3.1 Plane Problems

What follows is to be applicable to any two dimensional problem, so it is taken that $\sigma_{yz} = \sigma_{xz} = 0$, which is true of both plane stress and plane strain.

3.1.1 Governing Equations for Plane Problems

To recall, the equations governing the elastostatic problem are the *elastic stress-strain law* (Part I, Eqns. 4.2.11-14), the *strain-displacement relations* (Eqns. 1.2.5) and the *equations of equilibrium* (1.1.10)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right], \quad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right], \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$
$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

Plane Stress (3.1.1a)

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right], \quad \varepsilon_{yy} = \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right], \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$
$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

Plane Strain (3.1.1b)

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{aligned}$$

Strain-displacement relations (3.1.2)

$$\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\ &\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \\ &\frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{aligned}$$

Equations of Equilibrium (3.1.3)

One way of solving these equations is to re-write the stresses in 3.1.3 in terms of strains by using 3.1.1, and then using 3.1.2 to re-write the resulting equations in terms of displacements only. For example in the case of plane strain one arrives at

$$\frac{E}{2(1+\nu)(1-2\nu)}\left\{2(1-\nu)\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x \partial y}+(1-2\nu)\frac{\partial^2 u_x}{\partial y^2}\right\}+b_x=0$$
$$\frac{E}{2(1+\nu)(1-2\nu)}\left\{2(1-\nu)\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_x}{\partial x \partial y}+(1-2\nu)\frac{\partial^2 u_y}{\partial x^2}\right\}+b_y=0 \quad (3.1.4)$$

These are the 2D Navier's equations, analogous to the 1D version, Eqn. 2.1.2. This set of partial differential equations can be solved subject to boundary conditions on the displacement. Obviously, in the absence of body forces, any linear displacement field satisfies 3.1.4, for example the field

$$u_x=\frac{\sigma_o}{E}x+A-Cy, \qquad u_y=-\frac{\nu\sigma_o}{E}y+B+Cy \quad (3.1.5)$$

with A, B, C representing the possible rigid body motions; this corresponds to a simple tension $\sigma_{xx}=\sigma_o$.

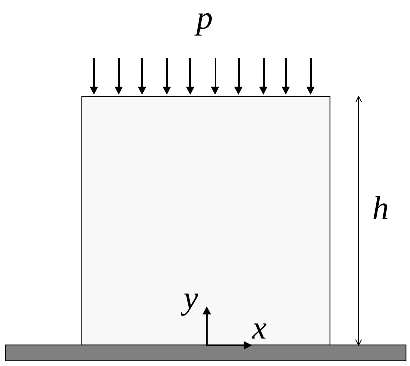
Solving Eqns. 3.1.4 directly for more complex cases is not an easy task. An alternative solution strategy for the plane elastostatic problem is the Airy stress function method described in the next section.

3.1.2 Problems

1. Derive the plane stress Navier equations analogous to 3.1.4.

2. Show that the displacement field $u_x=Ax^2$, $u_y=-4Axy/(1+\nu)$, in the absence of body forces, satisfies the plane stress governing equations derived in Problem 1 (this solution does not satisfy 3.1.4). Determine the corresponding stress field and verify that it satisfies the equilibrium equations.

3. Consider the *thin* plate shown below subjected to a uniform pressure *p* on the top and its own body weight. The plate is *perfectly bonded* to the base plate.
   (a) Does the stress distribution
   $$\sigma_{yy}(x,y)=-p+\rho g(y-h), \quad \sigma_{xx}=\sigma_{xy}=0$$
   satisfy the equations of equilibrium?
   (b) Does it satisfy the boundary conditions at the upper surface, and at the two free surfaces?
   (c) Suppose now that the plate was made out of elastic material. Show that, in that case, the stresses given above are actually not a correct solution to the problem.


p
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