6.4 Equilibrium and Lateral Loading

In this section, lateral loads are considered and these lead to shearing forces V_x, V_y, in the plate.

6.4.1 The Governing Differential Equation for Lateral Loads

In general, a plate will at any location be subjected to a lateral *pressure* q, bending moments M_x, M_y, M_{xy} and out-of-plane shear forces V_x and V_y; q is the normal pressure on the upper surface of the plate:

$$\sigma_{zz}(x,y) = \begin{cases} 0, & z = -h/2 \\ -q(x,y), & z = +h/2 \end{cases} \tag{6.4.1}$$

These quantities are related to each other through force equilibrium.

Force Equilibrium

Consider a differential plate element with one corner at $(x,y) = (0,0)$, Fig. 6.4.1, subjected to moments, pressure and shear force. Taking force equilibrium in the vertical direction (neglecting a possible small variation in *q*, since this will only introduce higher order terms):

$$\sum F\big|_z = +V_y \Delta x - \left(V_y + \frac{\partial V_y}{\partial y}\Delta y\right)\Delta x + V_x \Delta y - \left(V_x + \frac{\partial V_x}{\partial x}\Delta x\right)\Delta y - q\Delta x \Delta y = 0 \tag{6.4.2}$$

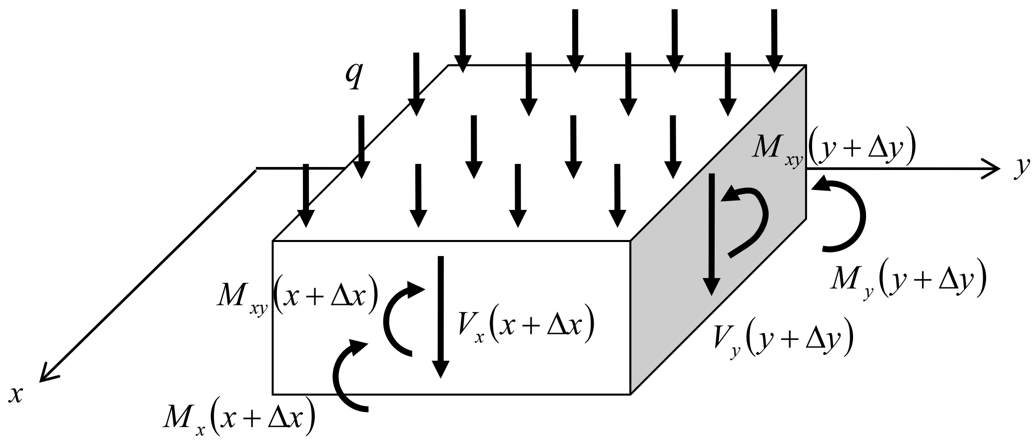

Fig. 6.4.1: a plate element subjected to moments, pressure and shear forces

Eqn. 6.4.2 gives the vertical equilibrium equation

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -q \tag{6.4.3}$$

This is analogous to the beam theory equation $p = dV / dx$ (see Book I, §7.4.3).

Next, taking moments about the x axis:

$$\begin{aligned}\sum M\big|_x &= M_{xy}\Delta y - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\Delta x\right)\Delta y - M_y \Delta x + \left(M_y + \frac{\partial M_y}{\partial y}\Delta y\right)\Delta x \\ &\quad + yV_y\Delta x - (y+\Delta y)\left(V_y + \frac{\partial V_y}{\partial y}\Delta y\right)\Delta x + (y + \Delta y/2)V_x\Delta y \\ &\quad - (y+\Delta y/2)\left(V_x + \frac{\partial V_x}{\partial x}\Delta x\right)\Delta y - (y+\Delta y/2)q\Delta x\Delta y = 0\end{aligned} \tag{6.4.4}$$

Using 6.4.3, this reduces to (and similarly for moments about the y-axis, the first eqn. here),

$$\begin{aligned} V_x &= +\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} \\ V_y &= -\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \end{aligned} \tag{6.4.5}$$

These are analogous to the beam theory equation $V = dM / dx$ (see Book I, §7.4.3).

Relations directly from the Equations of Equilibrium

The equilibrium relations 6.4.3, 6.4.5 can also be derived directly from the equations of equilibrium, Eqns. 1.1.9, which encompass the force balances:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \tag{6.4.6}$$

Taking the first of these (which ensures equilibrium of forces in the x direction), multiplying by z and integrating over the plate thickness, gives

$$\begin{aligned} &\int_{-h/2}^{+h/2} z\frac{\partial \sigma_{xx}}{\partial x}dz + \int_{-h/2}^{+h/2} z\frac{\partial \sigma_{yx}}{\partial y}dz + \int_{-h/2}^{+h/2} z\frac{\partial \sigma_{zx}}{\partial z}dz = 0 \\ \to\ &\frac{\partial}{\partial x}\left[\int_{-h/2}^{+h/2} z\sigma_{xx}dz\right] + \frac{\partial}{\partial y}\left[\int_{-h/2}^{+h/2} z\sigma_{yx}dz\right] + \left[z\sigma_{zx}\right]_{-h/2}^{+h/2} - \int_{-h/2}^{+h/2}\sigma_{zx}dz = 0 \end{aligned} \tag{6.4.7}$$

and, since the shear stress σ_{zx} must be zero over the top and bottom surfaces, one has Eqn. 6.4.5a. Applying a similar procedure to the second equilibrium equation gives Eqn. 6.4.5b. Finally, integrating directly the third equilibrium equation without multiplying across by z, one arrives at Eqn. 6.4.3.

Now, eliminating the shear forces from 6.4.3, 6.4.5 leads to the differential equation

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{6.4.8}$$

This equation is analogous to the equation $\partial^2 M / \partial x^2 = p$ in the beam theory. Finally, substituting in the moment-curvature equations 6.2.31 leads to[1]

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{q}{D}} \tag{6.4.9}$$

This is sometimes called **the equation of Sophie Germain** after the French investigator who first obtained it in 1815[2]. This partial differential equation is solved subject to the boundary conditions of the problem, i.e. the fixing conditions of the plate (see below). Again, when once an expression for $\omega(x, y)$ is obtained, the strains, stresses, forces and moments follow.

Note that the differential equation 6.4.8 with $q = 0$ is trivially satisfied in the simple pure bending and torsion problems considered earlier.

Eqn. 6.4.9 can be succinctly expressed as

$$\nabla^4 w = -\frac{q}{D} \tag{6.4.10}$$

where ∇^2 is the **Laplacian**, or "del" operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{6.4.11}$$

Note that the Laplacian operator (on w) gives the sum of the curvatures in two perpendicular directions and so it is independent of the directions chosen (see Eqn. 6.2.14).

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[1] note that the moment curvature relations were derived for the case of pure bending; here, as in the beam theory, the possible effect of the shearing forces on the curvature is neglected. This is a valid assumption provided the thickness of the plate is small in comparison with its other dimensions. A more exact theory taking into account the effect of the shear forces on deflection can be developed

[2] Germain submitted her work to the French Academy, which was awarding a prize for anyone who could solve the problem of the vibration of plates; Lagrange was on the Academy awarding committee and corrected some of her work, deriving Eqn. 6.4.9 in its final form

Shear Forces in terms of Deflection

From 6.4.5 and the moment-curvature equations, one has the useful relations

$$V_x = D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \qquad V_y = D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{6.4.12}$$

6.4.2 Stresses in the Plate

The normal and in-plane shear stresses have been expressed in terms of the moments, Eqns. 6.2.33. Note that these stresses are zero over the mid-surface and attain a maximum at the outer surfaces.

Expressions for the remaining stress components can be obtained from the equations of equilibrium as follows: the first of Eqns. 6.4.6 leads, with 6.4.5a, to

$$\begin{aligned} 0 &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ &= \frac{\partial}{\partial x}\left[-\frac{12z}{h^3}M_x\right] + \frac{\partial}{\partial y}\left[\frac{12z}{h^3}M_{xy}\right] + \frac{\partial \sigma_{zx}}{\partial z} \\ &= -\frac{12z}{h^3}\left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial \sigma_{zx}}{\partial z} \\ &= -\frac{12z}{h^3}V_x + \frac{\partial \sigma_{zx}}{\partial z} \end{aligned} \tag{6.4.13}$$

Integrating now gives (note that V_x is independent of z)

$$\sigma_{zx} = \frac{6}{h^3}V_x z^2 + C \tag{6.4.14}$$

This shear stress must be zero at the upper and lower (free –) surfaces, at $z = \pm h/2$. This condition can be used to determine the arbitrary constant C and one finds that (see Fig. 6.1.9)

$$\sigma_{zx} = -\frac{3V_x}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{6.4.15}$$

The other shear stress, σ_{zy}, can be evaluated in a similar manner: {▲Problem 1}

$$\sigma_{zy} = -\frac{3V_y}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{6.4.16}$$

In some analyses, these shear stresses are taken to be zero, although they can be quite significant.

The only remaining stress component is σ_{zz}. This will never exceed the intensity of the external load on the plate; the lateral load itself, however, is negligibly small in comparison with the in-plane stresses set up by the bending of the plate, and for this reason it is acceptable to disregard σ_{zz}, as has been done, in the plate theory.

6.4.3 Problems

1. Derive the expression for shear stress 6.4.16.