6.10 Limitations of Classical Plate Theory

The validity of the classical plate theory depends on a number of factors:
1. the curvatures are small
2. the in-plane plate dimensions are large compared to the thickness
3. membrane strains can be neglected

The second and third of these points are discussed briefly in what follows.

6.10.1 Moderately Thick Plates

As with beam theory, and as mentioned already, it turns out that the solutions based on the classical theory agree well with the full elasticity solutions (away from the edges of the plate), provided the plate thickness is small relative to its other linear dimensions. When the plate is relatively thick, one is advised to use a more exact theory, for example one of the shear deformation theories:

Shear deformation Theories

The **Mindlin plate theory (**or **moderately thick plate theory** or **shear deformation theory**) was developed in the mid-1900s to allow for possible transverse shear strains. In this theory, there is the added complication that vertical line elements before deformation do *not* have to remain perpendicular to the mid-surface after deformation, although they do remain straight. Thus shear strains ε_{yz} and ε_{zx} are generated, constant through the thickness of the plate.

The classical plate theory is inconsistent in the sense that elements are assumed to remain perpendicular to the mid-plane, yet equilibrium requires that stress components σ_{xz}, σ_{yz} still arise (which would cause these elements to deform). The theory of thick plates is more consistent, but it still makes the assumption that $\sigma_{zz} = 0$. Note that *both* are approximations of the exact three-dimensional equations of elasticity.

As an indication of the error involved in using the classical plate theory, consider the problem of a simply supported square plate subjected to a uniform pressure. According to Eqn. 6.5.52, the central deflection is $w/(qa^4/1000D) = 4.062$. The shear deformation theory predicts 4.060 (for $a/h = 100$), 4.070 (for $a/h = 50$), 4.111 (for $a/h = 20$) and 4.259 (for $a/h = 10$). This trend holds in general; the classical theory is good for thin plates but under-predicts deflections (and over-predicts buckling loads and natural frequencies) in relatively thick plates.

An important difference between the thin plate and thick plate theories is that in the former the moments are related to the curvatures through (using M_x for illustration)

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \qquad (6.10.1)$$

This is only an approximate relation (although it turns out to be exact in the case of pure bending). The thick plate theory predicts that, in the case of a uniform lateral load q, the relationship is given by

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + h^2 q \frac{8+\nu+\nu^2}{20(1-\nu)} \tag{6.10.2}$$

The thin plate expression will be approximately equal to the thick plate expression when the thickness h is very small, since in that case $h^2 \to 0$, or when the ratio of load to stiffness, q/D, is small.

Some solutions for circular plates using the various theories are presented next for comparison.

Comparison of Solutions for Circular Plates

1. Uniform load q, clamped:

Both thin and thick plate theories give

$$\omega = -\frac{q}{64D}\left(a^2 - r^2\right)^2 \tag{6.10.3}$$

2. Uniform load q, simply supported:

$$\omega = -\frac{q}{64D}\left(a^2 - r^2\right)\left(\frac{5+\nu}{1+\nu}a^2 - r^2 + \eta\right), \qquad \eta = \begin{cases} 0 & \text{thin} \\ +\frac{8}{5}\frac{8+\nu+\nu^2}{1-\nu^2}h^2 & \text{thick} \end{cases} \tag{6.10.4}$$

3. Concentrated central load, clamped (Eqn. 6.6.25):

$$\omega = -\frac{F}{16\pi D}\left(\left(a^2 - r^2\right) + 2r^2 \ln(r/a) + \eta\right)$$
$$\eta = \begin{cases} 0 & \text{thin} \\ 0 & \text{thick} \\ -\frac{4}{5}\frac{2-\nu}{1-\nu}h^2 \ln(r/a) & \text{exact} \end{cases} \tag{6.10.5}$$

4. Concentrated central load Q, simply supported:

$$\omega = -\frac{F}{16\pi D}\left(\frac{3+\nu}{1+\nu}\left(a^2 - r^2\right) + 2r^2 \ln(r/a) + \eta\right)$$

(6.10.6)

$$\eta = \begin{cases} 0 & \text{thin} \\ +\frac{2}{5}\frac{8+\nu}{1+\nu}\frac{h^2}{a^2}\left(a^2 - r^2\right) & \text{thick} \\ -\frac{4}{5}\frac{2-\nu}{1-\nu}h^2 \ln(r/a) - \frac{1}{5}\frac{2\nu}{1+\nu}\frac{h^2}{a^2}\left(a^2 - r^2\right) & \text{exact} \end{cases}$$

Higher-order deformation Theories

A further theory known as **third-order plate theory** has also been developed. This allows for the displacements to vary not only linearly (the previously described Mindlin theory is also called the **first order shear deformation theory**), but as cubic functions; the in-plane strains are cubic ($\sim z^3$) and the shear strains are quadratic. This allows the line elements normal to the mid-surface not only to rotate, but also to deform and not necessarily remain straight.

6.10.2 Large Deflections

Consider now the assumption that the membrane stresses may be neglected. To investigate the validity of this, consider an initially circular plate of diameter *d*, clamped at the edges, and deformed into a spherically shaped surface, Fig. 6.10.1.

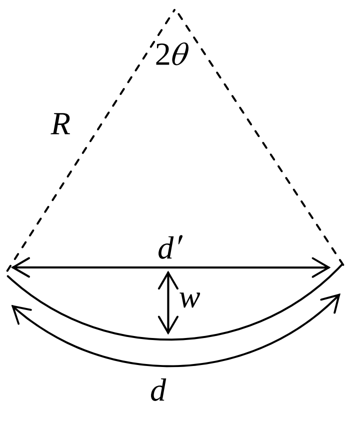

Figure 6.10.1: a deformed circular plate

Considering a beam, the length of the neutral axis before and after deformation can safely be taken to be equal, even when the beam deforms as in Fig. 6.10.1, i.e. one can take $d = d'$. The reason for this is that the "supports" are assumed to move slightly to accommodate any small deflection; thus the neutral axis of a beam remains strain-free and hence stress-free.

Consider next the plate. Suppose that the supports could move slightly to accommodate the deformation of the plate so that the curved length in Fig. 6.10.1 was equal in length to the original diameter. One then sees that a compressive circumferential strain is set up in the plate mid-surface, of magnitude $(d - d')/d$. To quantify this, note that $\theta = d/2R$ and $\sin\theta = d'/2R$. Thus $d'/2R = d/2R - (d/2R)^3/3! + (d/2R)^5/5! - \cdots$. Then

$$\varepsilon_{\theta\theta}^0 = \frac{1}{24}\left(\frac{d}{R}\right)^2 - \frac{1}{1920}\left(\frac{d}{R}\right)^2 + \cdots \tag{6.10.7}$$

With $\cos\theta = 1 - w/R$ and $\cos\theta = 1 - \theta^2/2! + \cdots$, one also has

$$R = \frac{1}{8}\frac{d^2}{w} - \frac{1}{384}\frac{1}{w}\frac{d^4}{R^2} + \cdots \tag{6.10.8}$$

so that, approximately,

$$\varepsilon_{\theta\theta}^0 = \frac{8w^2}{3d^2} \tag{6.10.9}$$

The maximum bending strain occurs at $z = h/2$, where $\varepsilon_{rr} = (h./2)(1/R)$, so

$$\varepsilon_{rr} = \frac{4hw}{d^2} \tag{6.10.10}$$

One can conclude from this rough analysis that, in order that the membrane strains can be safely ignored, the deflection *w* must be small when compared to the thickness *h* of the plate[1]. The corollary of this is that when there are large deflections, the middle surface will strain and take up the load as in a stretching membrane. For the bending of circular plates, one usually requires that $w < 0.5h$ in order that the membrane strains can be safely ignored without introducing considerable error. For example, a uniformly loaded clamped plate deflected to $w = h$ experiences a maximum membrane stress of approximately 20% of the maximum bending stress.

When the deflections are large, the membrane strains need to be considered. This means that the von Kármán strains, Eqns. 6.2.22, 6.2.25, must be used in the analysis. Further, the in-plane forces, for example in the bending Eqn. 6.7.6, are now an unknown of the problem. Some approximate solutions of the resulting equations have been worked out, for example for uniformly loaded circular and rectangular plates.

[1] except in some special cases, for example when a plate deforms into the surface of a cylinder