## 2.1 One-dimensional Elastostatics

Consider a bar or rod made of linearly elastic material subjected to some load. Static problems will be considered here, by which is meant it is not necessary to know how the load was applied, or how the material particles moved to reach the stressed state; it is necessary only that the load was applied slowly enough so that the accelerations are zero, or that it was applied sufficiently long ago that any vibrations have died away and movement has ceased.

The equations governing the static response of the rod are:

$$\frac{d\sigma}{dx} + b = 0 \qquad \textbf{Equation of Equilibrium} \qquad (2.1.1a)$$

$$\varepsilon = \frac{du}{dx} \qquad \textbf{Strain-Displacement Relation} \qquad (2.1.1b)$$

$$\sigma = E\varepsilon \qquad \textbf{Constitutive Equation} \qquad (2.1.1c)$$

where $E$ is the Young's modulus, $\rho$ is the density and $b$ is a body force (per unit volume). The unknowns of the problem are the stress $\sigma$, strain $\varepsilon$ and displacement $u$.

These equations can be combined to give a second order differential equation in $u$, called **Navier's Equation**:

$$\boxed{\frac{d^2u}{dx^2} + \frac{b}{E} = 0} \qquad \textbf{1-D Navier's Equation} \qquad (2.1.2)$$

One requires two **boundary conditions** to obtain a solution. Let the length of the rod be $L$ and the $x$ axis be positioned as in Fig. 2.1.1. The possible boundary conditions are then

1. displacement specified at both ends ("fixed-fixed")
   $u(0) = \bar{u}_0, \quad u(L) = \bar{u}_L$
2. stress specified at both ends ("free-free")
   $\sigma(0) = \bar{\sigma}_0, \quad \sigma(L) = \bar{\sigma}_L$
3. displacement specified at left-end, stress specified at right-end ("fixed-free"):
   $u(0) = \bar{u}_0, \quad \sigma(L) = \bar{\sigma}_L$
4. stress specified at left-end, displacement specified at right-end ("free-fixed"):
   $\sigma(0) = \bar{\sigma}_0, \quad u(L) = \bar{u}_L$


**Figure 2.1.1: an elastic rod**

Note that, from 2.1.1b-c, a stress boundary condition is a condition on the first derivative of *u*.

**Example**

Consider a rod in the absence of any body forces subjected to an applied stress $\sigma_o$, Fig. 2.1.2.

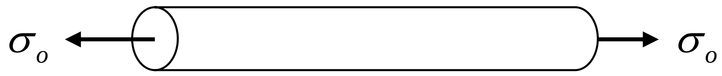


**Figure 2.1.2: an elastic rod subjected to stress**

The equation to solve is

$$\frac{d^2u}{dx^2} = 0 \qquad (2.1.3)$$

subject to the boundary conditions

$$\left.\frac{\partial u}{dx}\right|_{x=0} = \frac{\sigma_0}{E}, \qquad \left.\frac{\partial u}{dx}\right|_{x=L} = \frac{\sigma_0}{E} \qquad (2.1.4)$$

Integrating twice and applying the conditions gives the solution

$$u = \frac{\sigma_0}{E}x + B \qquad (2.1.5)$$

The stress is thus a constant $\sigma_0$ and the strain is $\sigma_o / E$. There is still an arbitrary constant *B* and this physically represents a possible rigid body translation of the rod. To remove this arbitrariness, one must specify the displacement at some point in the rod. For example, if $u(L/2) = 0$, the complete solution is

$$u = \frac{\sigma_o}{E}\left(x - \frac{L}{2}\right), \quad \varepsilon = \frac{\sigma_o}{E}, \quad \sigma = \sigma_o \qquad (2.1.6)$$

## 2.1.1 Problems

1. What are the displacements of material particles in an elastic bar of length *L* and density $\rho$ which hangs from a ceiling (see Fig. 1.1.2).

2. Consider a steel rod ($E = 210\ \text{GPa}$, $\rho = 7.85\ \text{g/cm}^3$) of length $30\ \text{cm}$, fixed at one end and subjected to a displacement $u = 1\text{mm}$ at the other. Solve for the stress, strain and displacement for the case of gravity acting along the rod. What is the solution in the absence of gravity. How significant is the effect of gravity on the stress?