6.2 The Moment-Curvature Equations

6.2.1 From Beam Theory to Plate Theory

In the beam theory, based on the assumptions of plane sections remaining plane and that one can neglect the transverse strain, the strain varies linearly through the thickness. In the notation of the beam, with y positive up, $\varepsilon_{xx} = -y/R$, where R is the **radius of curvature**, R positive when the beam bends "up" (see Book I, Eqn. 7.4.16). In terms of the **curvature** $\partial^2 v/\partial x^2 = 1/R$, where v is the deflection (see Book I, Eqn. 7.4.36), one has

$$\varepsilon_{xx} = -y\frac{\partial^2 v}{\partial x^2} \qquad (6.2.1)$$

The beam theory assumptions are essentially the same for the plate, leading to strains which are proportional to distance from the neutral (mid-plane) surface, z, and expressions similar to 6.2.1. This leads again to linearly varying stresses σ_{xx} and σ_{yy} (σ_{zz} is also taken to be zero, as in the beam theory).

6.2.2 Curvature and Twist

The plate is initially undeformed and flat with the mid-surface lying in the $x-y$ plane. When deformed, the mid-surface occupies the surface $w = w(x, y)$ and w is the elevation above the $x-y$ plane, Fig. 6.2.1.

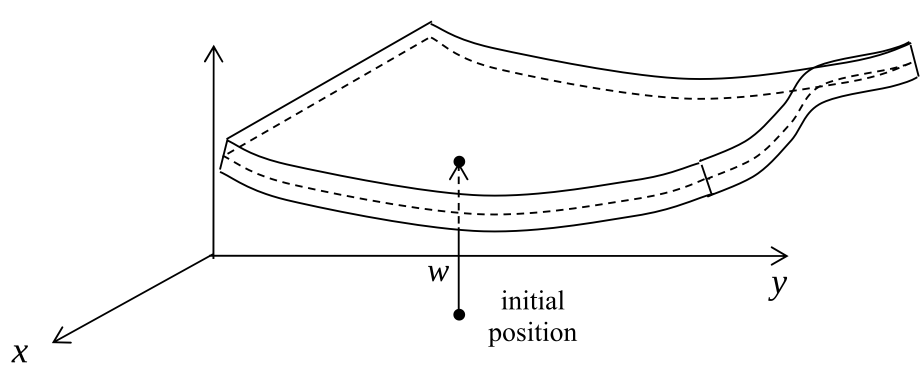

Fig. 6.2.1: Deformed Plate

The slopes of the plate along the x and y directions are $\partial w/\partial x$ and $\partial w/\partial y$.

Curvature

Recall from Book I, §7.4.11, that the curvature in the *x* direction, κ_x, is the rate of change of the slope angle ψ with respect to arc length *s*, Fig. 6.2.2, $\kappa_x = d\psi / ds$. One finds that

$$\kappa_x = \frac{\partial^2 w / \partial x^2}{\left[1 + \left(\partial w / \partial x\right)^2\right]^{3/2}} \tag{6.2.2}$$

Also, the radius of curvature R_x, Fig. 6.2.2, is the reciprocal of the curvature, $R_x = 1/\kappa_x$.

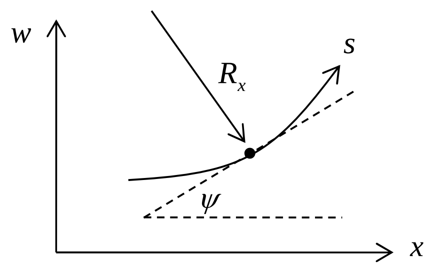

Fig. 6.2.2: Angle and arc-length used in the definition of curvature

As with the beam, *when the slope is small*, one can take $\psi \approx \tan\psi = \partial w / \partial x$ and $d\psi / ds \approx \partial \psi / \partial x$ and Eqn. 6.2.2 reduces to (and similarly for the curvature in the *y* direction)

$$\kappa_x = \frac{1}{R_x} = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = \frac{1}{R_y} = \frac{\partial^2 w}{\partial y^2} \tag{6.2.3}$$

This important assumption of small slope, $\partial w / \partial x, \partial w / \partial y << 1$, means that the theory to be developed will be valid when the deflections are small compared to the overall dimensions of the plate.

The curvatures 6.2.3 can be interpreted as in Fig. 6.2.3, as the unit increase in slope along the *x* and *y* directions.

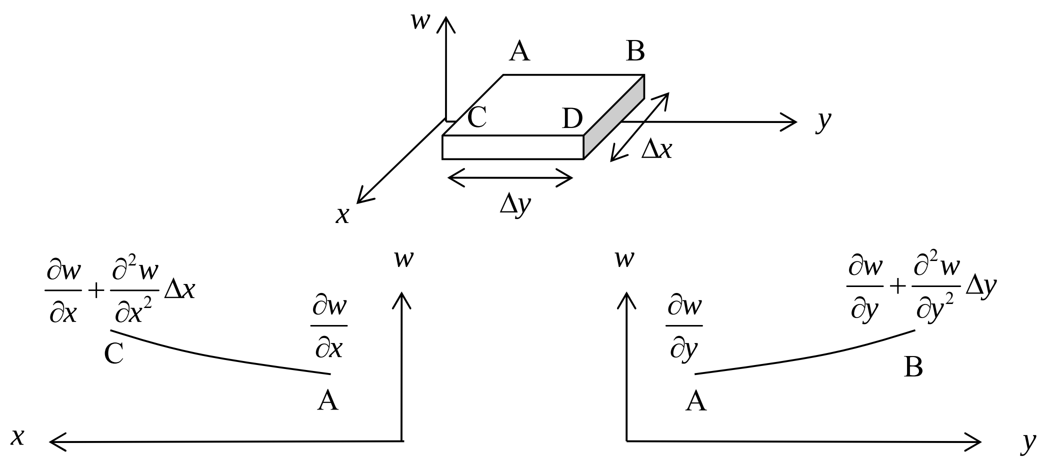

Figure 6.2.3: Physical meaning of the curvatures

Twist

Not only does a plate curve up or down, it can also twist (see Fig. 6.1.3). For example, shown in Fig. 6.2.4 is a plate undergoing a *pure* twisting (constant applied twisting moments and no bending moments).

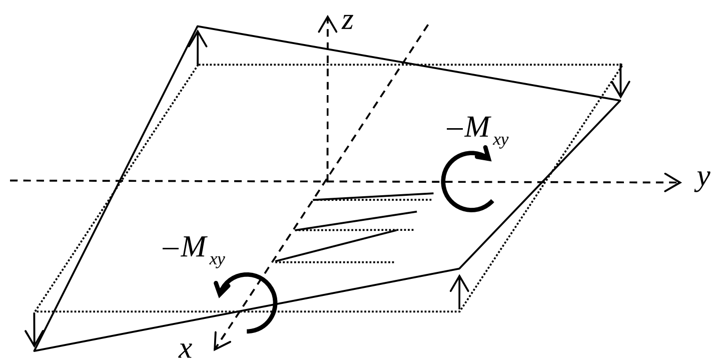

Figure 6.2.4: A twisting plate

If one takes a row of line elements lying in the *y* direction, emanating from the *x* axis, the further one moves along the *x* axis, the more they twist, Fig. 6.2.4. Some of these line elements are shown in Fig. 6.2.5 (bottom right), as veiwed looking down the *x* axis towards the origin (elements along the *y* axis are shown bottom left). If a line element at position *x* has slope $\partial w / \partial y$, the slope at $x + \Delta x$ is $\partial w / \partial y + \Delta x \partial (\partial w / \partial y) / \partial x$. This motivates the definition of the **twist**, defined analogously to the curvature, and denoted by $1/T_{xy}$; it is a measure of the "twistiness" of the plate:

$$\frac{1}{T_{xy}} = \frac{\partial^2 w}{\partial x \partial y} \tag{6.2.4}$$

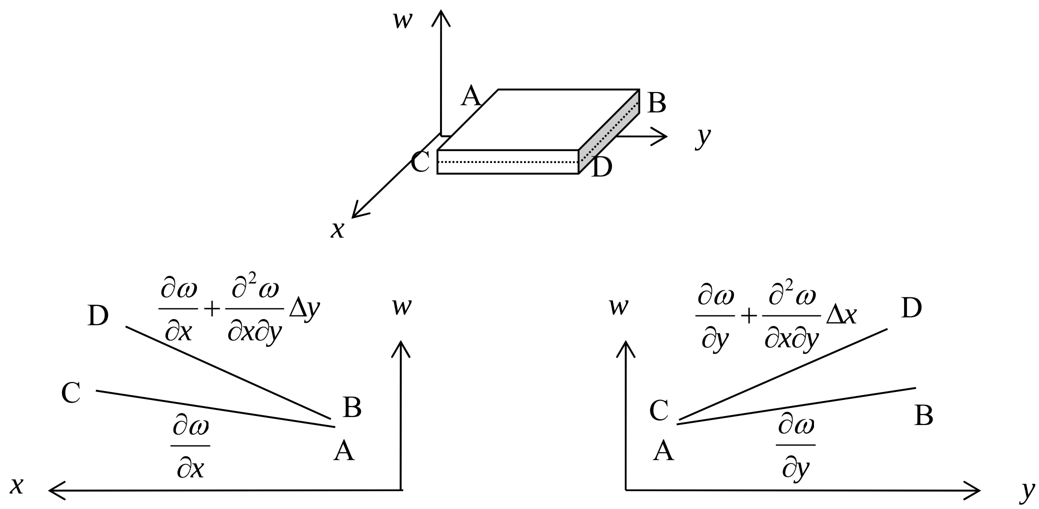

Figure 6.2.5: Physical meaning of the twist

The signs of the moments, radii of curvature and curvatures are illustrated in Fig. 6.2.6. Note that the deflection w may or may not be of the same sign as the curvature. Note also that when $M_x > 0$, $\partial^2 \omega / \partial x^2 > 0$, when $M_y > 0$, $\partial^2 \omega / \partial y^2 > 0$.

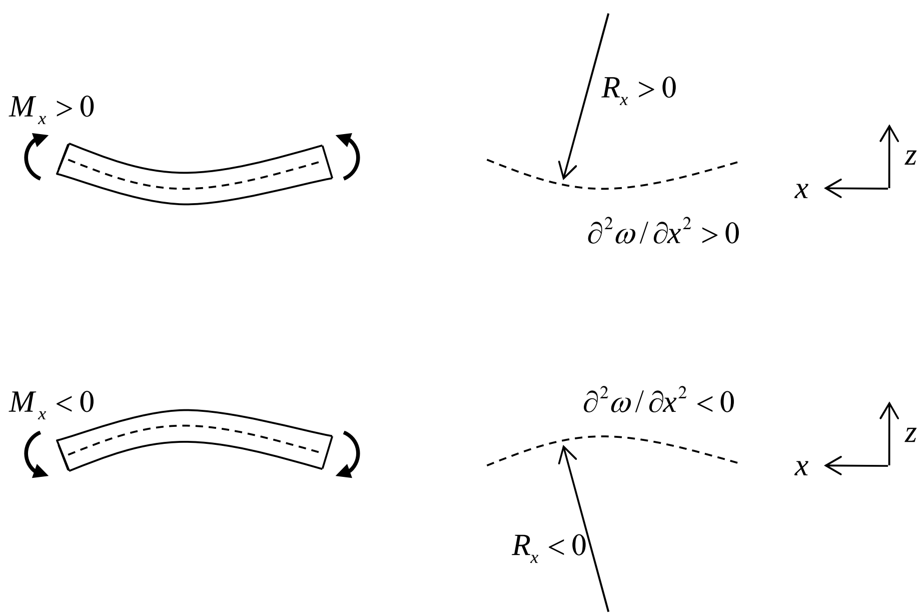

Figure 6.2.6: sign convention for curvatures and moments

On the other hand, for the twist, with the sign convention being used, when $M_{xy} > 0$, $\partial^2 \omega / \partial x \partial y < 0$, as depicted in Fig. 6.2.4.

Principal Curvatures

Consider the two Cartesian coordinate systems shown in Fig. 6.2.7, the second ($t - n$) obtained from the first ($x - y$) by a positive rotation θ. The partial derivatives arising in the curvature expressions can be expressed in terms of derivatives with respect to *t* and *n* as follows: with $w = w(x, y)$, an increment in *w* is

$$\Delta w = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y \tag{6.2.5}$$

Also, referring to Fig. 6.2.7, with $\Delta n = 0$,

$$\Delta x = \Delta t \cos\theta, \qquad \Delta y = \Delta t \sin\theta \tag{6.2.6}$$

Thus

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta \tag{6.2.7}$$

Similarly, for an increment Δn, one finds that

$$\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial x}\sin\theta + \frac{\partial w}{\partial y}\cos\theta \tag{6.2.8}$$

Equations 6.2.7-8 can be inverted to get the inverse relations

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial t}\cos\theta - \frac{\partial w}{\partial n}\sin\theta \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial t}\sin\theta + \frac{\partial w}{\partial n}\cos\theta \end{aligned} \tag{6.2.9}$$

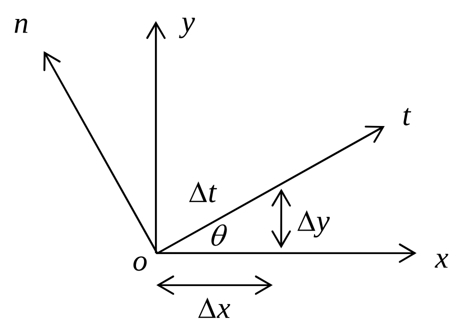

Figure 6.2.7: Two different Cartesian coordinate systems

The relationship between second derivatives can be found in the same way. For example,

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \left(\frac{\partial}{\partial t}\cos\theta - \frac{\partial}{\partial n}\sin\theta\right)\left(\frac{\partial w}{\partial t}\cos\theta - \frac{\partial w}{\partial n}\sin\theta\right) \\ &= \cos^2\theta\frac{\partial^2 w}{\partial t^2} + \sin^2\theta\frac{\partial^2 w}{\partial n^2} - \sin 2\theta\frac{\partial^2 w}{\partial t \partial n} \end{aligned} \tag{6.2.10}$$

In summary, one has

$$
\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= \cos^2\theta \frac{\partial^2 \omega}{\partial t^2} + \sin^2\theta \frac{\partial^2 \omega}{\partial n^2} - \sin 2\theta \frac{\partial^2 \omega}{\partial t \partial n} \\
\frac{\partial^2 w}{\partial y^2} &= \sin^2\theta \frac{\partial^2 \omega}{\partial t^2} + \cos^2\theta \frac{\partial^2 \omega}{\partial n^2} + \sin 2\theta \frac{\partial^2 \omega}{\partial t \partial n} \\
\frac{\partial^2 w}{\partial x \partial y} &= -\sin\theta\cos\theta \left(\frac{\partial^2 \omega}{\partial n^2} - \frac{\partial^2 \omega}{\partial t^2} \right) + \cos 2\theta \frac{\partial^2 \omega}{\partial t \partial n}
\end{aligned}
\tag{6.2.11}
$$

and the inverse relations

$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= \cos^2\theta \frac{\partial^2 \omega}{\partial x^2} + \sin^2\theta \frac{\partial^2 \omega}{\partial y^2} + \sin 2\theta \frac{\partial^2 \omega}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial n^2} &= \sin^2\theta \frac{\partial^2 \omega}{\partial x^2} + \cos^2\theta \frac{\partial^2 \omega}{\partial y^2} - \sin 2\theta \frac{\partial^2 \omega}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial t \partial n} &= \sin\theta\cos\theta \left(\frac{\partial^2 \omega}{\partial y^2} - \frac{\partial^2 \omega}{\partial x^2} \right) + \cos 2\theta \frac{\partial^2 \omega}{\partial x \partial y}
\end{aligned}
\tag{6.2.12}
$$

or[1]

$$
\begin{aligned}
\frac{1}{R_t} &= \cos^2\theta \frac{1}{R_x} + \sin^2\theta \frac{1}{R_y} + \sin 2\theta \frac{1}{T_{xy}} \\
\frac{1}{R_n} &= \sin^2\theta \frac{1}{R_x} + \cos^2\theta \frac{1}{R_y} - \sin 2\theta \frac{1}{T_{xy}} \\
\frac{1}{T_{tn}} &= \sin\theta\cos\theta \left(\frac{1}{R_y} - \frac{1}{R_x} \right) + \cos 2\theta \frac{1}{T_{xy}}
\end{aligned}
\tag{6.2.13}
$$

These equations which transform between curvatures in different coordinate systems have the same structure as the stress transformation equations (and the strain transformation equations), Book I, Eqns. 3.4.8. As with principal stresses/strains, there will be some angle θ for which the twist is zero; at this angle, one of the curvatures will be the minimum and one will be the maximum at that point in the plate. These are called the **principal curvatures**. Similarly, just as the sum of the normal stresses is an invariant (see Book I, Eqn. 3.5.1), the sum of the curvatures is an invariant[2]:

$$
\frac{1}{R_x} + \frac{1}{R_y} = \frac{1}{R_t} + \frac{1}{R_n} \tag{6.2.14}
$$

If the principal curvatures are equal, the curvatures are the same at all angles, the twist is always zero and so the plate deforms locally into the surface of a sphere.

[1] these equations are valid for any continuous surface; Eqns. 6.2.12 are restricted to nearly-flat surfaces.
[2] this is known as Euler's theorem for curvatures

6.2.3 Strains in a Plate

The strains arising in a plate are next examined. A comprehensive strain-state will be first examined and this will then be simpilfied down later to various approximate solutions. Consider a line element parallel to the x axis, of length Δx. Let the element displace as shown in Fig. 6.2.8. Whereas w was used in the previous section on curvatures to denote displacement of the mid-surface, here, for the moment, let $w(x, y, z)$ be the general vertical displacement of any particle in the plate. Let u and v be the corresponding displacements in the x and y directions. Denote the original and deformed length of the element by dS and ds respectively.

The unit change in length of the element (that is, the exact normal strain) is, using Pythagoras' theorem,

$$\varepsilon_{xx} = \frac{ds - dS}{dS} = \frac{|p'q'| - |pq|}{|pq|} = \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \qquad (6.2.15)$$

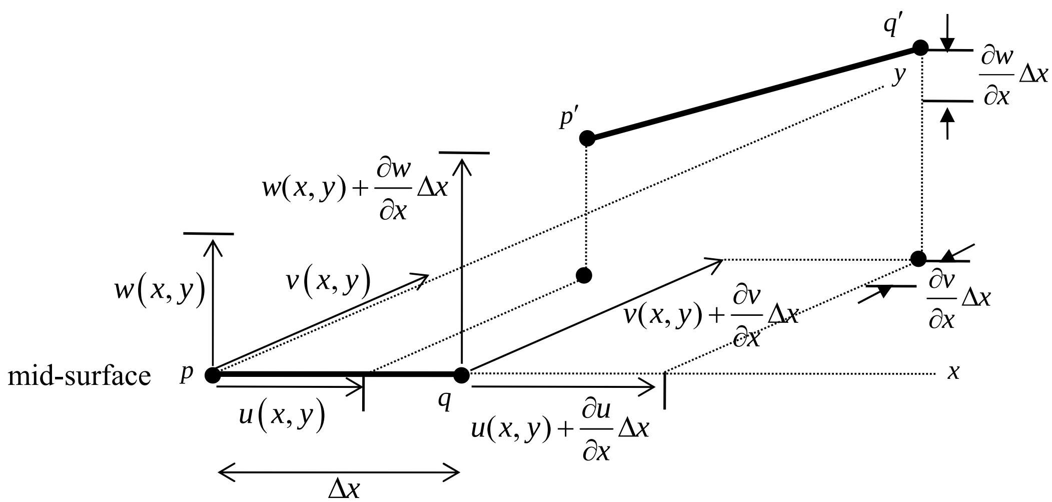

Figure 6.2.8: deformation of a material fibre in the *x* direction

In the plate theory, it will be assumed that the displacement gradients are small:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial z}, \quad \frac{\partial w}{\partial z},$$

of order $\varepsilon << 1$ say, so that squares and products of these terms may be neglected.

However, for the moment, the squares and products of the slopes will be retained, as they may be significant, i.e. of the same order as the strains, under certain circumstances:

$$\left(\frac{\partial w}{\partial x}\right)^2, \quad \left(\frac{\partial w}{\partial y}\right)^2, \quad \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Eqn. 6.2.15 now reduces to

$$\varepsilon_{xx} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \tag{6.2.16}$$

With $\sqrt{1+x} \approx 1 + x/2$ for $x << 1$, one has (and similarly for the other normal strains)

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned} \tag{6.2.17}$$

Consider next the angle change for line elements initially lying parallel to the axes, Fig. 6.2.9. Let θ be the angle $\angle r'p'q'$, so that $\gamma = \pi/2 - \theta$ is the change in the initial right angle $\angle rpq$.

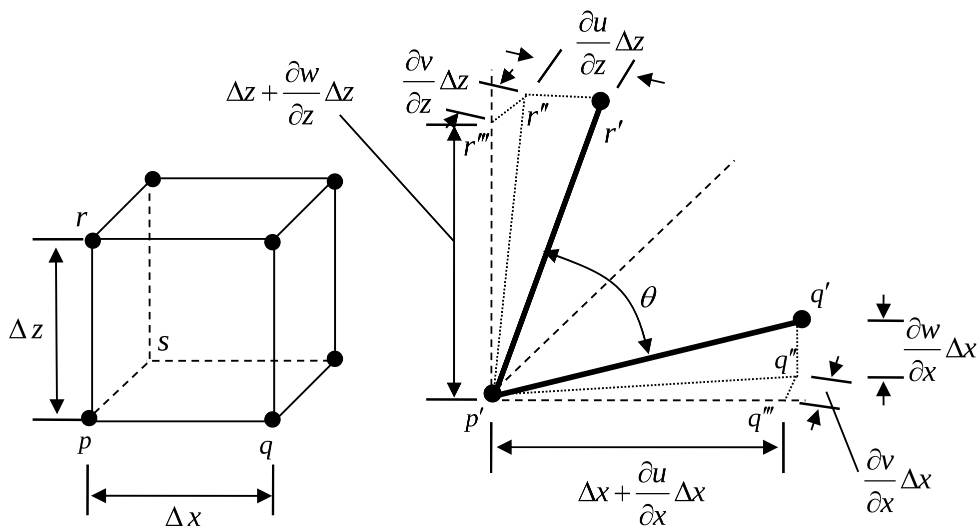

Figure 6.2.9: the deformation of Fig. 6.2.8, showing shear strains

Taking the dot product of the of the vector elements $\overrightarrow{p'q'}$ and $\overrightarrow{p'r'}$:

$$\cos\theta = \frac{|p'q'''||r''r'| + |q'''q''||r'''r''| + |q''q'||p'r'''|}{|p'q'||p'r'|}$$
$$= \frac{\left(\Delta x + \frac{\partial u}{\partial x}\Delta x\right)\left(\frac{\partial u}{\partial z}\Delta z\right) + \left(\frac{\partial v}{\partial x}\Delta x\right)\left(\frac{\partial v}{\partial z}\Delta z\right) + \left(\frac{\partial w}{\partial x}\Delta x\right)\left(\Delta z + \frac{\partial w}{\partial z}\Delta z\right)}{\Delta x\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\,\Delta z\sqrt{\left(1+\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}} \qquad (6.2.18)$$

Again, with the displacement gradients $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial z$, $\partial v/\partial z$, $\partial w/\partial z$ of order $\varepsilon << 1$ (and the squares $(\partial w/\partial x)^2$ at most of order $\varepsilon << 1$),

$$\cos\theta = \frac{\Delta x\left(\frac{\partial u}{\partial z}\Delta z\right) + \left(\frac{\partial v}{\partial x}\Delta x\right)\left(\frac{\partial v}{\partial z}\Delta z\right) + \left(\frac{\partial w}{\partial x}\Delta x\right)\Delta z}{\Delta x \Delta z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} \qquad (6.2.19)$$

For small γ, $\gamma \approx \sin\gamma = \cos\theta$, so (and similarly for the other shear strains)

$$\begin{aligned} \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\ \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{aligned} \qquad (6.2.20)$$

The normal strains 6.2.17 and the shear strains 6.2.20 are non-linear. They are the starting point for the various different plate theories.

Von Kármán Strains

Introduce now the assumptions of the classical plate theory. The assumption that line elements normal to the mid-plane remain inextensible implies that

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \qquad (6.2.21)$$

This implies that $w = w(x, y)$ so that all particles at a given (x, y) through the thickness of the plate experinece the same vertical displacement. The assumption that line elements perpendicular to the mid-plane remain normal to the mid-plane after deformation then implies that $\varepsilon_{xz} = \varepsilon_{yz} = 0$.

The strains now read

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\varepsilon_{zz} &= 0 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= 0 \\
\varepsilon_{yz} &= 0
\end{aligned} \qquad (6.2.22)$$

These are known as the **Von Kármán strains**.

Membrane Strains and Bending Strains

Since $\varepsilon_{xz} = 0$ and $w = w(x, y)$, one has from Eqn. 6.2.20b,

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} \quad \rightarrow \quad u(x, y, z) = -z\frac{\partial w}{\partial x} + u_0(x, y) \qquad (6.2.23)$$

It can be seen that the function $u_0(x, y)$ is the displacement in the mid-plane. In terms of the mid-surface displacements u_0, v_0, w_0, then,

$$u = u_0 - z\frac{\partial w_0}{\partial x}, \quad v = v_0 - z\frac{\partial w_0}{\partial y}, \quad w = w_0 \qquad (6.2.24)$$

and the strains 6.2.22 may be expressed as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right) - z\frac{\partial^2 w_0}{\partial x \partial y}
\end{aligned} \qquad (6.2.25)$$

The first terms are the usual small-strains, for the mid-surface. The second terms, involving squares of displacement gradients, are non-linear, and need to be considered when the plate bending is fairly large (when the rotations are about 10 – 15 degrees). These first two terms together are called the **membrane strains**. The last terms, involving second derivatives, are the **flexural** (**bending**) **strains**. They involve the curvatures.

When the bending is not too large (when the rotations are below about 10 degrees), one has (dropping the subscript "0" from *w*)

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{\partial^2 w}{\partial x \partial y}
\end{aligned} \tag{6.2.26}$$

Some of these strains are illustrated in Figs. 6.2.10 and 6.2.11; the physical meaning of ε_{xx} is shown in Fig. 6.2.10 and some terms from ε_{xy} are shown in Fig. 6.2.11.

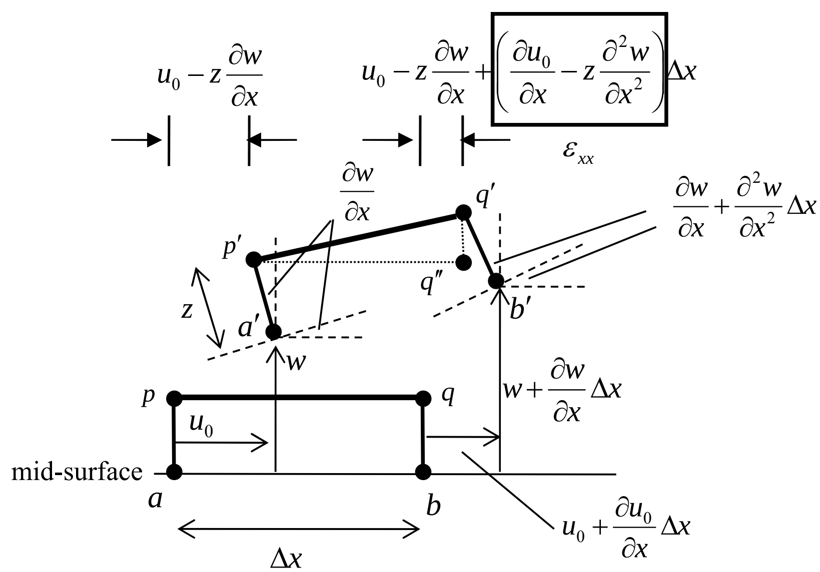

Figure 6.2.10: deformation of material fibres in the *x* direction

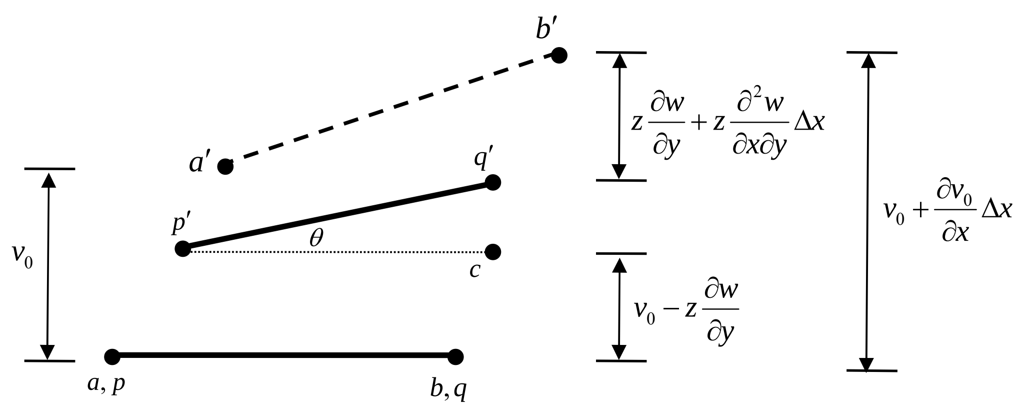

Figure 6.2.11: the deformation of 6.2.10 viewed "from above"; a′, b′ are the deformed positions of the mid-surface points a, b

Finally, when the mid-surface strains are neglected, according to the final assumption of the classical plate theory, one has

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2}, \qquad \varepsilon_{xy} = -z\frac{\partial^2 w}{\partial x \partial y} \tag{6.2.27}$$

In summary, when the plate bends "up", the curvature is positive, and points "above" the mid-surface experience negative normal strains and points "below" experience positive normal strains; there is zero shear strain. On the other hand, when the plate undergoes a positive pure twist, so the twisting moment is negative, points "above" the mid-surface experience negative shear strain and points "below" experience positive shear strain; there is zero normal strain. A pure shearing of the plate in the $x-y$ plane is illustrated in Fig. 6.2.12.

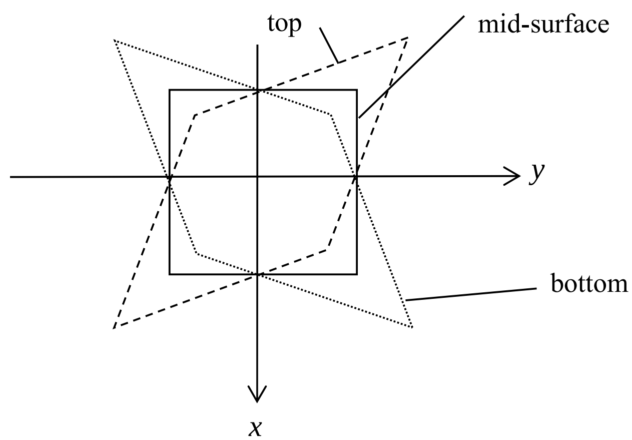

Figure 6.2.12: Shearing of the plate due to a positive twist (neative twisting moment)

Compatibility

Note that the strain fields arising in the plate satisfy the 2D compatibility relation Eqn. 1.3.1:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{6.2.28}$$

This can be seen by substituting Eqn. 6.25 (or Eqns 6.26-27) into Eqn. 6.2.28.

6.2.4 The Moment-Curvature equations

Now that the strains have been related to the curvatures, the moment-curvature relations, which play a central role in plate theory, can be derived.

Stresses and the Curvatures/Twist in a Linear Elastic Plate

From Hooke's law, taking $\sigma_{zz} = 0$,

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}, \quad \varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}, \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} \tag{6.2.29}$$

so, from 6.2.27, and solving 6.2.29a-b for the normal stresses,

$$\begin{aligned} \sigma_{xx} &= -\frac{E}{1-\nu^2} z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_{yy} &= -\frac{E}{1-\nu^2} z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ \sigma_{xy} &= -\frac{E}{1+\nu} z\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{6.2.30}$$

The Moment-Curvature Equations

Substituting Eqns. 6.2.30 into the definitions of the moments, Eqns. 6.1.1, 6.1.2, and integrating, one has

$$\boxed{\begin{aligned} M_x &= D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{aligned}} \tag{6.2.31}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{6.2.32}$$

Equations 6.2.31 are the **moment-curvature equations** for a plate. The moment-curvature equations are analogous to the beam moment-deflection equation $\partial^2 v / \partial x^2 = M / EI$. The factor D is called the **plate stiffness** or **flexural rigidity** and plays the same role in the plate theory as does the flexural rigidity term EI in the beam theory.

Stresses and Moments

From 6.30-6.31, the stresses and moments are related through

$$\sigma_{xx} = -\frac{M_x z}{h^3/12}, \quad \sigma_{yy} = -\frac{M_y z}{h^3/12}, \quad \sigma_{xy} = +\frac{M_{xy} z}{h^3/12} \tag{6.2.33}$$

Note the similarity of these relations to the beam formula $\sigma = -My/I$ with $I = h^3/12$ times the width of the beam.

6.2.5 Principal Moments

It was seen how the curvatures in different directions are related, through Eqns. 6.2.11-12. It comes as no surprise, examining 6.2.31, that the moments are related in the same way.

Consider a small differential element of a plate, Fig. 6.2.13a, subjected to stresses σ_{xx}, σ_{yy}, σ_{xy}, and corresponding moments M_x, M_y, M_{xy} given by 6.1.1-2. On any perpendicular planes rotated from the orginal $x - y$ axes by an angle θ, one can find the new stresses σ_{tt}, σ_{nn}, σ_{tn}, Fig. 6.2.13b (see Fig. 6.2.5), through the stress transformatrion equations (Book I, Eqns. 3.4.8). Then

$$\begin{aligned} M_t &= -\int z\sigma_{tt}dz = \cos^2\theta\left[-\int z\sigma_{xx}dz\right] + \sin^2\theta\left[-\int z\sigma_{yy}dz\right] + \sin 2\theta\left[-\int z\sigma_{xy}dz\right] \\ &= \cos^2\theta M_x + \sin^2\theta M_y - \sin 2\theta M_{xy} \end{aligned} \tag{6.2.34}$$

and similarly for the other moments, leading to

$$\begin{aligned} M_t &= \cos^2\theta M_x + \sin^2\theta M_y - \sin 2\theta M_{xy} \\ M_n &= \sin^2\theta M_x + \cos^2\theta M_y + \sin 2\theta M_{xy} \\ M_{tn} &= -\cos\theta\sin\theta\left(M_y - M_x\right) + \cos 2\theta M_{xy} \end{aligned} \tag{6.2.35}$$

Also, there exist principal planes, upon which the shear stress is zero (right through the thickness). The moments acting on these planes, M_1 and M_2, are called the **principal moments**, and are the greatest and least bending moments which occur at the element. On these planes, the twisting moment is zero.

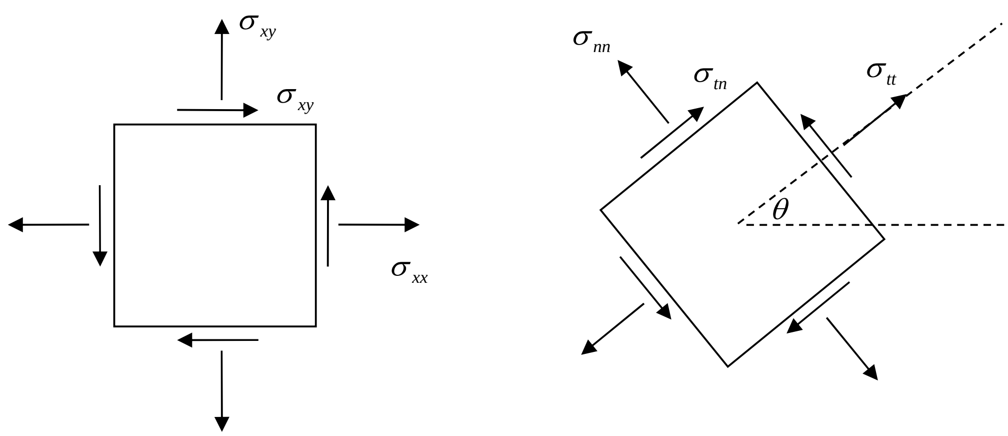

Figure 6.2.13: Plate Element; (a) stresses acting on element, (b) rotated element

Moments in Different Coordinate Systems

From the moment-curvature equations 6.2.31, {▲Problem 1}

$$\begin{aligned} M_t &= D\left(\frac{\partial^2 \omega}{\partial t^2} + \nu \frac{\partial^2 \omega}{\partial n^2}\right) \\ M_n &= D\left(\frac{\partial^2 \omega}{\partial n^2} + \nu \frac{\partial^2 \omega}{\partial t^2}\right) \\ M_{tn} &= -D(1-\nu)\frac{\partial^2 \omega}{\partial t \partial n} \end{aligned} \qquad (6.2.36)$$

showing that the moment-curvature relations 6.2.31 hold in all Cartesian coordinate systems.

6.2.6 Problems

1. Use the curvature transformation relations 6.2.11 and the moment transformation relations 6.2.35 to derive the moment-curvature relations 6.2.36.