# 4.3 Plane Axisymmetric Problems

In this section are considered **plane axisymmetric problems**. These are problems in which both the geometry *and* loading are axisymmetric.

## 4.3.1 Plane Axisymmetric Problems

Some three dimensional (not necessarily plane) examples of axisymmetric problems would be the thick-walled (hollow) cylinder under internal pressure, a disk rotating about its axis[1], and the two examples shown in Fig. 4.3.1; the first is a complex component loaded in a complex way, but exhibits axisymmetry in both geometry and loading; the second is a sphere loaded by concentrated forces along a diameter.

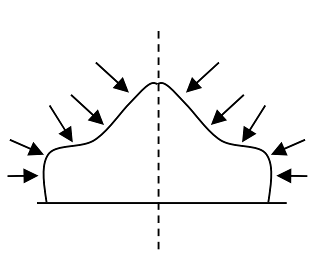


**Figure 4.3.1: axisymmetric problems**

A two-dimensional (plane) example would be one plane of the thick-walled cylinder under internal pressure, illustrated in Fig. 4.3.2[2].

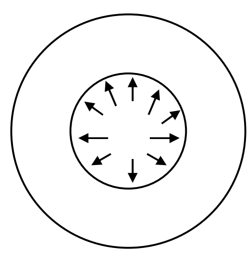


**Figure 4.3.2: a cross section of an internally pressurised cylinder**

It should be noted that many problems involve axisymmetric geometries but non-axisymmetric loadings, and *vice versa*. These problems are *not* axisymmetric. An example is shown in Fig. 4.3.3 (the problem involves a plane axisymmetric geometry).

[1] the rotation induces a stress in the disk
[2] the rest of the cylinder is coming out of, and into, the page

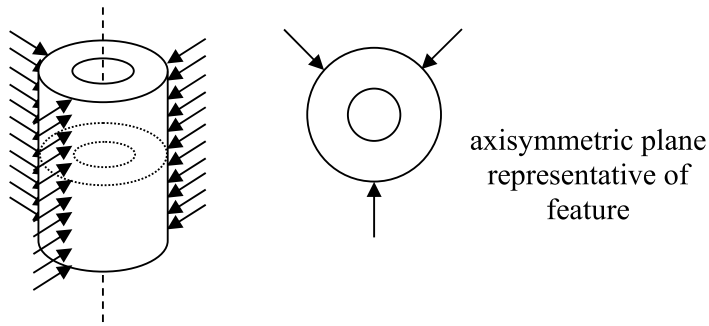



**Figure 4.3.3: An axially symmetric geometry but with a non-axisymmetric loading**

The important characteristic of these axisymmetric problems is that all quantities, be they stress, displacement, strain, or anything else associated with the problem, *must be independent of the circumferential variable* $\theta$. As a consequence, any term in the differential equations of §4.2 involving the derivatives $\partial/\partial\theta, \partial^2/\partial\theta^2$, etc. can be immediately set to zero.

### 4.3.2 Governing Equations for Plane Axisymmetric Problems

The two-dimensional strain-displacement relations are given by Eqns. 4.2.4 and these simplify in the axisymmetric case to

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}\\ \varepsilon_{\theta\theta} &= \frac{u_r}{r}\\ \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\end{aligned} \tag{4.3.1}$$

Here, it will be assumed that the displacement $u_\theta = 0$. Cases where $u_\theta \neq 0$ but where the stresses and strains are still independent of $\theta$ are termed **quasi-axisymmetric problems**; these will be examined in a later section. Then 4.3.1 reduces to

$$\boxed{\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{r\theta} = 0} \tag{4.3.2}$$

It follows from Hooke's law that $\sigma_{r\theta} = 0$. The non-zero stresses are illustrated in Fig. 4.3.4.

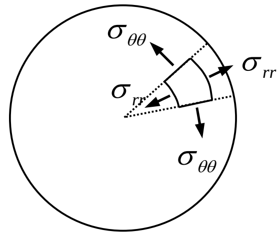



**Figure 4.3.4: stress components in plane axisymmetric problems**

## 4.3.3 Plane Stress and Plane Strain

Two cases arise with plane axisymmetric problems: in the plane stress problem, the feature is very thin and unloaded on its larger free-surfaces, for example a thin disk under external pressure, as shown in Fig. 4.3.5. Only two stress components remain, and Hooke's law 4.2.5a reads

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right] \\ \varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right] \end{aligned} \quad \text{or} \quad \begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu^2}\left[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right] \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left[\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right] \end{aligned} \tag{4.3.3}$$

with $\varepsilon_{zz} = \frac{-\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})$, $\varepsilon_{zr} = \varepsilon_{z\theta} = 0$ and $\sigma_{zz} = 0$.

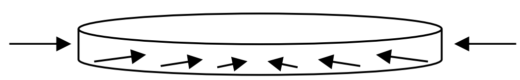


**Figure 4.3.5: plane stress axisymmetric problem**

In the plane strain case, the strains $\varepsilon_{zz}$, $\varepsilon_{z\theta}$ and $\varepsilon_{zr}$ are zero. This will occur, for example, in a hollow cylinder under internal pressure, with the ends fixed between immovable platens, Fig. 4.3.6.

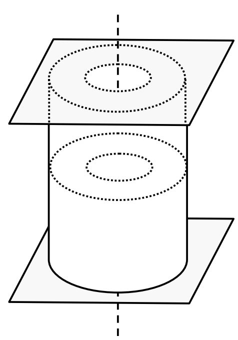


**Figure 4.3.6: plane strain axisymmetric problem**

Hooke's law 4.2.5b reads

$$\begin{aligned}\varepsilon_{rr} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right] \\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right]\end{aligned} \quad \text{or} \quad \begin{aligned}\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{\theta\theta} + (1-\nu)\varepsilon_{rr}\right] \\ \sigma_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{rr} + (1-\nu)\varepsilon_{\theta\theta}\right]\end{aligned} \tag{4.3.4}$$

with $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$.

Shown in Fig. 4.3.7 are the stresses acting in the axisymmetric plane body (with $\sigma_{zz}$ zero in the plane stress case).

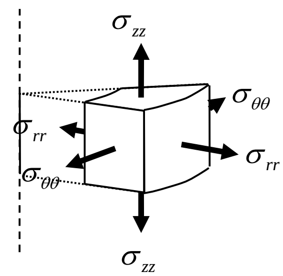



**Figure 4.3.7: stress components in plane axisymmetric problems**

## 4.3.4 Solution of Plane Axisymmetric Problems

The equations governing the plane axisymmetric problem are the equations of equilibrium 4.2.3 which reduce to the single equation

$$\boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0}, \tag{4.3.5}$$

the strain-displacement relations 4.3.2 and the stress-strain law 4.3.3-4.

Taking the plane stress case, substituting 4.3.2 into the second of 4.3.3 and then substituting the result into 4.3.5 leads to (with a similar result for plane strain)

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u = 0 \tag{4.3.6}$$

This is Navier's equation for plane axisymmetry. It is an "Euler-type" ordinary differential equation which can be solved exactly to get (see Appendix to this section, §4.3.8)

$$u = C_1 r + C_2 \frac{1}{r} \tag{4.3.7}$$

With the displacement known, the stresses and strains can be evaluated, and the full solution is

$$
\begin{gathered}
u = C_1 r + C_2 \frac{1}{r} \\
\varepsilon_{rr} = C_1 - C_2 \frac{1}{r^2}, \quad \varepsilon_{\theta\theta} = C_1 + C_2 \frac{1}{r^2} \\
\sigma_{rr} = \frac{E}{1-\nu} C_1 - \frac{E}{1+\nu} C_2 \frac{1}{r^2}, \quad \sigma_{\theta\theta} = \frac{E}{1-\nu} C_1 + \frac{E}{1+\nu} C_2 \frac{1}{r^2}
\end{gathered}
\tag{4.3.8}
$$

For problems involving stress boundary conditions, it is best to have simpler expressions for the stress so, introducing new constants $A = -EC_2/(1+\nu)$ and $C = EC_1/2(1-\nu)$, the solution can be re-written as

$$
\boxed{
\begin{gathered}
\sigma_{rr} = +A\frac{1}{r^2} + 2C, \quad \sigma_{\theta\theta} = -A\frac{1}{r^2} + 2C \\
\varepsilon_{rr} = +\frac{(1+\nu)A}{E}\frac{1}{r^2} + \frac{2(1-\nu)C}{E}, \quad \varepsilon_{\theta\theta} = -\frac{(1+\nu)A}{E}\frac{1}{r^2} + \frac{2(1-\nu)C}{E}, \quad \varepsilon_{zz} = -\frac{4\nu C}{E} \\
u = -\frac{(1+\nu)A}{E}\frac{1}{r} + \frac{2(1-\nu)C}{E} r
\end{gathered}
}
\tag{4.3.9}
$$

**Plane stress axisymmetric solution**

Similarly, the plane strain solution turns out to be again 4.3.8a-b only the stresses are now {▲Problem 1}

$$
\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[-(1-2\nu)C_2\frac{1}{r^2} + C_1\right], \quad \sigma_{\theta\theta} = \frac{E}{(1+\nu)(1-2\nu)}\left[+(1-2\nu)C_2\frac{1}{r^2} + C_1\right]
\tag{4.3.10}
$$

Then, with $A = -EC_2/(1+\nu)$ and $C = EC_1/2(1+\nu)(1-2\nu)$, the solution can be written as

$$
\boxed{
\begin{gathered}
\sigma_{rr} = +A\frac{1}{r^2} + 2C, \quad \sigma_{\theta\theta} = -A\frac{1}{r^2} + 2C, \quad \sigma_{zz} = 4\nu C \\
\varepsilon_{rr} = \frac{1+\nu}{E}\left[+A\frac{1}{r^2} + 2(1-2\nu)C\right], \quad \varepsilon_{\theta\theta} = \frac{1+\nu}{E}\left[-A\frac{1}{r^2} + 2(1-2\nu)C\right] \\
u = \frac{1+\nu}{E}\left[-A\frac{1}{r} + 2(1-2\nu)Cr\right]
\end{gathered}
}
\tag{4.3.11}
$$

**Plane strain axisymmetric solution**

The solutions 4.3.9, 4.3.11 involve two constants. When there is a solid body with one boundary, $A$ must be zero in order to ensure finite-valued stresses and strains; $C$ can be determined from the boundary condition. When there are two boundaries, both $A$ and $C$ are determined from the boundary conditions.

### 4.3.5 Example: Expansion of a thick circular cylinder under internal pressure

Consider the problem of Fig. 4.3.8. The two unknown constants $A$ and $C$ are obtained from the boundary conditions

$$\begin{aligned}\sigma_{rr}(a) &= -p \\ \sigma_{rr}(b) &= 0\end{aligned} \tag{4.3.12}$$

which lead to

$$\sigma_{rr}(a) = \frac{A}{a^2} + 2C = -p, \quad \sigma_{rr}(b) = \frac{A}{b^2} + 2C = 0 \tag{4.3.13}$$

so that

$$\boxed{\sigma_{rr} = -p\frac{b^2/r^2 - 1}{b^2/a^2 - 1}, \quad \sigma_{\theta\theta} = +p\frac{b^2/r^2 + 1}{b^2/a^2 - 1}, \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})} \tag{4.3.14}$$

**Cylinder under Internal Pressure**

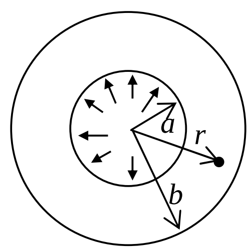


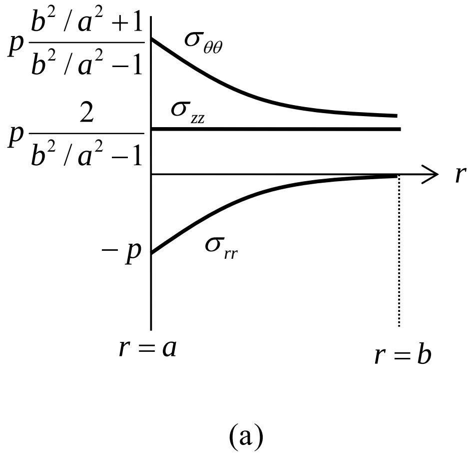

**Figure 4.3.8: an internally pressurised cylinder**

The stresses through the thickness of the cylinder walls are shown in Fig. 4.3.9a. The maximum principal stress is the $\sigma_{\theta\theta}$ stress and this attains a maximum at the inner face. For this reason, internally pressurized vessels often fail there first, with microcracks perpendicular to the inner edge been driven by the tangential stress, as illustrated in Fig. 4.3.9b.

Note that by setting $b = a + t$ and taking the wall thickness to be very small, $t, t^2 << a$, and letting $a = r$, the solution 4.3.14 reduces to:

$$\sigma_{rr} = -p, \quad \sigma_{\theta\theta} = +p\frac{r}{t}, \quad \sigma_{zz} = \nu p\frac{r}{t} \tag{4.3.15}$$

which is equivalent to the thin-walled pressure-vessel solution, Part I, §4.5.2 (if $\nu = 1/2$, i.e. incompressible).



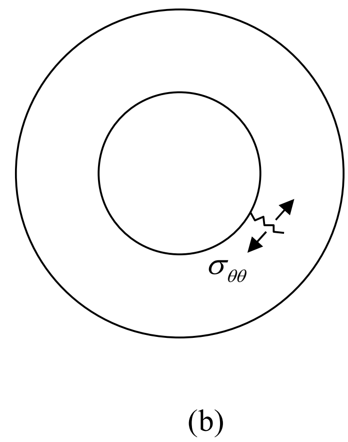




**Figure 4.3.9: (a) stresses in the thick-walled cylinder, (b) microcracks driven by tangential stress**

### Generalised Plane Strain and Other Solutions

The solution for a pressurized cylinder in plane strain was given above, i.e. where $\varepsilon_{zz}$ was enforced to be zero. There are two other useful situations:

(1) The cylinder is free to expand in the axial direction. In this case, $\varepsilon_{zz}$ is not forced to zero, but allowed to be a constant along the length of the cylinder, say $\bar{\varepsilon}_{zz}$. The $\sigma_{zz}$ stress is zero, as in plane stress. This situation is called **generalized plane strain**.

(2) The cylinder is closed at its ends. Here, the axial stresses $\sigma_{zz}$ inside the walls of the tube are counteracted by the internal pressure *p* acting on the closed ends. The force acting on the closed ends due to the pressure is $p\pi a^2$ and the balancing axial force is $\sigma_{zz}\pi\left(b^2-a^2\right)$, assuming $\sigma_{zz}$ to be constant through the thickness. For equilibrium

$$\sigma_{zz}=\frac{p}{b^2/a^2-1} \tag{4.3.16}$$

Returning to the full three-dimensional stress-strain equations (Part I, Eqns. 4.2.9), set $\varepsilon_{zz}=\bar{\varepsilon}_{zz}$, a constant, and $\varepsilon_{xz}=\varepsilon_{yz}=0$. Re-labelling $x,y,z$ with $r,\theta,z$, and again with $\sigma_{r\theta}=0$, one has

$$\begin{aligned}
\sigma_{rr}&=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{rr}+\nu\left(\varepsilon_{\theta\theta}+\bar{\varepsilon}_{zz}\right)\right]\\
\sigma_{\theta\theta}&=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{\theta\theta}+\nu\left(\varepsilon_{rr}+\bar{\varepsilon}_{zz}\right)\right]\\
\sigma_{zz}&=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\bar{\varepsilon}_{zz}+\nu\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}\right)\right]
\end{aligned} \tag{4.3.17}$$

Substituting the strain-displacement relations 4.3.2 into 4.3.16a-b, and, as before, using the axisymmetric equilibrium equation 4.3.5, again leads to the differential equation 4.3.6 and the solution $u = C_1 r + C_2 / r$, with $\varepsilon_{rr} = C_1 - C_2 / r^2$, $\varepsilon_{\theta\theta} = C_1 + C_2 / r^2$, but now the stresses are

$$\begin{aligned}
\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left[C_1 - (1-2\nu)C_2\frac{1}{r^2} + \nu\bar{\varepsilon}_{zz}\right] \\
\sigma_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left[C_1 + (1-2\nu)C_2\frac{1}{r^2} + \nu\bar{\varepsilon}_{zz}\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[2\nu C_1 + (1-\nu)\nu\bar{\varepsilon}_{zz}\right]
\end{aligned} \tag{4.3.18}$$

As before, to make the solution more amenable to stress boundary conditions, we let $A = -EC_2 /(1+\nu)$ and $C = EC_1 / 2(1+\nu)(1-2\nu)$, so that the solution is

$$\boxed{\begin{gathered}
\sigma_{rr} = +A\frac{1}{r^2} + 2C + \frac{\nu E}{(1+\nu)(1-2\nu)}\bar{\varepsilon}_{zz}, \quad \sigma_{\theta\theta} = -A\frac{1}{r^2} + 2C + \frac{\nu E}{(1+\nu)(1-2\nu)}\bar{\varepsilon}_{zz} \\
\sigma_{zz} = 4\nu C + \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\bar{\varepsilon}_{zz} \\
\varepsilon_{rr} = \frac{1+\nu}{E}\left[+A\frac{1}{r^2} + 2(1-2\nu)C\right], \quad \varepsilon_{\theta\theta} = \frac{1+\nu}{E}\left[-A\frac{1}{r^2} + 2(1-2\nu)C\right] \\
u = \frac{1+\nu}{E}\left[-A\frac{1}{r} + 2(1-2\nu)Cr\right]
\end{gathered}} \tag{4.3.19}$$

**Generalised axisymmetric solution**

For internal pressure $p$, the solution to 4.3.19 gives the same solution for radial and tangential stresses as before, Eqn. 4.3.14. The axial displacement is $u_z = z\bar{\varepsilon}_{zz}$ (to within a constant).

In the case of the cylinder with open ends (generalized plane strain), $\sigma_{zz} = 0$, and one finds from Eqn. 4.3.19 that $\bar{\varepsilon}_{zz} = -2\nu p / E\left(b^2 / a^2 - 1\right) < 0$. In the case of the cylinder with closed ends, one finds that $\bar{\varepsilon}_{zz} = (1-2\nu)\, p / E\left(b^2 / a^2 - 1\right) > 0$.

## A Transversely isotropic Cylinder

Consider now a transversely isotropic cylinder. The strain-displacement relations 4.3.2 and the equilibrium equation 4.3.5 are applicable to any type of material. The stress-strain law can be expressed as (see Part I, Eqn. 6.2.14)

$$\begin{aligned}
\sigma_{rr} &= C_{11}\varepsilon_{rr} + C_{12}\varepsilon_{\theta\theta} + C_{13}\varepsilon_{zz} \\
\sigma_{\theta\theta} &= C_{12}\varepsilon_{rr} + C_{11}\varepsilon_{\theta\theta} + C_{13}\varepsilon_{zz} \\
\sigma_{zz} &= C_{13}\varepsilon_{rr} + C_{13}\varepsilon_{\theta\theta} + C_{33}\varepsilon_{zz}
\end{aligned} \tag{4.3.20}$$

Here, take $\varepsilon_{zz} = \bar{\varepsilon}_{zz}$, a constant. Then, using the strain-displacement relations and the equilibrium equation, one again arrives at the differential equation 4.3.6 so the solution for displacement and strain is again 4.3.8a-b. With $A = C_2 / (C_{12} - C_{11})$ and $C = C_1 / 2(C_{11} + C_{12})$, the stresses can be expressed as

$$
\begin{aligned}
\sigma_{rr} &= +A\frac{1}{r^2} + 2C + C_{13}\bar{\varepsilon}_{zz} \\
\sigma_{\theta\theta} &= -A\frac{1}{r^2} + 2C + C_{13}\bar{\varepsilon}_{zz} \\
\sigma_{zz} &= 4C\frac{C_{13}}{C_{11} + C_{12}} + C_{33}\bar{\varepsilon}_{zz}
\end{aligned}
\tag{4.3.21}
$$

The plane strain solution then follows from $\bar{\varepsilon}_{zz} = 0$ and the generalized plane strain solution from $\sigma_{zz} = 0$. These solutions reduce to 4.3.11, 4.3.19 in the isotropic case.

### 4.3.6 Stress Function Solution

An alternative solution procedure for axisymmetric problems is the stress function approach. To this end, first specialise equations 4.2.6 to the axisymmetric case:

$$
\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = 0 \tag{4.3.22}
$$

One can check that these equations satisfy the axisymmetric equilibrium equation 4.3.4.

The biharmonic equation in polar coordinates is given by Eqn. 4.2.7. Specialising this to the axisymmetric case, that is, setting $\partial / \partial\theta = 0$, leads to

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) = 0 \tag{4.3.23}
$$

or

$$
\frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0 \tag{4.3.24}
$$

Alternatively, one could have started with the compatibility relation 4.2.8, specialised that to the axisymmetric case:

$$
\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} = 0 \tag{4.3.25}
$$

and then combine with Hooke's law 4.3.3 or 4.3.4, and 4.3.22, to again get 4.3.24.

Eqn. 4.3.24 is an Euler-type ODE and has solution (see Appendix to this section, §4.3.8)

$$\phi = A\ln r + Br^2 \ln r + Cr^2 + D \qquad (4.3.26)$$

The stresses then follow from 4.3.22:

$$\begin{aligned} \sigma_{rr} &= +\frac{A}{r^2} + B(1 + 2\ln r) + 2C \\ \sigma_{\theta\theta} &= -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \end{aligned} \qquad (4.3.27)$$

The strains are obtained from the stress-strain relations. For plane strain, one has, from 4.3.4,

$$\begin{aligned} \varepsilon_{rr} &= \frac{1+\nu}{E}\left\{+\frac{A}{r^2} + B[1 - 4\nu + 2(1-2\nu)\ln r] + 2C(1-2\nu)\right\} \\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left\{-\frac{A}{r^2} + B[3 - 4\nu + 2(1-2\nu)\ln r] + 2C(1-2\nu)\right\} \end{aligned} \qquad (4.3.28)$$

Comparing these with the strain-displacement relations 4.3.2, and integrating $\varepsilon_{rr}$, one has

$$\begin{aligned} u_r &= \int \varepsilon_{rr}\, dr = \frac{1+\nu}{E}\left\{-\frac{A}{r} + Br[-1 + 2(1-2\nu)\ln r] + 2C(1-2\nu)r\right\} + F \\ u_r &= r\varepsilon_{\theta\theta} \quad = \frac{1+\nu}{E}\left\{-\frac{A}{r} + Br[+1 + 2(1-2\nu)\ln r] + 2C(1-2\nu)r\right\} \end{aligned} \qquad (4.3.27)$$

To ensure that one has a unique displacement $u_r$, one must have $B = 0$ and the constant of integration $F = 0$, and so one again has the solution 4.3.11[3].

## 4.3.7 Problems

1. Derive the solution equations 4.3.11 for axisymmetric plane strain.

2. A cylindrical rock specimen is subjected to a pressure *p* over its cylindrical face and is constrained in the axial direction. What are the stresses, including the axial stress, in the specimen? What are the displacements?

[3] the biharmonic equation was derived using the expression for compatibility of strains (4.3.23 being the axisymmetric version). In simply connected domains, i.e. bodies without holes, compatibility is assured (and indeed *A* and *B* must be zero in 4.3.26 to ensure finite strains). In multiply connected domains, however, for example the hollow cylinder, the compatibility condition is necessary but *not sufficient* to ensure compatible strains (see, for example, Shames and Cozzarelli (1997)), and this is why compatibility of strains must be explicitly enforced as in 4.3.25

3. A long hollow tube is subjected to internal pressure $p_i$ and external pressures $p_o$ and constrained in the axial direction. What is the stress state in the walls of the tube? What if $p_i = p_o = p$?

4. A long mine tunnel of radius *a* is cut in deep rock. Before the mine is constructed the rock is under a uniform pressure *p*. Considering the rock to be an infinite, homogeneous elastic medium with elastic constants *E* and $\nu$, determine the radial displacement at the surface of the tunnel due to the excavation. What radial stress $\sigma_{rr}(a) = -P$ should be applied to the wall of the tunnel to prevent any such displacement?

5. A long hollow elastic tube is fitted to an inner rigid (immovable) shaft. The tube is perfectly bonded to the shaft. An external pressure *p* is applied to the tube. What are the stresses and strains in the tube?

6. Repeat Problem 3 for the case when the tube is free to expand in the axial direction. How much does the tube expand in the axial direction (take $u_z = 0$ at $z = 0$)?

## 4.3.8 Appendix

### Solution to Eqn. 4.3.6

The differential equation 4.3.6 can be solved by a change of variable $r = e^t$, so that

$$r = e^t, \quad \log r = t, \quad \frac{1}{r} = \frac{dt}{dr} \tag{4.3.28}$$

and, using the chain rule,

$$\begin{aligned} \frac{du}{dr} &= \frac{du}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{du}{dt} \\ \frac{d^2u}{dr^2} &= \frac{d}{dr}\left(\frac{du}{dt}\frac{dt}{dr}\right) = \frac{d^2u}{dt^2}\frac{dt}{dr}\frac{dt}{dr} + \frac{du}{dt}\frac{d^2t}{dr^2} = \frac{1}{r^2}\frac{d^2u}{dt^2} - \frac{1}{r^2}\frac{du}{dt} \end{aligned} \tag{4.3.29}$$

The differential equation becomes

$$\frac{d^2u}{dt^2} - u = 0 \tag{4.3.30}$$

which is an ordinary differential equation with constant coefficients. With $u = e^{\lambda t}$, one has the characteristic equation $\lambda^2 - 1 = 0$ and hence the solution

$$\begin{aligned} u &= C_1 e^{+t} + C_2 e^{-t} \\ &= C_1 r + C_2 \frac{1}{r} \end{aligned} \tag{4.3.31}$$

## Solution to Eqn. 4.3.24

The solution procedure for 4.3.24 is similar to that given above for 4.3.6. Using the substitution $r = e^t$ leads to the differential equation with constant coefficients

$$\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0 \tag{4.3.32}$$

which, with $\phi = e^{\lambda t}$, has the characteristic equation $\lambda^2(\lambda - 2)^2 = 0$. This gives the repeated roots solution

$$\phi = At + Bte^{2t} + Ce^{2t} + D \tag{4.3.33}$$

and hence 4.3.24.