4.1 Cylindrical and Polar Coordinates

4.1.1 Geometrical Axisymmetry

A large number of practical engineering problems involve geometrical features which have a natural **axis of symmetry**, such as the solid cylinder, shown in Fig. 4.1.1. The axis of symmetry is an **axis of revolution**; the feature which possesses **axisymmetry** (axial symmetry) can be generated by revolving a surface (or line) about this axis.

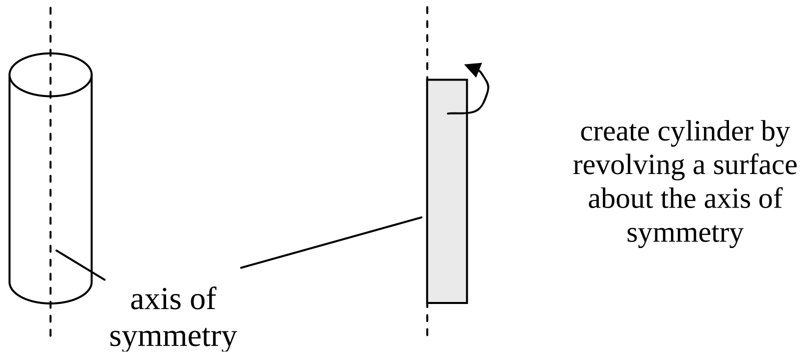


Figure 4.1.1: a cylinder

Some other axisymmetric geometries are illustrated Fig. 4.1.2; a frustum, a disk on a shaft and a sphere.

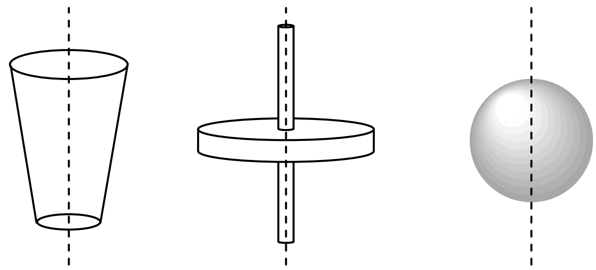


Figure 4.1.2: axisymmetric geometries

Some features are not only axisymmetric – they can be represented by a plane, which is similar to other planes right through the axis of symmetry. The hollow cylinder shown in Fig. 4.1.3 is an example of this **plane axisymmetry**.

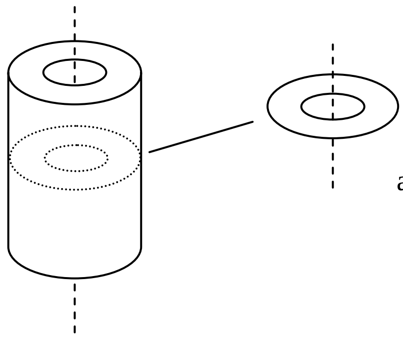


Figure 4.1.3: a plane axisymmetric geometries

Axially Non-Symmetric Geometries

Axially non-symmetric geometries are ones which have a natural axis associated with them, but which are not completely symmetric. Some examples of this type of feature, the curved beam and the half-space, are shown in Fig. 4.1.4; the half-space extends to "infinity" in the axial direction and in the radial direction "below" the surface – it can be thought of as a solid half-cylinder of infinite radius. One can also have plane axially non-symmetric features; in fact, both of these are examples of such features; a slice through the objects perpendicular to the axis of symmetry will be representative of the whole object.

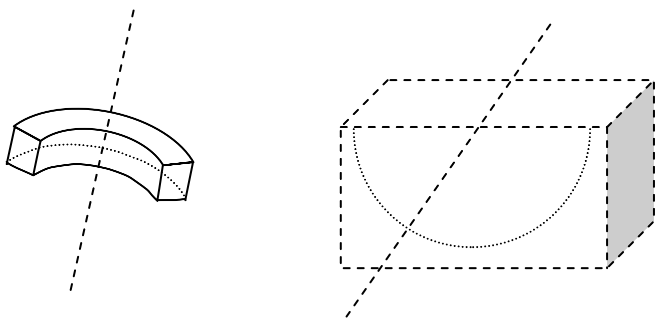


Figure 4.1.4: a plane axisymmetric geometries

4.1.2 Cylindrical and Polar Coordinates

The above features are best described using **cylindrical coordinates**, and the plane versions can be described using **polar coordinates**. These coordinates systems are described next.

Stresses and Strains in Cylindrical Coordinates

Using cylindrical coordinates, any point on a feature will have specific (r, θ, z) coordinates, Fig. 4.1.5:

r – the **radial** direction ("out" from the axis)
θ – the **circumferential** or **tangential** direction ("around" the axis – counterclockwise when viewed from the positive z side of the $z = 0$ plane)
z – the **axial** direction ("along" the axis)

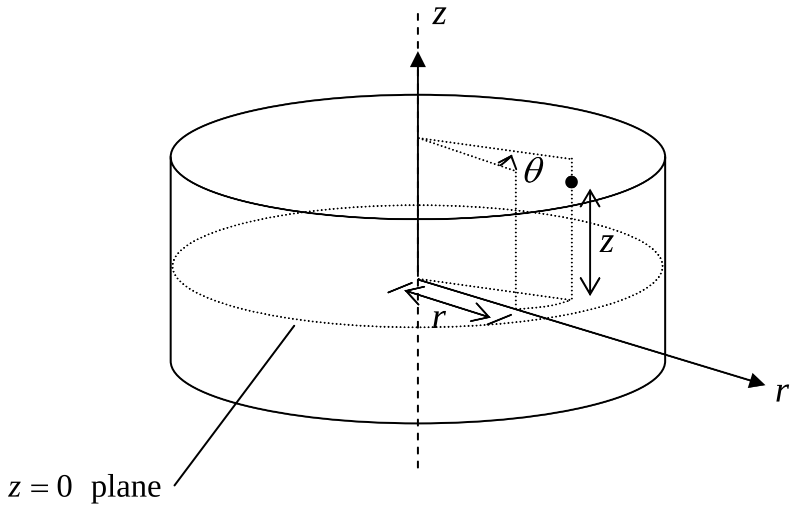


Figure 4.1.5: cylindrical coordinates

The displacement of a material point can be described by the three components in the radial, tangential and axial directions. These are often denoted by

$$u \equiv u_r, \; v \equiv u_\theta \text{ and } w \equiv u_z$$

respectively; they are shown in Fig. 4.1.6. Note that the displacement v is positive in the positive θ direction, i.e. the direction of increasing θ.

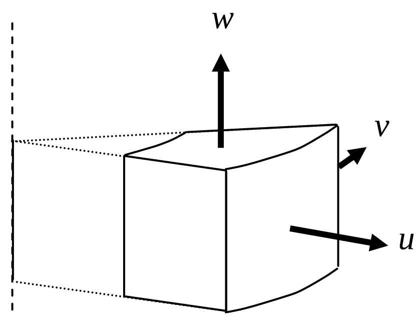


Figure 4.1.6: displacements in cylindrical coordinates

The stresses acting on a small element of material in the cylindrical coordinate system are as shown in Fig. 4.1.7 (the normal stresses on the left, the shear stresses on the right).

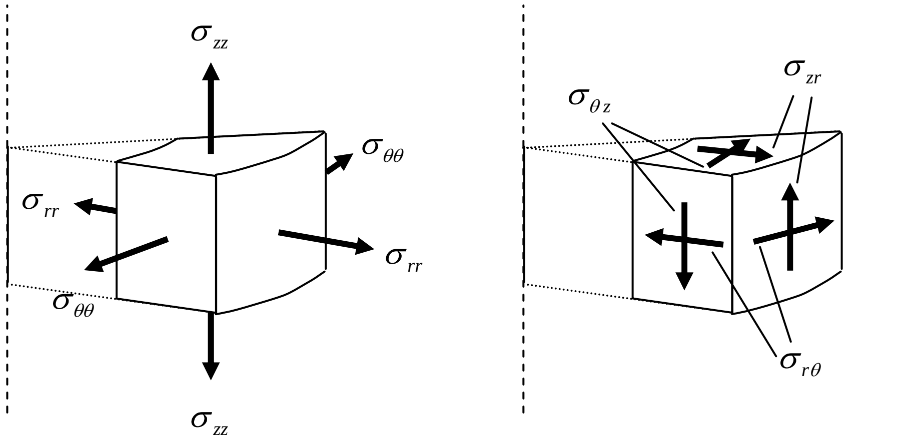
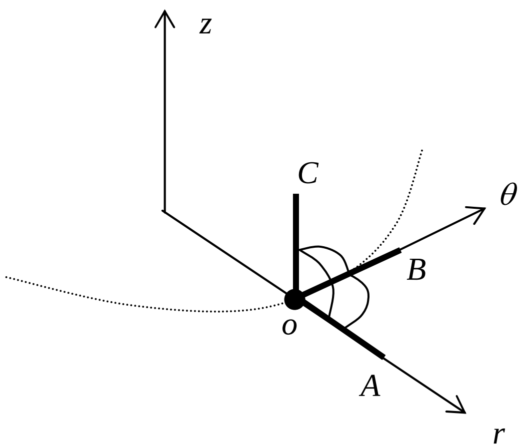


Figure 4.1.7: stresses in cylindrical coordinates

The normal strains ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{zz} are a measure of the elongation/shortening of material, per unit length, in the radial, tangential and axial directions respectively; the shear strains $\varepsilon_{r\theta}$, $\varepsilon_{\theta z}$ and ε_{zr} represent (half) the change in the right angles between line elements along the coordinate directions. The physical meaning of these strains is illustrated in Fig. 4.1.8.



<u>strain at point *o*</u>

ε_{rr} = unit elongation of *oA*
$\varepsilon_{\theta\theta}$ = unit elongation of *oB*
ε_{zz} = unit elongation of *oC*
$\varepsilon_{r\theta}$ = ½ change in angle $\angle AoB$
$\varepsilon_{\theta z}$ = ½ change in angle $\angle BoC$
ε_{zr} = ½ change in angle $\angle AoC$

Figure 4.1.8: strains in cylindrical coordinates

Plane Problems and Polar Coordinates

The stresses in any particular plane of an axisymmetric body can be described using the two-dimensional polar coordinates (r,θ) shown in Fig. 4.1.9.

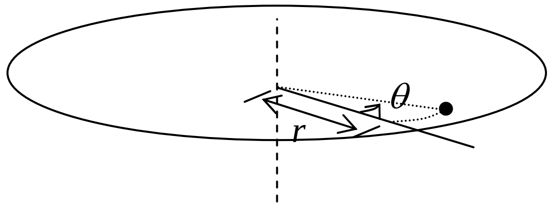


Figure 4.1.9: polar coordinates

There are *three* stress components acting *in* the plane $z = 0$: the radial stress σ_{rr}, the circumferential (tangential) stress $\sigma_{\theta\theta}$ and the shear stress $\sigma_{r\theta}$, as shown in Fig. 4.1.10. Note the direction of the (positive) shear stress – it is conventional to take the *z* axis out of the page and so the θ direction is counterclockwise. The three stress components which do not act in this plane, but which act *on* this plane (σ_{zz}, $\sigma_{\theta z}$ and σ_{zr}), may or may not be zero, depending on the particular problem (see later).

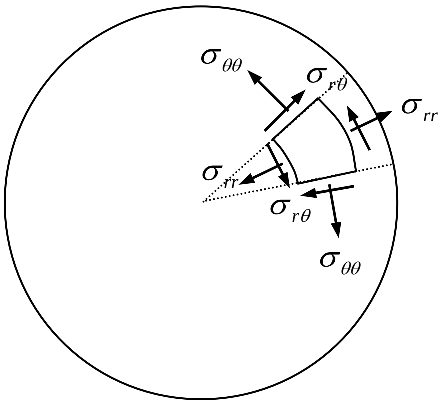


Figure 4.1.10: stresses in polar coordinates