

4.4 Rotating Discs

4.4.1 The Rotating Disc

Consider a thin disc rotating with constant angular velocity ω , Fig. 4.4.1. Material particles are subjected to a centripetal acceleration $a_r = -r\omega^2$. The subscript r indicates an acceleration in the radial direction and the minus sign indicates that the particles are accelerating towards the centre of the disc.

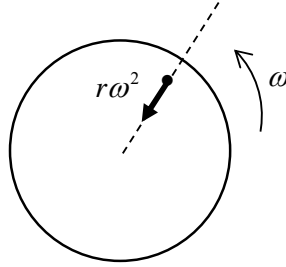


Figure 4.4.1: the rotating disc

The accelerations lead to an inertial force (per unit volume) $F_a = -\rho r\omega^2$ which in turn leads to stresses in the disc. The inertial force is an axisymmetric “loading” and so this is an axisymmetric problem. The axisymmetric equation of equilibrium is given by 4.3.5. Adding in the acceleration term gives the corresponding equation of motion:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = -\rho r\omega^2, \quad (4.4.1)$$

This equation can be expressed as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = 0, \quad (4.4.2)$$

where $b_r = \rho r\omega^2$. Thus the dynamic rotating disc problem has been converted into an equivalent static problem of a disc subjected to a known body force. Note that, in a general dynamic problem, and unlike here, one does not know what the accelerations are – they have to be found as part of the solution procedure.

Using the strain-displacement relations 4.3.2 and the plane stress Hooke’s law 4.3.3 then leads to the differential equation

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{1}{r^2} u = -\frac{1-\nu^2}{E} \rho r\omega^2 \quad (4.4.3)$$

This is Eqn. 4.3.6 with a non-homogeneous term. The solution is derived in the Appendix to this section, §4.4.3:

$$u = C_1 r + C_2 \frac{1}{r} - \frac{1}{8} \frac{1-\nu^2}{E} \rho r^3 \omega^2 \quad (4.4.4)$$

As in §4.3.4, let $A = -EC_2 / (1+\nu)$ and $C = EC_1 / 2(1-\nu)$, and the full general solution is, using 4.3.2 and 4.3.3, {▲ Problem 1}

$$\begin{aligned} \sigma_{rr} &= +A \frac{1}{r^2} + 2C - \frac{1}{8} (3+\nu) \rho \omega^2 r^2 \\ \sigma_{\theta\theta} &= -A \frac{1}{r^2} + 2C - \frac{1}{8} (1+3\nu) \rho \omega^2 r^2 \\ \varepsilon_{rr} &= \frac{1}{E} \left[+ (1+\nu) \frac{A}{r^2} + 2(1-\nu)C - \frac{3}{8} (1-\nu^2) \rho \omega^2 r^2 \right] \\ \varepsilon_{\theta\theta} &= \frac{1}{E} \left[- (1+\nu) \frac{A}{r^2} + 2(1-\nu)C - \frac{1}{8} (1-\nu^2) \rho \omega^2 r^2 \right] \\ u &= \frac{1}{E} \left[- (1+\nu) \frac{A}{r} + 2(1-\nu)Cr - \frac{1}{8} (1-\nu^2) \rho \omega^2 r^3 \right] \end{aligned} \quad (4.4.5)$$

which reduce to 4.3.9 when $\omega = 0$.

A Solid Disc

For a solid disc, A in 4.4.5 must be zero to ensure finite stresses and strains at $r = 0$. C is then obtained from the boundary condition $\sigma_{rr}(b) = 0$, where b is the disc radius:

$$A = 0, \quad C = \frac{1}{16} (3+\nu) \rho \omega^2 b^2 \quad (4.4.6)$$

The stresses and displacements are

$$\begin{aligned} \sigma_{rr}(r) &= \frac{3+\nu}{8} \rho \omega^2 [b^2 - r^2] \\ \sigma_{\theta\theta}(r) &= \frac{3+\nu}{8} \rho \omega^2 \left[b^2 - \frac{1+3\nu}{3+\nu} r^2 \right] \\ u(r) &= \frac{3+\nu}{8} \rho \omega^2 \frac{1-\nu}{E} r \left[b^2 - \frac{1+\nu}{3+\nu} r^2 \right] \end{aligned} \quad (4.4.7)$$

Note that the displacement is zero at the disc centre, as it must be, but the strains (and hence stresses) do not have to be, and are not, zero there.

Dimensionless stress and displacement are plotted in Fig. 4.4.2 for the case of $\nu = 0.3$. The maximum stress occurs at $r = 0$, where

$$\sigma_{rr}(0) = \sigma_{\theta\theta}(0) = \frac{3+\nu}{8} \rho \omega^2 b^2 \quad (4.4.8)$$

The disc expands by an amount

$$u(b) = \frac{1-\nu}{4E} \rho \omega^2 b^3 \quad (4.4.9)$$

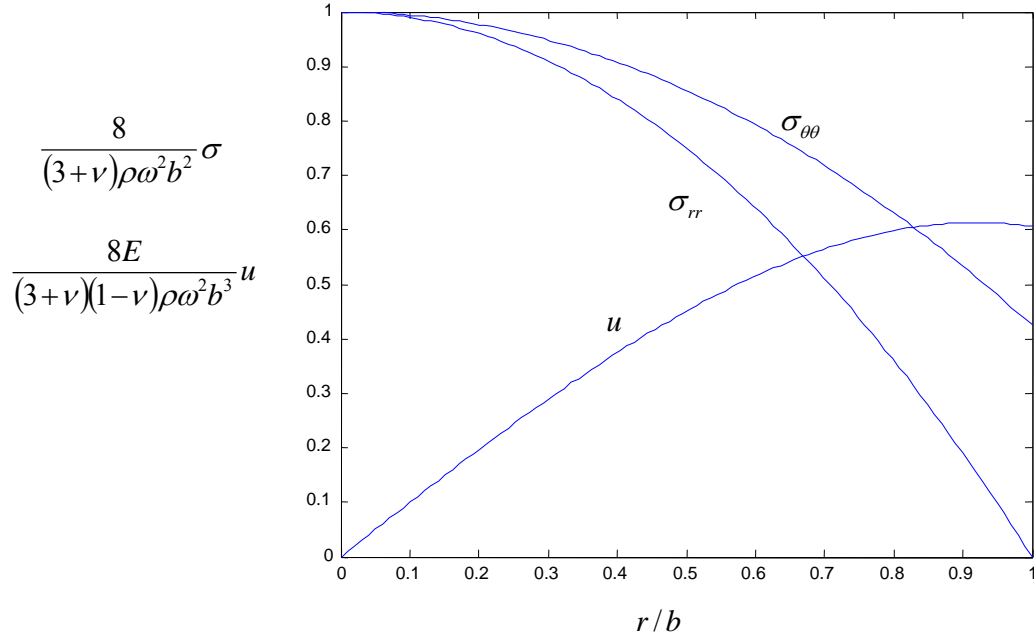


Figure 4.4.2: stresses and displacements in the solid rotating disc

A Hollow Disc

The boundary conditions for the hollow disc are

$$\sigma_{rr}(a) = 0, \quad \sigma_{rr}(b) = 0 \quad (4.4.10)$$

where a and b are the inner and outer radii respectively. It follows from 4.4.5 that

$$A = -\frac{1}{8}(3+\nu)\rho\omega^2 a^2 b^2, \quad C = \frac{1}{16}(3+\nu)\rho\omega^2 (a^2 + b^2) \quad (4.4.11)$$

and the stresses and displacement are

$$\begin{aligned} \sigma_{rr}(r) &= \frac{3+\nu}{8} \rho \omega^2 \left[a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2} \right] \\ \sigma_{\theta\theta}(r) &= \frac{3+\nu}{8} \rho \omega^2 \left[a^2 + b^2 - \frac{1+3\nu}{3+\nu} r^2 + \frac{a^2 b^2}{r^2} \right] \\ u(r) &= \frac{3+\nu}{8} \rho \omega^2 \frac{1-\nu}{E} r \left[a^2 + b^2 - \frac{1+\nu}{3+\nu} r^2 + \frac{1+\nu}{1-\nu} \frac{a^2 b^2}{r^2} \right] \end{aligned} \quad (4.4.12)$$

which reduce to 4.4.7 when $a = 0$.

Dimensionless stress and displacement are plotted in Fig. 4.4.3 for the case of $\nu = 0.3$ and $a/b = 0.2$. The maximum stress occurs at the inner surface, where

$$\sigma_{\theta\theta}(0) = \frac{3+\nu}{4} \rho \omega^2 b^2 \left[1 + \frac{1-\nu}{3+\nu} (a/b^2) \right] \quad (4.4.13)$$

which is approximately twice the solid-disc maximum stress.

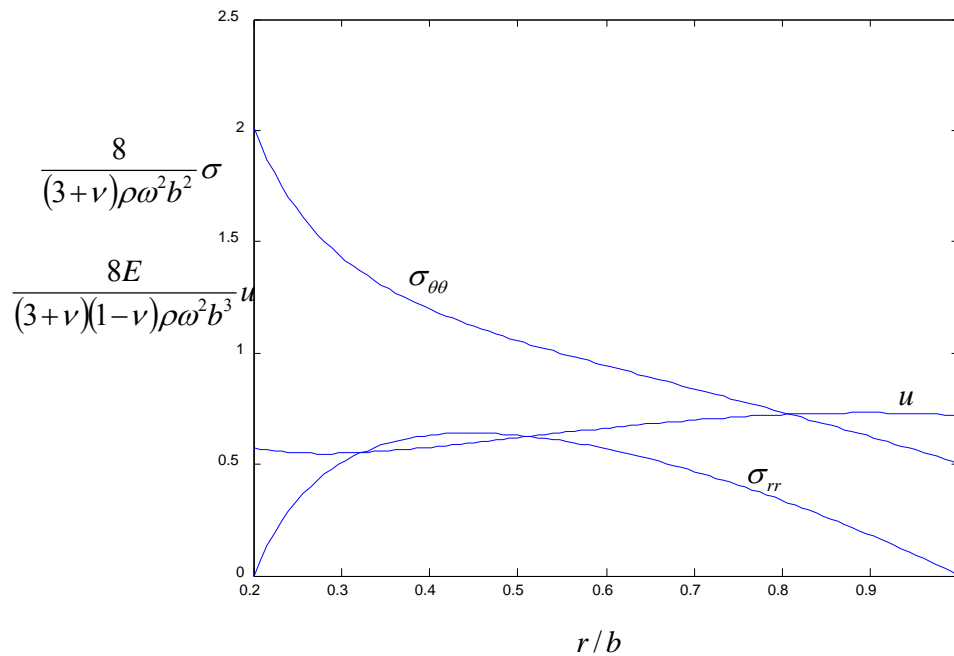


Figure 4.4.3: stresses and displacements in the hollow rotating disc

4.4.2 Problems

1. Derive the full solution equations 4.4.5 for the thin rotating disc, from the displacement solution 4.4.4.

4.4.3 Appendix: Solution to Eqn. 4.4.3

As in §4.3.8, transform Eqn. 4.4.3 using $r = e^t$ into

$$\frac{d^2 u}{dt^2} - u = -\frac{1-\nu^2}{E} \rho e^{3t} \omega^2 \quad (4.4.14)$$

The homogeneous solution is given by 4.3.31. Assume a particular solution of the form $u_p = Ae^{3t}$ which, from 4.4.14, gives

$$u_p = -\frac{1}{8} \frac{1-\nu^2}{E} \rho \omega^2 e^{3t} \quad (4.4.15)$$

Adding together the homogeneous and particular solutions and transforming back to r 's then gives 4.4.4.