6.1 Plate Theory

6.1.1 Plates

A **plate** is a flat structural element for which the thickness is small compared with the surface dimensions. The thickness is usually constant but may be variable and is measured normal to the **middle surface** of the plate, Fig. 6.1.1

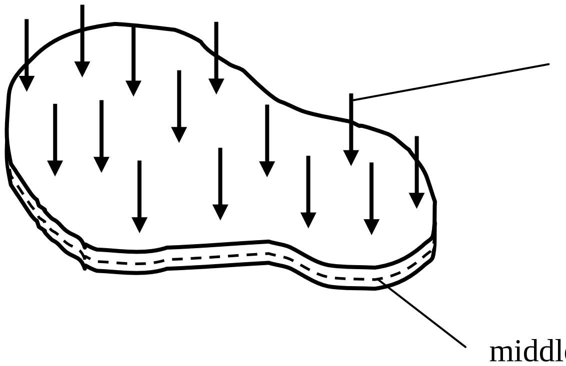

Fig. 6.1.1: A plate

6.1.2 Plate Theory

Plates subjected only to in-plane loading can be solved using two-dimensional plane stress theory[1] (see Book I, §3.5). On the other hand, **plate theory** is concerned mainly with **lateral loading**.

One of the differences between plane stress and plate theory is that in the plate theory the stress components are allowed to vary *through the thickness* of the plate, so that there can be bending moments, Fig. 6.1.2.

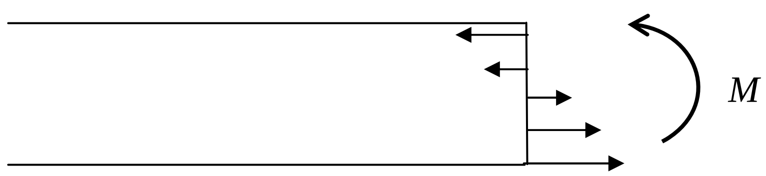

Fig. 6.1.2: Stress distribution through the thickness of a plate and resultant bending moment

Plate Theory and Beam Theory

Plate theory is an approximate theory; assumptions are made and the general three dimensional equations of elasticity are reduced. It is very like the **beam theory** (see Book

[1] although if the in-plane loads are compressive and sufficiently large, they can buckle (see §6.7)

I, §7.4) – only with an extra dimension. It turns out to be an accurate theory provided *the plate is relatively thin* (as in the beam theory) but also that *the deflections are small relative to the thickness*. This last point will be discussed further in §6.10.

Things are more complicated for plates than for the beams. For one, the plate not only bends, but torsion may occur (it can twist), as shown in Fig. 6.1.3

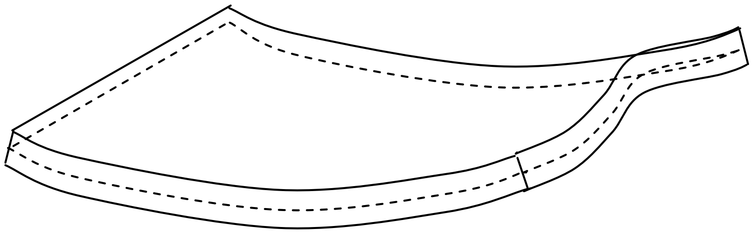

Fig. 6.1.3: torsion of a plate

Assumptions of Plate Theory

Let the plate mid-surface lie in the $x-y$ plane and the z – axis be along the thickness direction, forming a right handed set, Fig. 6.1.4.

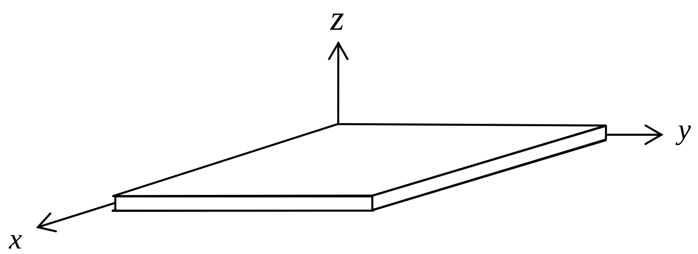

Fig. 6.1.4: Cartesian axes

The stress components acting on a typical element of the plate are shown in Fig. 6.1.5.

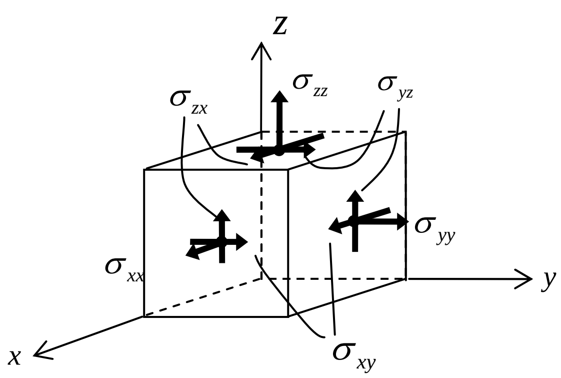

Fig. 6.1.5: stresses acting on a material element

The following assumptions are made:

(i) The mid-plane is a "neutral plane"

The middle plane of the plate remains free of in-plane stress/strain. Bending of the plate will cause material above and below this mid-plane to deform in-plane. The mid-plane plays the same role in plate theory as the neutral axis does in the beam theory.

(ii) Line elements remain normal to the mid-plane

Line elements lying perpendicular to the middle surface of the plate remain perpendicular to the middle surface during deformation, Fig. 6.1.6; this is similar the "plane sections remain plane" assumption of the beam theory.

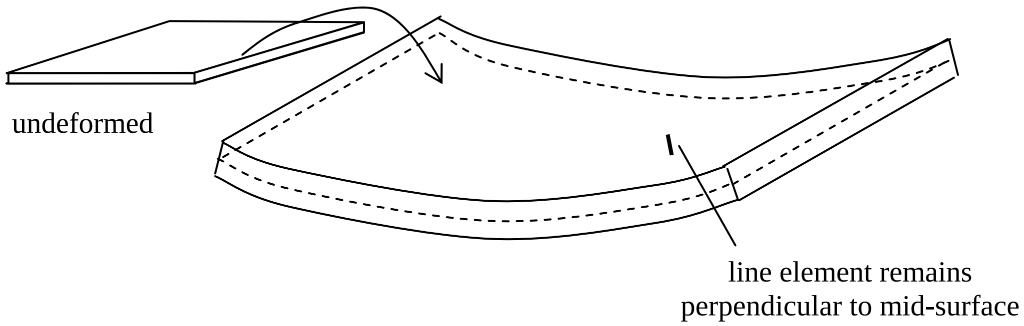

Fig. 6.1.6: deformed line elements remain perpendicular to the mid-plane

(iii) Vertical strain is ignored

Line elements lying perpendicular to the mid-surface do not change length during deformation, so that $\varepsilon_{zz} = 0$ throughout the plate. Again, this is similar to an assumption of the beam theory.

These three assumptions are the basis of the **Classical Plate Theory** or the **Kirchhoff Plate Theory**. The second assumption can be relaxed to develop a more exact theory (see §6.10).

6.1.3 Notation and Stress Resultants

The **stress resultants** are obtained by integrating the stresses through the thickness of the plate. In general there will be

moments M:	2 bending moments and 1 twisting moment
out-of-plane forces V:	2 shearing forces
in-plane forces N:	2 normal forces and 1 shear force

They are defined as follows:

In-plane normal forces and bending moments, Fig. 6.1.7:

$$\begin{aligned} N_x &= \int_{-h/2}^{+h/2} \sigma_{xx} dz, \qquad N_y = \int_{-h/2}^{+h/2} \sigma_{yy} dz \\ M_x &= -\int_{-h/2}^{+h/2} z\sigma_{xx} dz, \quad M_y = -\int_{-h/2}^{+h/2} z\sigma_{yy} dz \end{aligned} \tag{6.1.1}$$

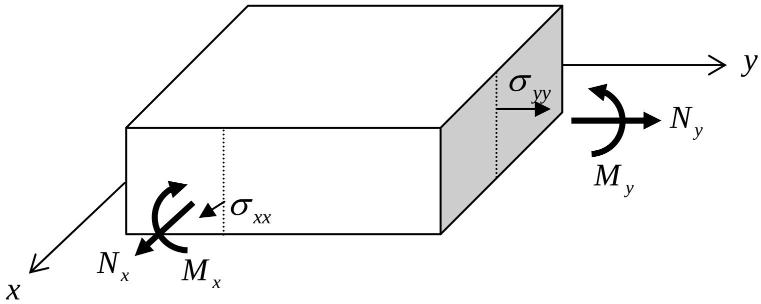

Fig. 6.1.7: in-plane normal forces and bending moments

In-plane shear force and twisting moment, Fig. 6.1.8:

$$N_{xy} = \int_{-h/2}^{+h/2} \sigma_{xy} dz, \qquad M_{xy} = \int_{-h/2}^{+h/2} z\sigma_{xy} dz \tag{6.1.2}$$

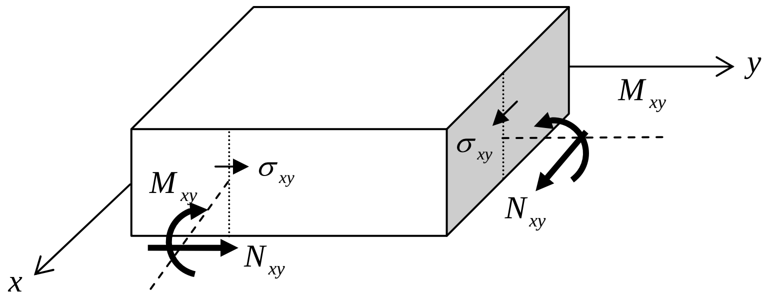

Fig. 6.1.8: in-plane shear force and twisting moment

Out-of-plane shearing forces, Fig. 6.1.9:

$$V_x = -\int_{-h/2}^{+h/2} \sigma_{zx} dz, \qquad V_y = -\int_{-h/2}^{+h/2} \sigma_{yz} dz \tag{6.1.3}$$

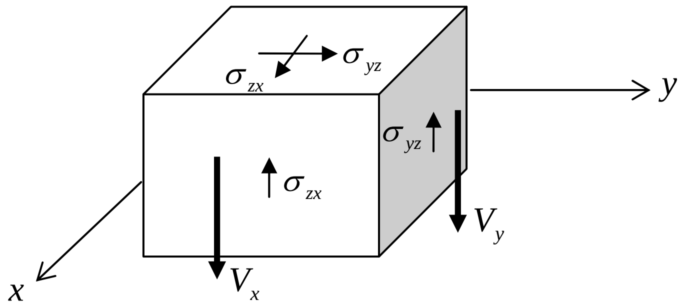

Fig. 6.1.9: out of plane shearing forces

Note that the above "forces" and "moments" are actually forces and moments *per unit length*. This allows one to have moments varying across any section – unlike in the beam theory, where the moments are for the complete beam cross-section. If one considers an element with dimensions Δx and Δy, the actual moments acting on the element are

$$M_x \Delta y, \quad M_y \Delta x, \quad M_{xy} \Delta x, \quad M_{xy} \Delta y \tag{6.1.4}$$

and the forces acting on the element are

$$V_x \Delta y, \quad V_y \Delta x, \quad N_x \Delta y, \quad N_y \Delta x, \quad N_{xy} \Delta x, \quad N_{xy} \Delta y \tag{6.1.5}$$

The in-plane forces, which are analogous to the axial forces of the beam theory, do not play a role in most of what follows. They are useful in the analysis of buckling of plates and it is necessary to consider them in more exact theories of plate bending (see later).