6.7 In-Plane Forces and Plate Buckling

In the previous sections, only bending and twisting moments and out-of-plane shear forces were considered. In this section, in-plane forces are considered also. The in-plane forces will give rise to in-plane membrane strains, but here it is assumed that these are uncoupled from the bending strains. In other words, the membrane strains can be found from a separate plane stress analysis of the mid-surface and the bending of the plate does not affect these membrane strains. The possible effect of the in-plane forces on the bending strains is the main concern here.

6.7.1 Equilibrium for In-plane Forces

Start again with the equations of equilibrium, Eqns. 6.4.6. Integrating the first and second through the thickness of the plate (this time without multiplying first by z), and using the definitions of the in-plane forces 6.1.1-6.1.2, leads to

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \end{aligned} \tag{6.7.1}$$

6.7.2 The Governing Differential Equation

Consider an element of the deflected plate, Fig. 6.7.1. Only a deflection in the y direction, $\partial\omega / \partial y$, is considered for clarity. Resolving the components of the in-plane forces into horizontal and vertical components:

$$\begin{aligned} \sum F_H &= -N_y \Delta x + \left(N_y + \frac{\partial N_y}{\partial y} \Delta y \right) \Delta x - N_{xy} \Delta y + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} \Delta x \right) \Delta y \\ \sum F_V &= -N_y \frac{\partial w}{\partial y} \Delta x + \left(N_y \frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y} \right) \Delta x \right) \Delta x \\ &\quad - N_{xy} \frac{\partial w}{\partial y} \Delta y + \left(N_{xy} \frac{\partial w}{\partial y} + \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y} \right) \Delta x \right) \Delta y \end{aligned} \tag{6.7.2}$$

These reduce to

$$\sum F_H = \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \Delta y \Delta x$$

$$\sum F_V = \left[\frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w}{\partial y} \right) \right] \Delta x \Delta y \qquad (6.7.3)$$

$$= \left[\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \frac{\partial w}{\partial y} + \left(N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \right] \Delta x \Delta y$$

Using 6.7.1, one has

$$\sum F_H = 0, \qquad \sum F_V = \left(N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \Delta x \Delta y \qquad (6.7.4)$$

Considering also a deflection $\partial \omega / \partial x$, one has for the resultant vertical force :

$$\sum F_V = \left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \Delta x \Delta y \qquad (6.7.5)$$

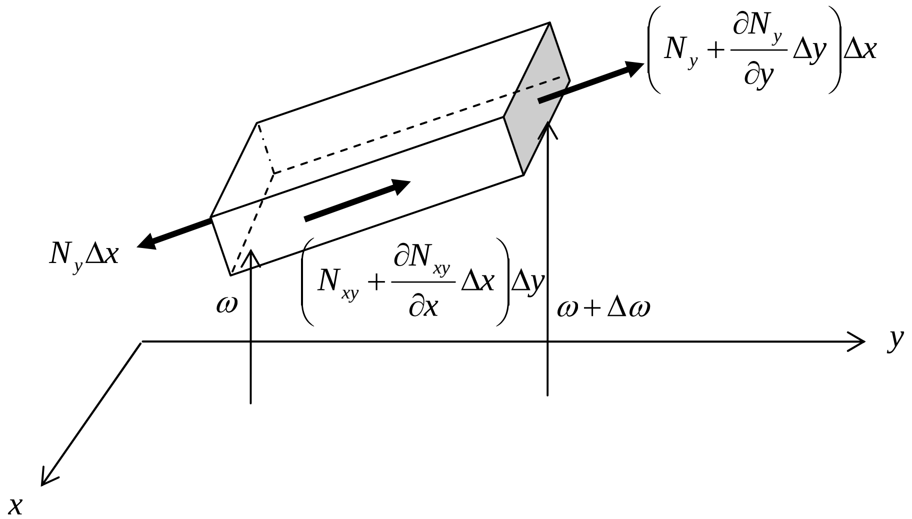

Figure 6.7.1: In-plane forces acting on a plate element

When the in-plane forces were neglected, the vertical stress resisted by bending and shear force was $\sigma_{zz} = -q$. Here, one has an additional stress given by 6.7.5, and so the governing differential equation 6.4.7 becomes

$$\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{1}{D} \left(-q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \qquad (6.7.6)$$

6.7.3 Buckling of Plates

When compressive in-plane forces are applied to a plate, the plate will at first remain flat and simply be compressed. However, when the in-plane forces reach a critical level, the plate will bend and the deflection will be given by the solution to 6.7.6. For example, consider the case of a simply supported plate subjected to a uniform in-plane compression N_x only, Fig. 6.7.2, in which case 6.7.6 reduces to

$$\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} \tag{6.7.7}$$

Following Navier's method from §6.5.5, assume a buckled shape

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.7.8}$$

so that 6.7.7 becomes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{N_x}{D}\frac{m^2\pi^2}{a^2}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0 \tag{6.7.9}$$

Disregarding the trivial $A_{mn} = 0$, this can be satisfied by taking

$$N_x = -\frac{Da^2\pi^2}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \tag{6.7.10}$$

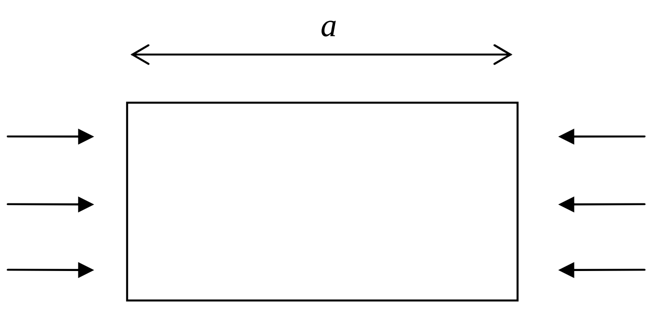

Figure 6.7.2: In-plane compression of a plate

The lowest in-plane force N_x which will deflect the plate is sought. Clearly, the smallest value on the right hand side of 6.7.10 will be when $n = 1$. This means that the buckling modes as given by 6.7.8 will be of the form

$$\sin\frac{m\pi x}{a}\sin\frac{\pi y}{b} \tag{6.7.11}$$

so that the plate will only ever buckle with one half-wave in the direction perpendicular to loading.

When $a \le b$, the smallest value occurs when $m = 1$, in which case the critical in-plane force is

$$\left(N_x\right)_{\mathrm{cr}} = -\frac{D\pi^2}{b^2}\left(\frac{b}{a} + \frac{a}{b}\right)^2 \tag{6.7.12}$$

When a / b is very small, the plate is loaded along the relatively long edges and the critical load is much higher than for a square plate.

The deflection (buckling mode) corresponding to this critical load is

$$w(x, y) = A_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{6.7.13}$$

Note that the amplitude A_{11} cannot be determined from the analysis[1].

As a / b increases above unity, the value of *m* at which the applied load is a minimum increases. When a / b reaches just over $\sqrt{2}$, the critical buckling load occurs for $m = 2$, for which

$$\left(N_x\right)_{cr} = -\frac{D\pi^2}{b^2}\left(\frac{2b}{a} + \frac{a}{2b}\right)^2 \tag{6.7.14}$$

and corresponding buckling more

$$w(x, y) = A_{21} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \tag{6.7.15}$$

The plate now buckles in two half-waves, as if the centre-line were simply supported and there were two smaller separate plates buckling similarly.

As a / b increases further, so too does *m*. For a very long, thin, plate, $m \approx a / b$, and so the plate subdivides approximately into squares, each buckling in a half-wave.

[1] this is a consequence of assuming small deflections; it can be determined when the deflections are not assumed to be small