The twisting moment is

$$
M_{xy}(x,y) = -D(1-\nu)\frac{\partial^2\omega}{\partial x\partial y} = -D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad (6.5.53)
$$

and the four corner forces required to hold the plate down are now

$$
\begin{aligned}
P_{00} &= +2M_{xy}(0,0) = -2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn} \\
P_{a0} &= -2M_{xy}(a,0) = +2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos m\pi \\
P_{0b} &= -2M_{xy}(0,b) = +2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos n\pi \\
P_{ab} &= +2M_{xy}(a,b) = -2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos m\pi\cos n\pi
\end{aligned}
\qquad (6.5.54)
$$

For a uniform load over a square plate, using 6.5.50, the corner forces reduce to

$$
\begin{aligned}
4P &= 4\frac{32q(1-\nu)a^2}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{\left(m^2+n^2\right)^2} \\
&= 4\frac{32q(1-\nu)a^2}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left((2m-1)^2+(2n-1)^2\right)^2} \\
&\approx 4\frac{32q(1-\nu)a^2}{\pi^4}\times 0.2825 \\
&\approx 0.26F_0
\end{aligned}
\qquad (6.5.55)
$$

(for $\nu = 0.3$) where $F_0 = qa^2$ is the resultant applied force.

6.5.6 Problems

1. Derive the expressions for the stress components in polar form, for the clamped circular plate under uniform lateral load, Eqn. 6.5.18.

6.6 Plate Problems in Polar Coordinates

6.6.1 Plate Equations in Polar Coordinates

To examine directly plate problems in polar coordinates, one can first transform the Cartesian plate equations considered in the previous sections into ones in terms of polar coordinates.

First, the definitions of the moments and forces are now

$$M_r = -\int_{-h/2}^{+h/2} z\sigma_{rr}dz, \quad M_\theta = -\int_{-h/2}^{+h/2} z\sigma_{\theta\theta}dz, \quad M_{r\theta} = \int_{-h/2}^{+h/2} z\sigma_{r\theta}dz \tag{6.6.1}$$

and

$$V_r = -\int_{-h/2}^{+h/2} \sigma_{zr}dz, \qquad V_\theta = -\int_{-h/2}^{+h/2} \sigma_{z\theta}dz \tag{6.6.2}$$

The strain-curvature relations, Eqns. 6.2.27, can be transformed to polar coordinates using the transformations from Cartesian to polar coordinates detailed in §4.2 (in particular, §4.2.6). One finds that {▲Problem 1}

$$\begin{aligned}
\varepsilon_{rr} &= -z\frac{\partial^2 w}{\partial r^2} \\
\varepsilon_{\theta\theta} &= -z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \\
\varepsilon_{r\theta} &= -z\left(-\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta}\right)
\end{aligned} \tag{6.6.3}$$

The moment-curvature relations 6.2.31 become {▲Problem 2}

$$\begin{aligned}
M_r &= D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \\
M_\theta &= D\left[\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + \nu\frac{\partial^2 w}{\partial r^2}\right] \\
M_{r\theta} &= -D(1-\nu)\left(-\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta}\right)
\end{aligned} \tag{6.6.4}$$

The governing differential equation 6.4.9 now reads

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w = -\frac{q}{D} \tag{6.6.5}$$

The shear forces in terms of deflection, Eqn 6.4.12, now read {▲Problem 3}

$$V_r = D\frac{\partial}{\partial r}\left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right], \quad V_\theta = D\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \quad (6.6.6)$$

Finally, the stresses are {▲Problem 4}

$$\sigma_{rr} = -\frac{12z}{h^3}M_r, \quad \sigma_{\theta\theta} = -\frac{12z}{h^3}M_\theta, \quad \sigma_{r\theta} = \frac{12z}{h^3}M_{r\theta} \quad (6.6.7)$$

and

$$\sigma_{zr} = -\frac{3V_r}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right], \qquad \sigma_{z\theta} = -\frac{3V_\theta}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \quad (6.6.8)$$

The differential equation 6.6.5 can be solved using a method similar to the Airy stress function method for problems in polar coordinates (the Mitchell solution), that is, a solution is sought in the form of a Fourier series. Here, however, only axisymmetric problems will be considered in detail.

6.6.2 Plate Equations for Axisymmetric Problems

When the loading and geometry of the plate are axisymmetric, the plate equations given above reduce to

$$\begin{aligned} M_r &= D\left[\frac{d^2 w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right] \\ M_\theta &= D\left[\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2 w}{dr^2}\right] \\ M_{r\theta} &= 0 \end{aligned} \quad (6.6.9)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = -\frac{q(r)}{D} \quad (6.6.10)$$

$$V_r = D\frac{d}{dr}\left[\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right], \quad V_\theta = 0 \quad (6.6.11)$$

$$\sigma_{rr} = -\frac{12z}{h^3}M_r, \quad \sigma_{\theta\theta} = -\frac{12z}{h^3}M_\theta, \quad \sigma_{r\theta} = 0 \quad (6.6.12)$$

and

$$\sigma_{zr} = -\frac{3V_r}{2h}\left[1-\left(\frac{z}{h/2}\right)^2\right], \qquad \sigma_{z\theta} = 0 \tag{6.6.13}$$

Note that there is no twisting moment, so the problem of dealing with non-zero twisting moments on free boundaries seen with rectangular plate does not arise here.

6.6.3 Axisymmetric Plate Problems

For uniform q, direct integration of 6.6.10 leads to

$$w = -\frac{qr^4}{64D} + \frac{1}{4}\overline{A}r^2(\ln r - 1) + \frac{1}{4}\overline{B}r^2 + \overline{C}\ln r + \overline{D} \tag{6.6.14}$$

with

$$\begin{aligned}
\frac{dw}{dr} &= -\frac{qr^3}{16D} + \frac{1}{4}\overline{A}r(2\ln r - 1) + \frac{1}{2}\overline{B}r + \overline{C}\frac{1}{r} \\
\frac{d^2w}{dr^2} &= -\frac{3qr^2}{16D} + \frac{1}{4}\overline{A}(2\ln r + 1) + \frac{1}{2}\overline{B} - \overline{C}\frac{1}{r^2} \\
\frac{d^3w}{dr^3} &= -\frac{3qr}{8D} + \frac{1}{2}\overline{A}\frac{1}{r} + 2\overline{C}\frac{1}{r^3}
\end{aligned} \tag{6.6.15}$$

and

$$V_r = D\left[\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right] = -\frac{qr}{2} + \frac{\overline{A}}{r}D \tag{6.6.16}$$

There are two classes of problem to consider, plates with a central hole and plates with no hole. For a plate with no hole in it, the condition that the stresses remain finite at the plate centre requires that d^2w/dr^2 remains finite, so $\overline{A} = \overline{C} = 0$. Thus immediately one has $V_r = -qr/2$. The boundary conditions at the outer edge $r = a$ give $\overline{B}$ and $\overline{D}$.

1. Solid Plate – Uniform Bending

The simplest case is pure bending of a plate, $M_r = M_0$, with no transverse pressure, $q = 0$. The plate is solid so $\overline{A} = \overline{C} = 0$ and one has $w = \overline{B}r^2/4 + \overline{D}$. The applied moment is

$$M_0 = D\left[\frac{d^2\omega}{dr^2} + \nu\frac{1}{r}\frac{d\omega}{dr}\right] = \frac{1}{2}\overline{B}D[1+\nu] \tag{6.6.17}$$

so $\overline{B} = 2M_0/D(1+\nu)$. Taking the deflection to be zero at the plate-centre, the solution is

$$w = \frac{M_0}{2D(1+\nu)} r^2 \tag{6.6.18}$$

2. Solid Plate Clamped – Uniform Load

Consider next the case of clamped plate under uniform loading. The boundary conditions are that $w = dw/dr = 0$ at $r = a$, leading to

$$\overline{B} = \frac{qa^2}{8D}, \qquad \overline{D} = -\frac{qa^4}{64D} \tag{6.6.19}$$

and hence

$$w = -\frac{q}{64D}\left(r^2 - a^2\right)^2 \tag{6.6.20}$$

which is the same as 6.5.10.

The reaction force at the outer rim is $V_r(a) = -qa/2$. This is a force per unit length; the force acting on an element of the outer rim is $-qa(a\Delta\theta)/2$ and the total reaction force around the outer rim is $-qa^2\pi$, which balances the same applied force.

3. Solid Plate Simply Supported – Uniform Load

For a simply supported plate, $w = 0$ and $M_r = 0$ at $r = a$. Using 6.6.9a, one then has {▲Problem 5}

$$\overline{B} = \frac{3+\nu}{1+\nu}\frac{qa^2}{8D}, \qquad \overline{D} = -\frac{5+\nu}{1+\nu}\frac{qa^4}{64D} \tag{6.6.21}$$

and hence

$$w = -\frac{q}{64D}\left(\frac{5+\nu}{1+\nu}a^2 - r^2\right)\left(a^2 - r^2\right) \tag{6.6.22}$$

The deflection for the clamped and simply supported cases are plotted in Fig. 6.6.1 (for $\nu = 0.3$).

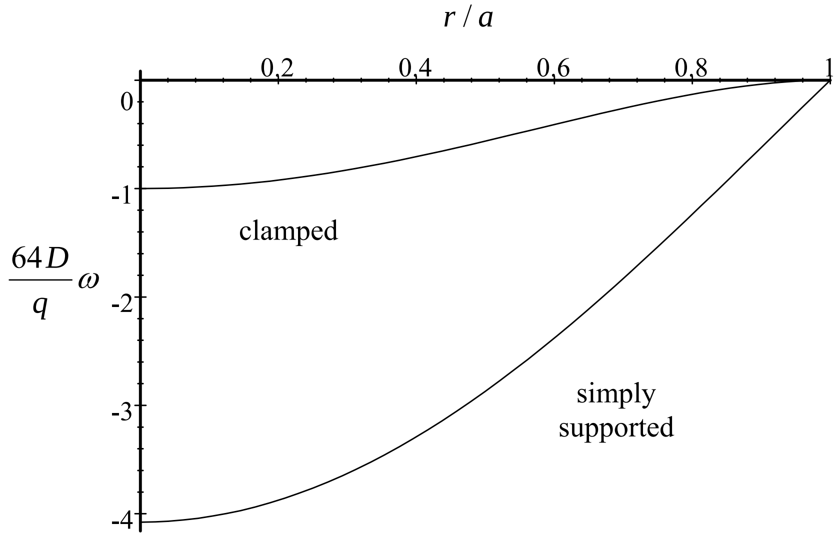


Figure 6.6.1: deflection for a circular plate under uniform loading

4. Solid Plate with a Central Concentrated Force

Consider now the case of a plate subjected to a single concentrated force *F* at $r = 0$. The resultant shear force acting on any cylindrical portion of the plate with radius *r* about the plate-centre is $2\pi r V_r(r)$. As $r \to 0$, one must have an infinite V_r so that this resultant is finite and equal to the applied force *F*. An infinite shear force implies infinite stresses. It is possible for the stresses at the centre of the plate to be infinite. However, although the stresses and strain might be infinite, the displacements, which are obtained from the strains through integration, can remain, and should remain, finite. Although the solution will be "unreal" at the plate-centre, one can again use Saint-Venant's principle to argue that the solution obtained will be valid everywhere except in a small region near where the force is applied.

Thus, seek a solution which has finite displacement in which case, by symmetry, the slope at $r = 0$ will be zero. From the general axisymmetric solution 6.6.15a,

$$\left.\frac{dw}{dr}\right|_{r=0} = \overline{C}\left.\frac{1}{r}\right|_{r=0} \tag{6.6.23}$$

so $\overline{C} = 0$.

From 6.6.16

$$2\pi r V_r\big|_{r=0} = 2\pi \overline{A} D \equiv F \tag{6.6.24}$$

Thus $\overline{A} = F / 2\pi D$ and the moments and shear force become infinite at the plate-centre.

The other two constants can be obtained from the boundary conditions. For a clamped plate, $w = dw/dr = 0$, and one finds that {▲Problem 7}

$$w = \frac{F}{16\pi D}\left[\left(a^2 - r^2\right) + 2r^2 \ln\left(r / a\right)\right] \tag{6.6.25}$$

This solution results in $\ln(r/a)$ terms in the expressions for moments, giving logarithmically infinite in-plane stresses at the plate-centre.

5. Plate with a Hole

For a plate with a hole in it, there will be four boundary conditions to determine the four constants in Eqn. 6.6.14. For example, for a plate which is simply supported around the outer edge $r = b$ and free on the inner surface $r = a$, one has

$$\begin{aligned} &M_r(a) = 0, \quad F_r(a) = 0 \\ &w(b) = 0, \quad\;\; M_r(b) = 0 \end{aligned} \tag{6.6.26}$$

6.6.4 Problems

1. Use the expressions 4.2.11-12, which relate second partial derivatives in the Cartesian and polar coordinate systems, together with the strain transformation relations 4.2.17, to derive the strain-curvature relations in polar coordinates, Eqn. 6.6.3.

2. Use the definitions of the moments, 6.6.1, and again relations 4.2.11-12, together with the stress transformation relations 4.2.18, to derive the moment-curvature relations in polar coordinates, Eqn. 6.6.4.

3. Derive Eqns. 6.6.6.

4. Use 6.2.33, 6.4.15-16 to derive the stresses in terms of moments and shear forces, Eqns. 6.6.7-8.

5. Solve the simply supported solid plate problem and hence derive the constants 6.6.21.

6. Show that the solution for a simply supported plate (with no hole), Eqn. 6.6.22, can be considered a superposition of the clamped solution, Eqn. 6.6.20, and a pure bending, by taking an appropriate deflection at the plate-centre in the pure bending case.

7. Solve for the deflection in the case of a clamped solid circular plate loaded by a single concentrated force, Eqn. 6.6.25.

6.7 In-Plane Forces and Plate Buckling

In the previous sections, only bending and twisting moments and out-of-plane shear forces were considered. In this section, in-plane forces are considered also. The in-plane forces will give rise to in-plane membrane strains, but here it is assumed that these are uncoupled from the bending strains. In other words, the membrane strains can be found from a separate plane stress analysis of the mid-surface and the bending of the plate does not affect these membrane strains. The possible effect of the in-plane forces on the bending strains is the main concern here.

6.7.1 Equilibrium for In-plane Forces

Start again with the equations of equilibrium, Eqns. 6.4.6. Integrating the first and second through the thickness of the plate (this time without multiplying first by *z*), and using the definitions of the in-plane forces 6.1.1-6.1.2, leads to

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \end{aligned} \tag{6.7.1}$$

6.7.2 The Governing Differential Equation

Consider an element of the deflected plate, Fig. 6.7.1. Only a deflection in the *y* direction, $\partial \omega / \partial y$, is considered for clarity. Resolving the components of the in-plane forces into horizontal and vertical components:

$$\begin{aligned} \sum F_H &= -N_y \Delta x + \left(N_y + \frac{\partial N_y}{\partial y} \Delta y \right) \Delta x - N_{xy} \Delta y + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} \Delta x \right) \Delta y \\ \sum F_V &= -N_y \frac{\partial w}{\partial y} \Delta x + \left(N_y \frac{\partial w}{\partial y} + \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} \right) \Delta x \right) \Delta x \\ &\quad - N_{xy} \frac{\partial w}{\partial y} \Delta y + \left(N_{xy} \frac{\partial w}{\partial y} + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w}{\partial y} \right) \Delta x \right) \Delta y \end{aligned} \tag{6.7.2}$$

These reduce to

$$\sum F_H = \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \Delta y \Delta x$$

$$\begin{aligned} \sum F_V &= \left[\frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y} \right) \right] \Delta x \Delta y \\ &= \left[\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \frac{\partial w}{\partial y} + \left(N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \right] \Delta x \Delta y \end{aligned} \tag{6.7.3}$$

Using 6.7.1, one has

$$\sum F_H = 0, \qquad \sum F_V = \left(N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \Delta x \Delta y \tag{6.7.4}$$

Considering also a deflection $\partial \omega / \partial x$, one has for the resultant vertical force :

$$\sum F_V = \left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \Delta x \Delta y \tag{6.7.5}$$

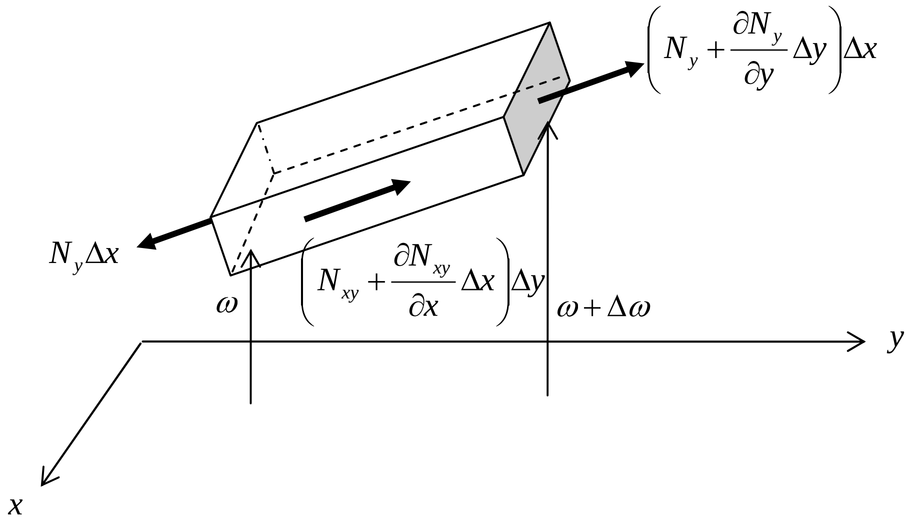


Figure 6.7.1: In-plane forces acting on a plate element

When the in-plane forces were neglected, the vertical stress resisted by bending and shear force was $\sigma_{zz} = -q$. Here, one has an additional stress given by 6.7.5, and so the governing differential equation 6.4.7 becomes

$$\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{1}{D}\left(-q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \tag{6.7.6}$$

6.7.3 Buckling of Plates

When compressive in-plane forces are applied to a plate, the plate will at first remain flat and simply be compressed. However, when the in-plane forces reach a critical level, the plate will bend and the deflection will be given by the solution to 6.7.6. For example, consider the case of a simply supported plate subjected to a uniform in-plane compression N_x only, Fig. 6.7.2, in which case 6.7.6 reduces to

$$\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} \tag{6.7.7}$$

Following Navier's method from §6.5.5, assume a buckled shape

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.7.8}$$

so that 6.7.7 becomes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{N_x}{D}\frac{m^2\pi^2}{a^2}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0 \tag{6.7.9}$$

Disregarding the trivial $A_{mn} = 0$, this can be satisfied by taking

$$N_x = -\frac{Da^2\pi^2}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \tag{6.7.10}$$

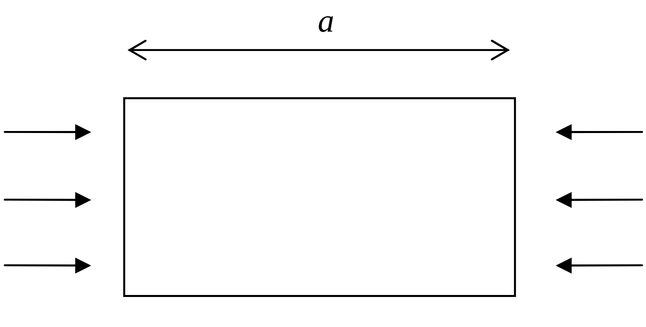


Figure 6.7.2: In-plane compression of a plate

The lowest in-plane force N_x which will deflect the plate is sought. Clearly, the smallest value on the right hand side of 6.7.10 will be when $n = 1$. This means that the buckling modes as given by 6.7.8 will be of the form

$$\sin\frac{m\pi x}{a}\sin\frac{\pi y}{b} \tag{6.7.11}$$

so that the plate will only ever buckle with one half-wave in the direction perpendicular to loading.

When $a \le b$, the smallest value occurs when $m = 1$, in which case the critical in-plane force is

$$\left(N_x\right)_{\rm cr} = -\frac{D\pi^2}{b^2}\left(\frac{b}{a} + \frac{a}{b}\right)^2 \tag{6.7.12}$$

When a/b is very small, the plate is loaded along the relatively long edges and the critical load is much higher than for a square plate.

The deflection (buckling mode) corresponding to this critical load is

$$w(x, y) = A_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{6.7.13}$$

Note that the amplitude A_{11} cannot be determined from the analysis[1].

As a/b increases above unity, the value of *m* at which the applied load is a minimum increases. When a/b reaches just over $\sqrt{2}$, the critical buckling load occurs for $m = 2$, for which

$$\left(N_x\right)_{cr} = -\frac{D\pi^2}{b^2}\left(\frac{2b}{a} + \frac{a}{2b}\right)^2 \tag{6.7.14}$$

and corresponding buckling more

$$w(x, y) = A_{21} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \tag{6.7.15}$$

The plate now buckles in two half-waves, as if the centre-line were simply supported and there were two smaller separate plates buckling similarly.

As a/b increases further, so too does *m*. For a very long, thin, plate, $m \approx a/b$, and so the plate subdivides approximately into squares, each buckling in a half-wave.

[1] this is a consequence of assuming small deflections; it can be determined when the deflections are not assumed to be small

6.8 Plate Vibrations

In this section, the problem of a vibrating circular plate will be considered. Vibrating plates will be re-examined again in the next section, using a strain energy formulation.

6.8.1 Vibrations of a Clamped Circular Plate

When a plate vibrates with velocity $\partial w / \partial t$, the third equation of equilibrium, Eqn. 6.6.2c becomes the equation of motion

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{6.8.1}$$

With this adjustment, the term q is replaced with $q + \rho h \partial^2 w / \partial t^2$ in the relevant equations; the acceleration term is treated as a transverse load of intensity $\rho h \partial^2 w / \partial t^2$.

Regarding the circular plate, one has from the axisymmetric governing equation 6.6.10 (with $q = 0$),

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right)^2 w = -\frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} \tag{6.8.2}$$

Assume a solution of the form

$$w(r,t) = W(r)\cos(\omega t + \phi) \tag{6.8.3}$$

Substituting into 6.8.2 gives

$$\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right)^2 - k^4 \right] W = 0 \tag{6.8.4}$$

where

$$k^2 = \omega \sqrt{\frac{\rho h}{D}} \tag{6.8.5}$$

Eqn. 6.8.4 gives the two differential equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 \right) W = 0, \qquad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 \right) W = 0 \tag{6.8.6}$$

The solution to these equations are

$$W = C_1 J_0(kr) + C_2 Y_0(kr), \qquad W = C_3 I_0(kr) + C_4 K_0(kr) \tag{6.8.7}$$

where J_0 and Y_0 are, respectively, the Bessel functions of order zero of the first kind and of the second kind; I_0 and K_0 are, respectively, the Modified Bessel functions of order zero of the first kind and of the second kind[1]. These functions are plotted in Fig. 6.8.1 below. For a solid plate with no hole at $r = 0$, one requires that $C_2 = C_4 = 0$, since Y_0 and K_0 become unbounded as $r \to 0$. The general solution is thus

$$W(r) = \overline{A} J_0(kr) + \overline{B} I_0(kr) \tag{6.8.8}$$

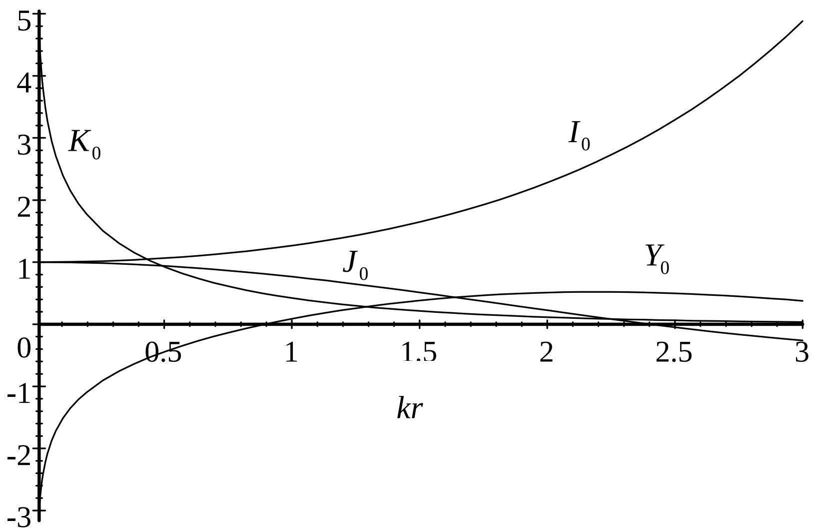


Figure 6.8.1: Bessel Functions

For a clamped plate, the boundary conditions give

$$\begin{aligned} W(a) &= \overline{A} J_0(ka) + \overline{B} I_0(ka) = 0 \\ \left.\frac{dW}{dr}\right|_{r=a} &= \overline{A} J_0'(ka) + \overline{B} I_0'(ka) = 0 \end{aligned} \tag{6.8.9}$$

where the dash means $J_0'(x) = dJ_0(x)/dx$ and $I_0'(x) = dI_0(x)/dx$. Using the relations

$$J_0'(x) = -J_1(x), \quad I_0'(x) = +I_1(x) \tag{6.8.10}$$

where J_1, I_1 are Bessel functions of order one, one has

$$\frac{J_0(ka)}{I_0(ka)} = -\frac{J_1(ka)}{I_1(ka)} \tag{6.8.11}$$

[1] by *definition*, these Bessel functions are the solution of the differential equations 6.8.6.

The roots ka give the frequencies of vibration of the plate. The function

$$J_0(ka)I_1(ka) + I_0(ka)J_1(ka) \tag{6.8.12}$$

is plotted in Fig. 6.8.2 below. The smallest root is found to be 3.1962. Eqn. 6.8.5 then gives for the frequency,

$$\omega = \alpha \frac{1}{a^2}\sqrt{\frac{D}{\rho h}} \tag{6.8.13}$$

where $\alpha = 10.2158$.

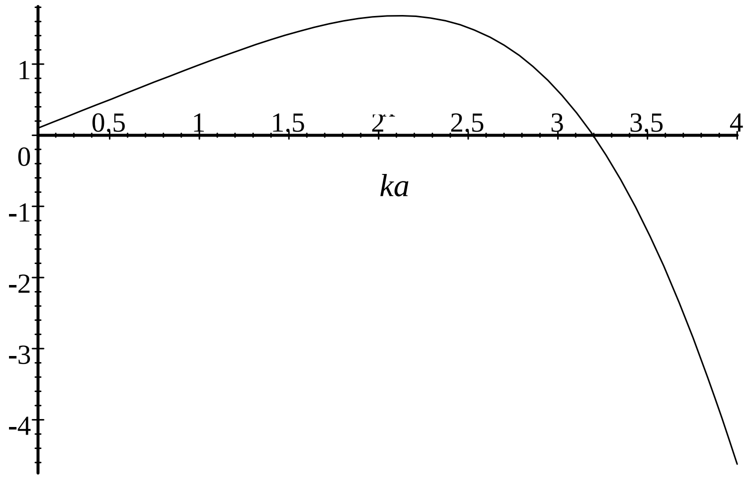


Figure 6.8.2: The Function 6.8.12

Further roots ka of 6.8.12 are given in Table 6.8.1. For each of these roots there is a corresponding frequency ω given by Eqn. 6.8.13, for which the value of α is also tabulated.

	ka	α	nodal circle
1	3.1962	10.2158	
2	6.3064	39.7711	0.3790
3	9.4395	89.1041	0.2548, 0.5833

Table 6.8.1: Roots of Eqn. 6.8.11, frequency factors and nodal circle roots

From 6.8.3, 6.8.8-9, the solution for the deflection is

$$w(r,t) = \overline{A}\left[J_0(kr) - \frac{J_0(ka)}{I_0(ka)} I_0(kr)\right]\cos(\omega t + \phi) \tag{6.8.14}$$

These are an infinite number of deflections, each one corresponding to a root ka. The actual deflection will be a superposition of these individual solutions.

The term inside the square brackets gives the mode shape of the plate during the vibration. The first three (normalized) mode shapes, corresponding to the first three roots, are shown in Fig. 6.8.3.

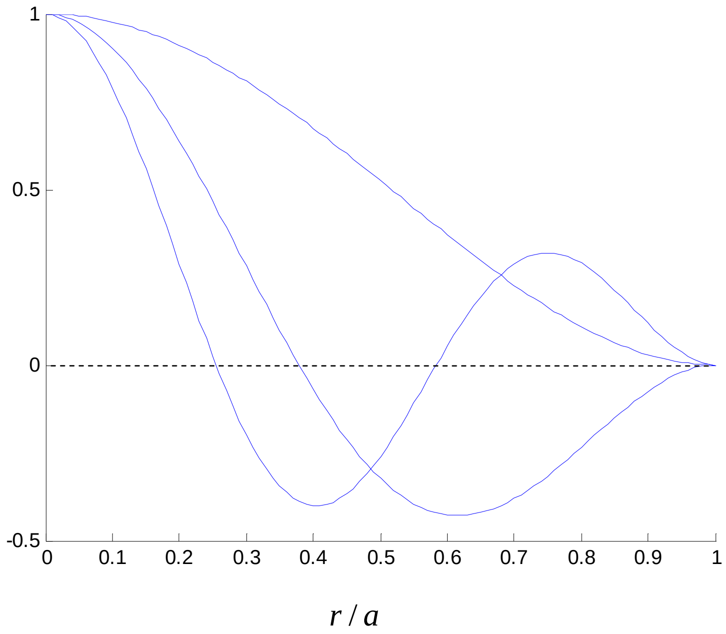


Figure 6.8.3: Mode shapes for the Clamped Circular Plate

The point r/a where these mode-shapes change sign are the positions of the so-called **nodal circles**. These roots of the mode shapes are given in the last column of Table 6.8.1

The General Problem

For circular plates not constrained to an axisymmetric response, one must use the more general differential equation 6.6.5

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w=-\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} \tag{6.8.15}$$

This time, instead of 6.8.3, assume a solution of the form

$$w(r,\theta,t)=\sum W_n(r)\cos(n\theta)\sin(\omega t+\phi) \tag{6.8.16}$$

Then 6.8.4-6 become

$$\left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{d}{\partial r}-\frac{n^2}{r^2}\right)^2-k^4\right]W=0 \tag{6.8.17}$$

where k is again given by 6.8.5, and 6.8.6 becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+k^2\right)W=0,\qquad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-k^2\right)W=0 \tag{6.8.18}$$

The solution to these equations are

$$W=C_1J_n(kr)+C_2Y_n(kr),\qquad W=C_3I_n(kr)+C_4K_n(kr) \tag{6.8.19}$$

where one now has Bessel functions of order n. Proceeding as before, one now needs to find roots of the equation

$$J_n(ka)I_{n+1}(ka)+I_n(ka)J_{n+1}(ka)=0 \tag{6.8.20}$$

and the deflection is

$$w(r,t)=\sum\overline{A}\left[J_n(kr)-\frac{J_n(ka)}{I_n(ka)}I_n(kr)\right]\cos(n\theta)\sin(\omega t+\phi) \tag{6.8.21}$$

The solution for $n=0$ has been given already. For other values of n, there are n so-called **nodal diameters**. For example, for $n=1$ there is one nodal diameter along $\theta=\pm\pi/2$, along which the deflection is zero. The roots of 6.8.20 for this case are given in Table 6.8.2, together with the nodal circle locations.

	ka	α	nodal circle
1	4.6109	21.2604	
2	7.7993	60.8287	0.4897
3	10.9581	120.0792	0.3497, 0.6390

Table 6.8.2: Roots of Eqn. 6.8.20 (*n*=1), frequency factors and nodal circle roots

The mode shapes for half the plate for this case of one nodal diameter are shown in Fig. 6.8.4, corresponding to the first two roots in Table 6.8.2. The frequencies corresponding to these solutions are again given by 6.8.13 with the frequency factor α given in the table.

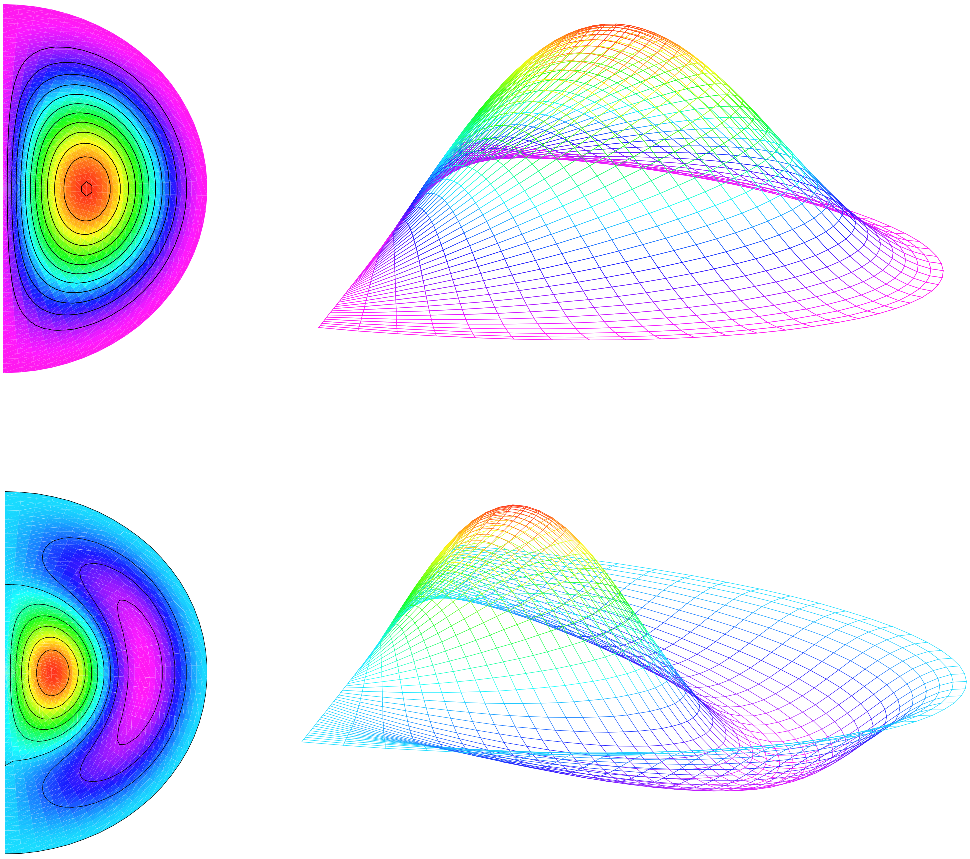


Figure 6.8.4: Mode shapes for the case of one nodal diameter

6.9 Strain Energy in Plates

6.9.1 Strain Energy due to Plate Bending and Torsion

Here, the elastic strain energy due to plate bending and twisting is considered.

Consider a plate element bending in the *x* direction, Fig. 6.9.1. The radius of curvature is $R = \partial^2 w / \partial x^2$. The strain energy due to bending through an angle $\Delta\theta$ by a moment $M_x \Delta y$ is

$$\Delta U = \frac{1}{2}(M_x \Delta y)\frac{\partial^2 w}{\partial x^2}\Delta x \tag{6.9.1}$$

Considering also contributions from M_y and M_{xy}, one has

$$\Delta U = \frac{1}{2}\left(M_x \frac{\partial^2 w}{\partial x^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} + M_y \frac{\partial^2 w}{\partial y^2}\right)\Delta x \Delta y \tag{6.9.2}$$

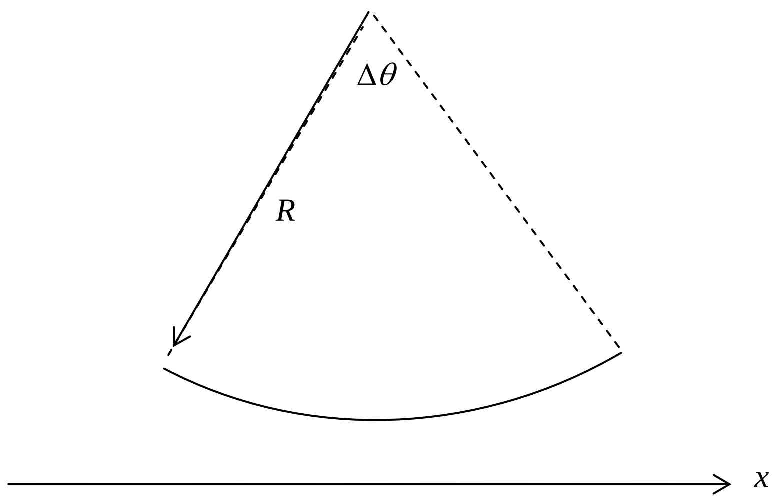


Figure 6.9.1: a bending plate element

Using the moment-curvature relations, one has

$$\begin{aligned}\Delta U &= \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\Delta x \Delta y \\ &= \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right)\right]\Delta x \Delta y\end{aligned} \tag{6.9.3}$$

This can now be integrated over the complete plate surface to obtain the total elastic strain energy.

6.9.2 The Principle of Minimum Potential Energy

Plate problems can be solved using the principle of minimum potential energy (see Book I, §8.6). Let $V = -W_{ext}$ be the potential energy of the loads, equivalent to the negative of the work done by those loads, and so the potential energy of the system is $\Pi(w) = U(w) + V(w)$. The solution is then the deflection which minimizes $\Pi(w)$.

When the load is a uniform lateral pressure q, one has

$$\Delta V = -\Delta W_{ext} = +q\, w(x,y)\Delta x \Delta x \tag{6.9.4}$$

and

$$\Delta\Pi = \left\{ \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right] + qw \right\} \Delta x \Delta y \tag{6.9.5}$$

As an example, consider again the simply supported rectangular plate subjected to a uniform load q. Use the same trial function 6.5.38 which satisfies the boundary conditions:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.9.6}$$

Substituting into 6.9.5 and integrating over the plate gives

$$\begin{aligned} \Pi = \int_0^b\int_0^a & \left\{ \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b} \right. \right. \\ & \left. - 2(1-\nu)\left(\frac{m^2 n^2 \pi^4}{a^2 b^2} \left(\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} - \cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b} \right) \right) \right] \\ & \left. + q\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \right\} dxdy \end{aligned} \tag{6.9.7}$$

Carrying out the integration leads to

$$\Pi = \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \times \frac{1}{4}ab \right] + q \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} A_{mn} \times \frac{4ab}{mn\pi^2} \tag{6.9.8}$$

To minimize the total potential energy, one sets

$$\frac{\partial \Pi}{\partial A_{mn}} = DA_{mn}\frac{ab\pi^4}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + q\frac{4ab}{mn\pi^2} = 0$$
$$\rightarrow \quad A_{mn} = -\frac{16q}{\pi^6 Dmn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2} \tag{6.9.9}$$

which is the same result as 6.5.50.

6.9.3 Strain Energy in Polar Coordinates

For circular plates, one can transform the strain energy expression 6.9.3 into polar coordinates, giving {▲Problem 1}

$$\Delta U = \frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 - 2(1-\nu)\times \left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)-\left(\frac{1}{r^2}\frac{\partial w}{\partial \theta}-\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right)^2\right]\right\}\Delta x \Delta y \tag{6.9.10}$$

For an axisymmetric problem, the strain energy is

$$\Delta U = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]\Delta x \Delta y \tag{6.9.11}$$

6.9.4 Vibration of Plates

For vibrating plates, one needs to include the kinetic energy of the plate. The kinetic energy of a plate element of dimensions $\Delta x, \Delta y$ and moving with velocity $\partial \omega / \partial t$ is

$$\Delta K = \frac{1}{2}\rho h\left(\frac{\partial w}{\partial t}\right)^2 \Delta x \Delta y \tag{6.9.12}$$

According to Hamilton's principle, then, the quantity to be minimized is now $U(w)+V(w)-K(w)$.

Consider again the problem of a circular plate undergoing axisymmetric vibrations. The potential energy function is

$$D\pi\int_0^a\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]rdr - \pi\rho h\int_0^a\left(\frac{\partial w}{\partial t}\right)^2 rdr \tag{6.9.13}$$

Assume a solution of the form

$$w(r,t) = W(r)\cos(\omega t + \phi) \tag{6.9.14}$$

Substituting this into 6.9.13 leads to

$$D\pi\int_0^a \left[\left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right)^2 - 2(1-\nu)\frac{d^2W}{dr^2}\left(\frac{1}{r}\frac{dW}{dr}\right)\right] rdr - \pi\rho h\omega\int_0^a W^2 rdr \tag{6.9.15}$$

Examining the clamped plate, assume a solution, an assumption based on the known static solution 6.6.20, of the form

$$W(r) = A(a^2 - r^2)^2 \tag{6.9.16}$$

Substituting this into 6.9.15 leads to

$$\begin{aligned} &32D\pi A^2 \int_0^a \left[2(a^4 - 4a^2r^2 + 4r^4) - (1-\nu)(a^4 - 4a^2r^2 + 3r^4)\right] rdr \\ &\qquad - \pi\rho\omega h A^2 \int_0^a r(a^2 - r^2)^4 dr \end{aligned} \tag{6.9.17}$$

Evaluating the integrals leads to

$$\pi A^2\left(\frac{32}{3}Da^6 - \frac{1}{10}\rho h\omega a^{10}\right) \tag{6.9.18}$$

Minimising this function, setting $\partial/\partial A\{\ \} = 0$, then gives

$$\omega = \alpha\frac{1}{a^2}\sqrt{\frac{D}{\rho h}}, \qquad \alpha = \sqrt{\frac{320}{3}} \approx 10.328 \tag{6.9.19}$$

This simple one-term solution is very close to the exact result given in Table 6.8.1, 10.2158. The result 6.9.19 is of course greater than the actual frequency.

6.9.5 Problems

1. Derive the strain energy expression in polar coordinates, Eqn. 6.9.10.

6.10 Limitations of Classical Plate Theory

The validity of the classical plate theory depends on a number of factors:

1. the curvatures are small
2. the in-plane plate dimensions are large compared to the thickness
3. membrane strains can be neglected

The second and third of these points are discussed briefly in what follows.

6.10.1 Moderately Thick Plates

As with beam theory, and as mentioned already, it turns out that the solutions based on the classical theory agree well with the full elasticity solutions (away from the edges of the plate), provided the plate thickness is small relative to its other linear dimensions. When the plate is relatively thick, one is advised to use a more exact theory, for example one of the shear deformation theories:

Shear deformation Theories

The **Mindlin plate theory (**or **moderately thick plate theory** or **shear deformation theory**) was developed in the mid-1900s to allow for possible transverse shear strains. In this theory, there is the added complication that vertical line elements before deformation do *not* have to remain perpendicular to the mid-surface after deformation, although they do remain straight. Thus shear strains ε_{yz} and ε_{zx} are generated, constant through the thickness of the plate.

The classical plate theory is inconsistent in the sense that elements are assumed to remain perpendicular to the mid-plane, yet equilibrium requires that stress components σ_{xz}, σ_{yz} still arise (which would cause these elements to deform). The theory of thick plates is more consistent, but it still makes the assumption that $\sigma_{zz} = 0$. Note that *both* are approximations of the exact three-dimensional equations of elasticity.

As an indication of the error involved in using the classical plate theory, consider the problem of a simply supported square plate subjected to a uniform pressure. According to Eqn. 6.5.52, the central deflection is $w/(qa^4/1000D) = 4.062$. The shear deformation theory predicts 4.060 (for $a/h = 100$), 4.070 (for $a/h = 50$), 4.111 (for $a/h = 20$) and 4.259 (for $a/h = 10$). This trend holds in general; the classical theory is good for thin plates but under-predicts deflections (and over-predicts buckling loads and natural frequencies) in relatively thick plates.

An important difference between the thin plate and thick plate theories is that in the former the moments are related to the curvatures through (using M_x for illustration)

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{6.10.1}$$

This is only an approximate relation (although it turns out to be exact in the case of pure bending). The thick plate theory predicts that, in the case of a uniform lateral load *q*, the relationship is given by

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + h^2 q\frac{8+\nu+\nu^2}{20(1-\nu)} \qquad (6.10.2)$$

The thin plate expression will be approximately equal to the thick plate expression when the thickness *h* is very small, since in that case $h^2 \to 0$, or when the ratio of load to stiffness, q/D, is small.

Some solutions for circular plates using the various theories are presented next for comparison.

Comparison of Solutions for Circular Plates

1. Uniform load *q*, clamped:

Both thin and thick plate theories give

$$\omega = -\frac{q}{64D}\left(a^2 - r^2\right)^2 \qquad (6.10.3)$$

2. Uniform load *q*, simply supported:

$$\omega = -\frac{q}{64D}\left(a^2 - r^2\right)\left(\frac{5+\nu}{1+\nu}a^2 - r^2 + \eta\right), \qquad \eta = \begin{cases} 0 & \text{thin} \\ +\frac{8}{5}\frac{8+\nu+\nu^2}{1-\nu^2}h^2 & \text{thick} \end{cases} \qquad (6.10.4)$$

3. Concentrated central load, clamped (Eqn. 6.6.25):

$$\omega = -\frac{F}{16\pi D}\left(\left(a^2 - r^2\right) + 2r^2\ln(r/a) + \eta\right)$$

$$\eta = \begin{cases} 0 & \text{thin} \\ 0 & \text{thick} \\ -\frac{4}{5}\frac{2-\nu}{1-\nu}h^2\ln(r/a) & \text{exact} \end{cases} \qquad (6.10.5)$$

4. Concentrated central load *Q*, simply supported:

$$\omega = -\frac{F}{16\pi D}\left(\frac{3+\nu}{1+\nu}\left(a^2 - r^2\right) + 2r^2 \ln(r/a) + \eta\right)$$

$$\eta = \begin{cases} 0 & \text{thin} \\ +\tfrac{2}{5}\dfrac{8+\nu}{1+\nu}\dfrac{h^2}{a^2}\left(a^2 - r^2\right) & \text{thick} \\ -\tfrac{4}{5}\dfrac{2-\nu}{1-\nu}h^2 \ln(r/a) - \tfrac{1}{5}\dfrac{2\nu}{1+\nu}\dfrac{h^2}{a^2}\left(a^2 - r^2\right) & \text{exact} \end{cases} \tag{6.10.6}$$

Higher-order deformation Theories

A further theory known as **third-order plate theory** has also been developed. This allows for the displacements to vary not only linearly (the previously described Mindlin theory is also called the **first order shear deformation theory**), but as cubic functions; the in-plane strains are cubic ($\sim z^3$) and the shear strains are quadratic. This allows the line elements normal to the mid-surface not only to rotate, but also to deform and not necessarily remain straight.

6.10.2 Large Deflections

Consider now the assumption that the membrane stresses may be neglected. To investigate the validity of this, consider an initially circular plate of diameter *d*, clamped at the edges, and deformed into a spherically shaped surface, Fig. 6.10.1.

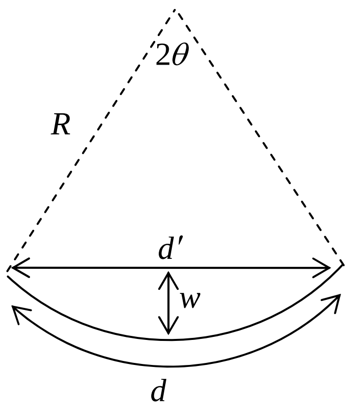


Figure 6.10.1: a deformed circular plate

Considering a beam, the length of the neutral axis before and after deformation can safely be taken to be equal, even when the beam deforms as in Fig. 6.10.1, i.e. one can take $d = d'$. The reason for this is that the "supports" are assumed to move slightly to accommodate any small deflection; thus the neutral axis of a beam remains strain-free and hence stress-free.

Consider next the plate. Suppose that the supports could move slightly to accommodate the deformation of the plate so that the curved length in Fig. 6.10.1 was equal in length to the original diameter. One then sees that a compressive circumferential strain is set up in the plate mid-surface, of magnitude $(d-d')/d$. To quantify this, note that $\theta = d/2R$ and $\sin\theta = d'/2R$. Thus $d'/2R = d/2R - (d/2R)^3/3! + (d/2R)^5/5! - \cdots$. Then

$$\varepsilon_{\theta\theta}^0 = \frac{1}{24}\left(\frac{d}{R}\right)^2 - \frac{1}{1920}\left(\frac{d}{R}\right)^2 + \cdots \tag{6.10.7}$$

With $\cos\theta = 1 - w/R$ and $\cos\theta = 1 - \theta^2/2! + \cdots$, one also has

$$R = \frac{1}{8}\frac{d^2}{w} - \frac{1}{384}\frac{1}{w}\frac{d^4}{R^2} + \cdots \tag{6.10.8}$$

so that, approximately,

$$\varepsilon_{\theta\theta}^0 = \frac{8w^2}{3d^2} \tag{6.10.9}$$

The maximum bending strain occurs at $z = h/2$, where $\varepsilon_{rr} = (h./2)(1/R)$, so

$$\varepsilon_{rr} = \frac{4hw}{d^2} \tag{6.10.10}$$

One can conclude from this rough analysis that, in order that the membrane strains can be safely ignored, the deflection *w* must be small when compared to the thickness *h* of the plate[1]. The corollary of this is that when there are large deflections, the middle surface will strain and take up the load as in a stretching membrane. For the bending of circular plates, one usually requires that $w < 0.5h$ in order that the membrane strains can be safely ignored without introducing considerable error. For example, a uniformly loaded clamped plate deflected to $w = h$ experiences a maximum membrane stress of approximately 20% of the maximum bending stress.

When the deflections are large, the membrane strains need to be considered. This means that the von Kármán strains, Eqns. 6.2.22, 6.2.25, must be used in the analysis. Further, the in-plane forces, for example in the bending Eqn. 6.7.6, are now an unknown of the problem. Some approximate solutions of the resulting equations have been worked out, for example for uniformly loaded circular and rectangular plates.

[1] except in some special cases, for example when a plate deforms into the surface of a cylinder