6.3 Plates subjected to Pure Bending and Twisting

6.3.1 Pure Bending of an Elastic Plate

Consider a plate subjected to bending moments $M_x = M_1 > 0$ and $M_y = M_2 > 0$, with no other loading, as shown in Fig. 6.3.1.

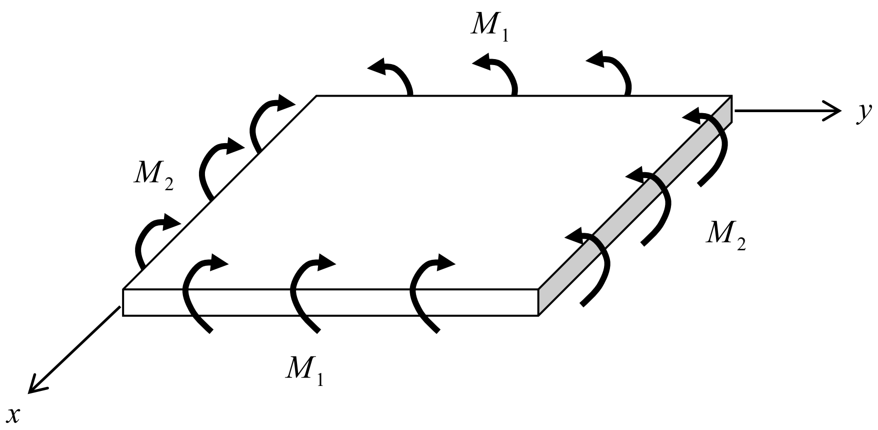

Figure 6.3.1: A plate under Pure Bending

From equilibrium considerations, these moments act at all points within the plate – they are constant throughout the plate. Thus, from the moment-curvature equations 6.2.31, one has the set of coupled partial differential equations

$$\frac{M_1}{D} = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad \frac{M_2}{D} = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad 0 = \frac{\partial^2 w}{\partial x \partial y} \tag{6.3.1}$$

Solving for the derivatives,

$$\frac{\partial^2 w}{\partial x^2} = \frac{M_1 - \nu M_2}{D(1-\nu^2)}, \quad \frac{\partial^2 w}{\partial y^2} = \frac{M_2 - \nu M_1}{D(1-\nu^2)}, \quad \frac{\partial^2 w}{\partial x \partial y} = 0 \tag{6.3.2}$$

Integrating the first two equations twice gives[1]

$$w = \frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)} x^2 + f_1(y)x + f_2(y), \quad w = \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)} y^2 + g_1(x)y + g_2(x) \tag{6.3.3}$$

and integrating the third shows that two of these four unknown functions are constants:

$$\frac{\partial w}{\partial x} = G(x), \quad \frac{\partial w}{\partial y} = F(y) \qquad \rightarrow \qquad f_1(y) = A, \;\; g_1(x) = B \tag{6.3.4}$$

[1] this analysis is similar to that used to evaluate displacements in plane elastostatic problems, §1.2.4

Equating both expressions for w in 6.3.3 gives

$$\frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)}x^2 + Ax - g_2(x) = \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)}y^2 + By - f_2(y) \tag{6.3.5}$$

For this to hold, both sides here must be a constant, $-C$ say. It follows that

$$w = \frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)}x^2 + \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)}y^2 + Ax + By + C \tag{6.3.6}$$

The three unknown constants represent an arbitrary rigid body motion. To obtain values for these, one must fix three degrees of freedom in the plate. If one supposes that the deflection w and slopes $\partial w/\partial x, \partial w/\partial y$ are zero at the origin $x = y = 0$ (so the origin of the axes are at the plate-centre), then $A = B = C = 0$; all deformation will be measured relative to this reference. It follows that

$$w = \frac{M_2}{2D(1-\nu^2)}\left[\left[(M_1/M_2) - \nu\right]x^2 + \left[1 - \nu(M_1/M_2)\right]y^2\right] \tag{6.3.7}$$

Once the deflection w is known, all other quantities in the plate can be evaluated – the strain from 6.2.27, the stress from Hooke's law or directly from 6.2.30, and moments and forces from 6.1.1-3.

In the special case of equal bending moments, with $M_1 = M_2 = M_o$ say, one has

$$w = \frac{M_o}{2D(1+\nu)}\left(x^2 + y^2\right) \tag{6.3.8}$$

This is the equation of a sphere. In fact, from the relationship between the curvatures and the radius of curvature R,

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{M_o}{D(1+\nu)} \quad \rightarrow \quad R = \frac{D(1+\nu)}{M_o} = \text{constant} \tag{6.3.9}$$

and so the mid-surface of the plate in this case deforms into the surface of a sphere with radius given by 6.3.9, as illustrated in Fig. 6.3.2.

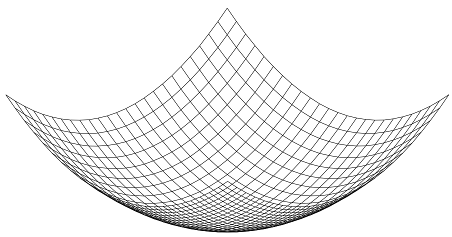

Figure 6.3.2: Deformed plate under Pure Bending with equal moments

The character of the deformed plate is plotted in Fig. 6.3.3 for various ratios M_2 / M_1 (for $\nu = 0.3$).

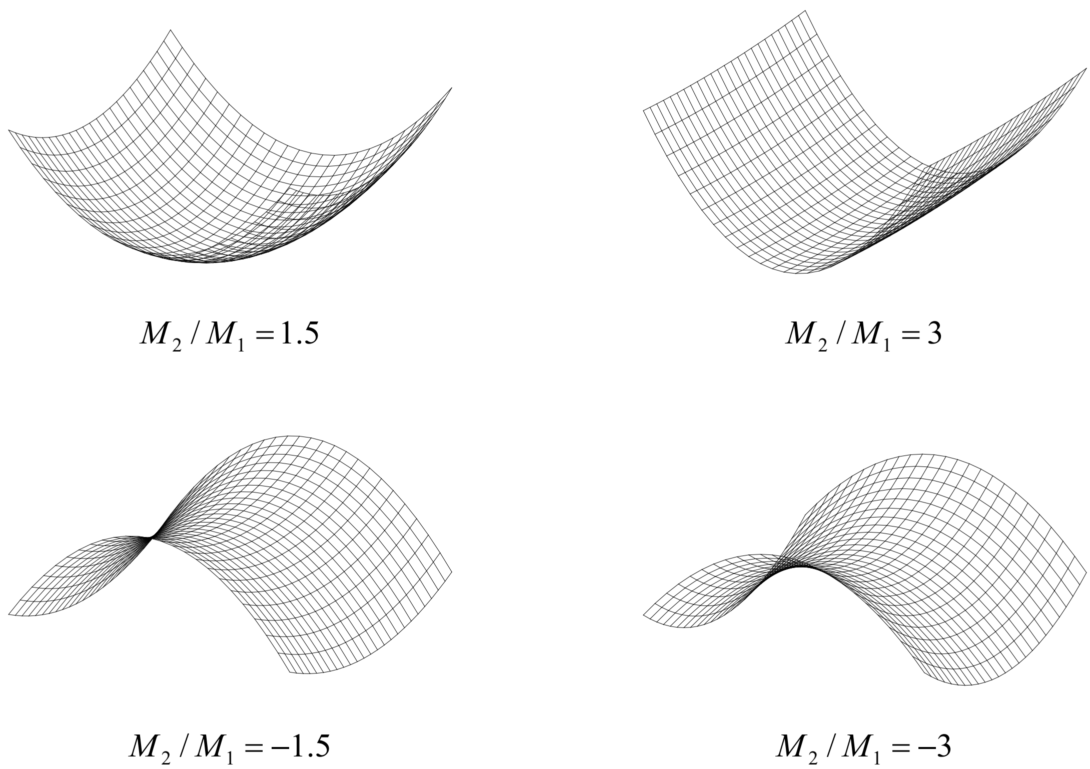

Figure 6.3.3: Bending of a Plate

When the curvatures $\partial^2 w / \partial x^2$ and $\partial^2 w / \partial y^2$ are of the same sign[2], the deformation is called **synclastic**. When the curvatures are of opposite sign, as in the lower plots of Fig. 6.3.3, the deformation is said to be **anticlastic**.

Note that when there is only one moment, $M_y = 0$ say, there is still curvature in both directions. In this case, one can solve the moment-curvature equations to get

$$\frac{\partial^2 w}{\partial x^2} = \frac{M_x}{\left(1-\nu^2\right)D}, \quad \frac{\partial^2 w}{\partial y^2} = -\nu \frac{\partial^2 w}{\partial x^2}, \qquad w = \frac{M_x}{2\left(1-\nu^2\right)D}\left(x^2 - \nu y^2\right) \tag{6.3.10}$$

which is an anticlastic deformation.

In order to get a pure cylindrical deformation, $w = f(x)$ say, one needs to apply moments M_x and $M_y = \nu M_x$, in which case, from 6.3.6,

[2] or principal curvatures in the case of a more complex general loading

$$w = \frac{M_x}{2D} x^2 \tag{6.3.11}$$

The deformation for $M_2 / M_1 = 3$ in Fig. 6.3.3 is very close to cylindrical, since there $M_x \approx \nu M_y$ for typical values of ν.

6.3.2 Pure Torsion of an Elastic Plate

In pure torsion, one has the twisting moment $M_{xy} = M > 0$ with no other loading, Fig. 6.3.4. From the moment-curvature equations,

$$0 = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad 0 = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad -\frac{M}{D(1-\nu)} = \frac{\partial^2 w}{\partial x \partial y} \tag{6.3.12}$$

so that

$$\frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad \frac{\partial^2 w}{\partial x \partial y} = -\frac{M}{D(1-\nu)} \tag{6.3.13}$$

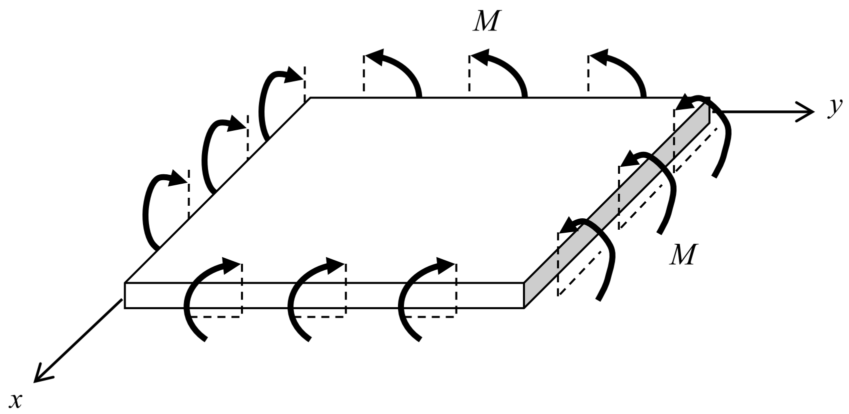

Figure 6.3.4: Twisting of a Plate

Using the same arguments as before, integrating these equations leads to

$$w = -\frac{M}{D(1-\nu)} xy \tag{6.3.14}$$

The middle surface is deformed as shown in Fig. 6.3.5, for a negative M_{xy}. Note that there is no deflection along the lines $x = 0$ or $y = 0$.

The principal curvatures will occur at 45° to the axes (see Eqns. 6.2.13):

$$\frac{1}{R_1} = +\frac{M}{D(1-\nu)}, \qquad \frac{1}{R_2} = -\frac{M}{D(1-\nu)} \tag{6.3.15}$$

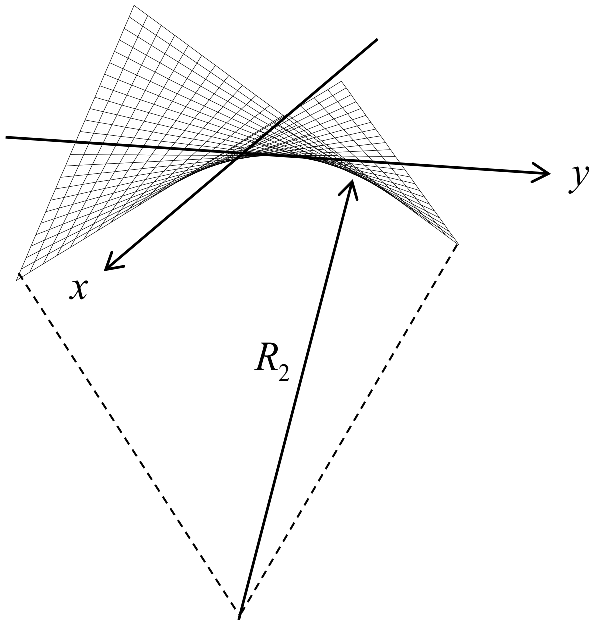

Figure 6.3.5: Deformation for a (negative) twisting moment